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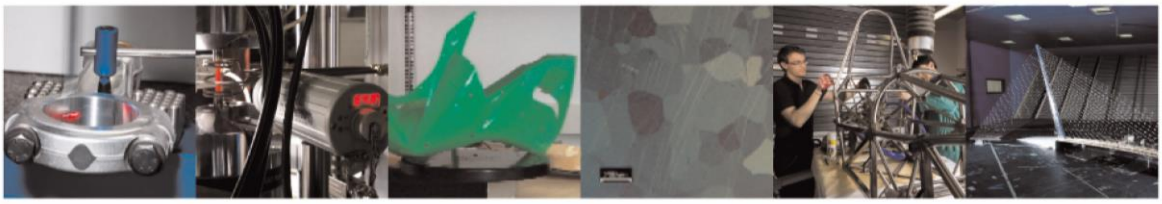

Group decision making in manufacturing systems: An approach using spatial preference information and indifference zone

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Group decision making in manufacturing systems: An approach using spatial preference information and indifference zone

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Abstract

Decision making is an essential activity in manufacturing systems when designing production lines, scheduling, etc. Many decision making problems are characterized by multiple conflicting criteria and a large number of alternatives. For these complex decision making problems, it is rational to involve a group of decision makers (DM) for considering different aspects of the problem. This paper proposes an approach for supporting the decision making group to reduce disagreement in the group and obtain a common solution. The proposed approach allows the DMs to specify a region of acceptance, known as indifference zone, in the objective space as preference inputs. This makes the proposed approach applicable to problems with a large number of alternatives. The use of indifference zone concept captures the uncertain nature of preference articulation. Moreover, the indifference zone is shown beneficial in reducing the difficulty of reaching a group common solution. The properties of the proposed method are investigated analytically and with numerical experiments. Finally, the usefulness of the proposed method is shown by tackling a real-world packaging line configuration problem with a large alternative set.

Keywords: Multi-criterion decision making; group decision making; consensus reaching; indifference zone; manufacturing systems; production line configuration.

1. Introduction

1.1. Motivation

Manufacturing can be viewed as the application of mechanical, physical, and chemical processes to modify the geometry and properties of the raw material to make the finished products. In manufacturing environments, the decision makers (DM) frequently face the problem of assessing a wide range of alternatives, and selecting one based on multiple criteria. These decision making problems arise across the entire manufacturing cycle, from vendor selection in a supply chain to the selection of machining methods and material handling equipment [1]. Generally, good decisions lead to more efficiency, profits and competitiveness of the manufacturing companies.

Making decision when taking into account multiple criteria is no trivial task. In many manufacturing scenarios, due to the conflicting nature of different criteria, the optimal alternative is generally not unique but a set of options non-dominated by each other. For example, to design a robotic assembly line, DM has to decide which type of robot to purchase and the allocation of robots to workstations. The optimal decisions correspond to a compromise among cycle time, robot setup cost and robot purchase cost [2]. Bukchin and Masin [3] studied the lot splitting problem in the flowshop system of single product. They showed that the best decisions on the

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sublot size are trading off between the system inventory level and the system throughput. Conflicting criteria exist also in production scheduling [4, 5], production-distribution planning [6], supplier selection [7, 8], product design [9], manufacturing cell design [10] and many others. To solve multi-criteria problems, making use of personal preference, which represents the understanding of the circumstance, experience and expertise of the DM, is inevitable. Take the scheduling problem as a concrete example, the multi-criteria flow shop scheduling problem is either solved in the "a priori" approach by providing the preference, e.g., weights of criteria, before running the optimization procedure; or it is tackled in the "a posteriori" approach, where the optimization algorithm generates a non-dominated set for the posteriori decision-making based on DM's preference [11]. To be specific, in this research we consider the posteriori decision-making problem.

Another feature is that the DMs in manufacturing environment are often facing a large number of alternatives. These are typically the outputs of factorial experiments or multi-objective optimization models [12]. In the tire tread compound problem, Derringer and Suich [13] used an experimental design to obtain twenty alternatives with different levels of hydrated silica, silane coupling agent and sulfur. For the assembly line balancing problem, Yoosefelahi et al. [2] generated thirty-five non-dominated solutions with the proposed multi-objective evolutionary algorithm. While in production scheduling, the number of alternatives could be even larger. As reported in Arroyo and Armentano [14], the number of non-dominated solutions found by the proposed genetic local search varies from hundreds to one thousand for a dual-objective flowshop problem with twenty to eighty jobs. Such cardinality no doubt makes the decision making difficult.

Although the given alternatives are non-dominated by each other, solutions having similar objective values are not significantly different to the DM. For this reason, the DM may prefer not only one single solution, but a set of solutions located in a certain region of the objective space. This region is known as the indifference zone (IZ) [15]. It defines the range of "close performance" where the alternatives are considered as indifferent. Actually, the indifference zone is a common concept applied in engineering problems. For examples, Boesel et al. [16] used the indifference zone of \$10 (per-unit cost) in evaluating different configurations of an automobile engine assembly line; Nazzal et al. [17] adopted the indifference zone of \$5000 on the investment cost when designing a flexible manufacturing facility. Gray and Goldman [18] applied an indifference zone of 0.365 minutes in choosing the best airspace configuration for the airport in terms of airspace route delay.

The quality of decision-making depends on the DM's preference. Hence, for complex problems, it would be too risky to rely on the opinion of only one DM due to the difficulty of considering all the relevant aspects of the problem. As a result, a group of DMs characterized by different background and knowledge are involved in the decision making process, to prevent prejudice and reduce opinion biases. Through discussion and negotiation, they reduce the level of disagreement and derive a common decision. This procedure is known as Group decision making (GDM) [19]. GDM is an essential topic for manufacturing, and it has been successfully applied to some problems like selecting enterprise resource planning (ERP) system [20], computer integrated manufacturing system [21] and advance manufacturing technology [22].

The problem investigated in this article is the group decision making procedure in manufacturing environment. We consider the industrial scenario where the DMs are facing with multiple conflicting criteria and a large number of alternatives. Each DM has his/her own preference and a set of preferred solutions located in a specified region of the objective space. We develop an approach to support the group to reach consensus and choose the final solution efficiently.

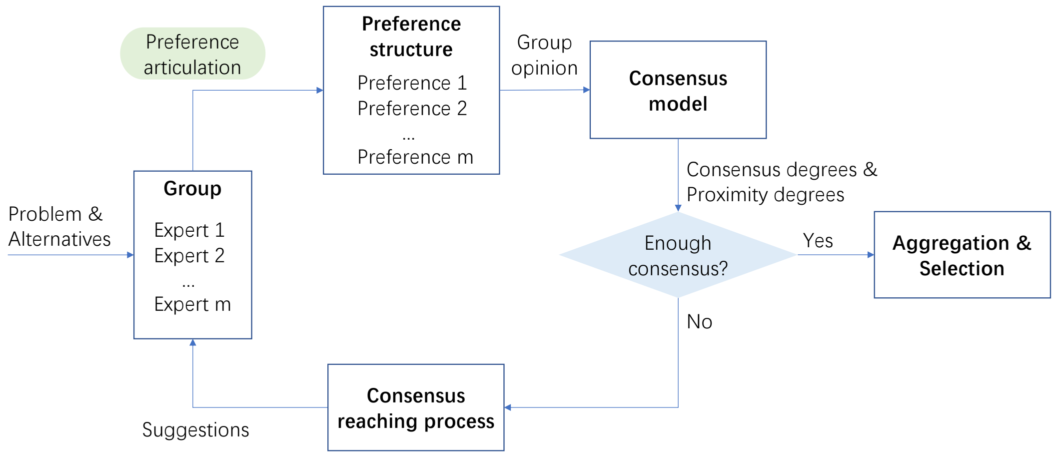


Figure 1: Framework of solving a group decision making problem

1.2. Short literature analysis

The group decision making problem has attracted many attention during the last decades [19, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The goal is to gather opinions from multiple decision makers and select a final solution. The GDM approaches can be described by the general framework depicted in Figure 1. First, the DMs express their opinions on the alternatives with certain preference structure. This procedure is also known as preference articulation $\mathscr{P} : o_k \to s_k$, which maps DM_k's opinion o_k to the preference structure s_k. For example, s_k can be a ranking of the alternatives, or can be a pairwise comparison matrix whose elements indicating the preference relation of two alternatives. The preference information are processed by a consensus model which reveals the group consensus status. Then, two processes are adopted. The *consensus reaching process* (CRP) iteratively increases the group consensus level. This process is generally facilitated by a moderator, or an algorithm, that gives feedback to the experts for opinion adjustment. After the group consensus reaches a satisfactory level, a *selection process* is undertaken to select the final solution based on the aggregated group preference. In some approaches, the group preference may be aggregated directly without using a CRP, as in [23, 25, 24].

Early researches on the GDM problem are as Herrera et al. [19]. The authors proposed a consensus model for supporting the GDM, where the DMs use linguistic labels as preference inputs. Based on several linguistic consensus degrees, the model can indicate the consensus information of the group at three levels, i.e., preference, alternative and relation, to facilitate the job of the moderator. Cabrerizo et al. [28] extended this model for unbalanced fuzzy linguistic terms. Besides linguistic information, consensus models adopting other preference structures were also proposed. For example, Xu [26] used the intuitionistic fuzzy preference relations which allow the DMs to express their uncertainties on the preference. For other preference structures, a detailed classification is described in the next paragraph. For aggregating the opinions to derive the final solution, Herrera et al. [23] developed a method to transform different preference structures into the multiplicative preference relations. Xu [24] developed several linguistic aggregation operators. Aggregation process was also developed in Herrera et al. [25] for unifying preference information in the format of numerical, interval-valued and linguistic. Although these aggregation methods can be directly applied to yield the final solution, they may sometimes lead to the solutions which are not well accepted by some DMs. To overcome this problem, the DMs should carry out a CRP, where the DMs discuss and negotiate in order to achieve a sufficient agreement[36]. Cabrerizo et al. [37]

introduced two CRPs; one is based on some identification rules and direction rules; the other one is an adaptive approach originally proposed in Mata et al. [27], which adjusts the preference change according to the group consensus state. Pérez et al. [30] developed a feedback mechanism taking into account the unequal importance of DMs. Dong et al. [33] proposed a dynamic peer-to-peer feedback mechanism that generates suggestions based on the relative positions of experts' opinions. Liao et al. [34] proposed a CRP which handles the DMs with minor views by removing some critical opinions. Dong et al. [35] used a twofold feedback mechanism to handle the non-cooperative experts by modifying their importance during the CRP. We refer the readers to Herrera et al. [36] for a comprehensive review of the GDM approaches.

The GDM approaches can be categorized by the preference structures used to represent DM opinions. These include preference orderings, utility values and preference relations [38]. With preference ordering, the opinion of DM_k is represented by a permutation $s_k = \{1, 2, ..., n\}$ which indicates the ranking of the alternatives. Using utility values, s_k is a vector $\{u_1, \ldots, u_n\}$, where $u_i \in [0, 1]$ represents the utility value of the alternative x_i. Preference relations describe the opinion by a matrix $P = [p_{ij}]_{n\times n}$, where the element p_{ij} represents the degree of how much the i-th alternative is preferred over the j-th alternative. Different types of preference relations are proposed according to the domain used to evaluate the intensity of the preference [36]: fuzzy, multiplicative, linguistic and intuitionistic fuzzy. In fuzzy preference relations [29, 39, 40, 41, 30], $p_{ij} \in [0, 1]$ and it is assumed that $p_{ij} + p_{ji} = 1, \forall i, j$. $p_{ij} = 1/2$ indicates indifference between x_i and x_j, and $p_{ij} > 1/2$ indicates x_i is preferred to x_j. In multiplicative preference relations [23, 35, 33], $p_{ij} = 1/p_{ji}$ and p_{ij} belongs to Saaty's 1-9 fundamental scale [42]. In linguistic preference relations [19, 27, 24, 25], $p_{ij} = S$, where S is a linguistic term set $S = \{s_0, \ldots, s_g\}$ with odd cardinality $g + 1$ and $s_{g/2}$ being a neutral label. In intuitionistic fuzzy preference relations [34, 26, 32, 31], to express the inaccurate cognitions of experts, the element p_{ij} is defined as a tuple of $(\mu_{ij}, v_{ij}, \pi_{ij})$, where $\mu_{ij} \in [0, 1]$ represents the preference degree of x_i to x_j, $v_{ij} \in [0, 1]$ indicates the non-preference degree of x_i to x_j and $\pi_{ij} = 1 - \mu_{ij} - v_{ij}$ is interpreted as a hesitancy degree.

It is discovered that the majority of these GDM approaches adopt the preference relations as preference input. Indeed, the preference relations retain as much as information and enable a more accurate articulation [43]. This structure has been applied to some small problems, like the scholarship candidate selection problem [34] and the brake pads suppliers selection problem [35] where only four or five alternatives are considered. Yet, as the alternative cardinality increases, the number of required pairwise judgments raises quickly. Hence, it is impractical for adopting these approaches on the problems where the alternative set is large, as in manufacturing scenarios. Moreover, it is almost inevitable to get inconsistent preference matrix due to the influence of the limited ability of human thinking. Extra efforts are required to solve the inconsistency issue in the preference matrix [44], and such efforts increase with the size of the alternative set and the preference structure complexity.

When faced with many alternatives, it is more convenient for the experts to express their preferences in the objective space. This can be done by specifying the aspiration levels of criteria directly, or by using some posterior preference articulation approaches for more accuracy. Lee et al. [45] proposed an interval selection approach to facilitate the selection of the best solution among a large number of non-dominated alternatives in the dual-response-surface. At each iteration, the front is partitioned into several intervals, and the DM chooses one from them. The selected interval is then partitioned into some sub-intervals for further selection. This procedure repeats until the alternatives within the interval are considered indifferent. To handle higher dimensional response surface, Lee et al.[46] proposed a framework based on the interactive method developed by Köksalan and Sagala [47]. This approach first partitions the objective space into some equal-sized cells. Then, it iteratively asks the DM to make the comparison between the current best alternative given by an estimated utility model and the ideal point of the least promising cell, until all non-interested cells are eliminated. Based on the space partitioning mechanism, these approaches can efficiently locate the DM's preferred region in the objective space regardless of the alternative set size.

One important feature in the problem is that the preference of DM is represented by an IZ instead of a single point. There are two advantages. First, IZ captures the preference uncertainty. Due to the limited rationality of humans, an exact preference elicitation is difficult and infeasible. It is hence more reasonable to use interval-value for opinions representation [48, 32, 49]. Second, IZ facilitates consensus making. IZ has been widely applied in the ranking & selection problem [50, 51, 52]. The goal of ranking & selection is to choose the best from a set of solutions. It is not easy due to the presence of simulation noise. However, by introducing the IZ as a tolerance, the DM can guarantee a quasi-best solution without excessive simulation efforts. Same rationality goes for the GDM procedure, in which DM can use IZ as tolerance of opinion incompatibility to reduce the difficulty of reaching group consensus.

In summary, the large alternative set is faced in many manufacturing decision making problems. Common GDM approaches based on pairwise comparison are found difficult to be applied due to the great efforts in preference articulation and potential preference inconsistency. The objective space preference information, supported by some posterior articulation techniques, provides an opportunity to tackle this issue. Yet, to the best of our knowledge, there is no GDM approach that is designed to make use of such preference structure.

1.3. Contribution

The paper proposes an approach to solve the group decision making problem in manufacturing systems. The research contribution of this paper is twofold:

- We study a group decision making problem where the preference information is provided as an indifference zone in the objective space, and propose a consensus model to measure the group consensus level and guide the consensus reaching process. Unlike common GDM models which adopt preference relations, objective space preference information can be efficiently abstracted. This enables the model to handle decision making problem with a much larger alternative set. Moreover, the indifference zone concept is integrated into group decision making to, first, represent the preference uncertainty from preference articulation; second, reduce the difficulty of reaching group common solution.
- We develop a consensus reaching process based on the peer-to-peer opinion adjustment strategy [33] for improving group consensus. We adjust the feedback mechanism to allow its implementation in the objective space with finite alternative points. More specifically, to generate feedback to DMs, we find the shortest opinion transition path on which solutions are proposed to the DMs for an efficient negotiation.

The proposed approach can be applied to many decision making problems in manufacturing systems like product design, assembly line balancing and production scheduling to improve the quality and efficiency of the decision making process.

1.4. Outline

The remaining of the paper is organized as follows. Section 2 provides a problem definition. Section 3 describes the proposed consensus model and section 4 presents the adopted consensus reaching process. In section 5, some numerical results on the efficiency and properties of the proposed approach are reported. In section 6 we apply our method to a line configuration problem, and in section 7 the conclusion is made.

2. Problem definition

To choose a final solution from an alternative set $X = \{x_1, ..., x_n\}$, a group of experts or decision makers $E = \{DM_1, ..., DM_m\}$ are involved in the decision making procedure. Each alternative x is evaluated by z quantitative criteria, and $x = (x(1), \ldots, x(z))$ where $x(i)$ is the i-th objective function value. Decision makers have equal importance but distinct preferences. As an initial step, preference articulation procedure is performed to select the most preferred alternative $x^*_k, \forall k \in 1, \ldots, m$. Then, each decision maker specifies an indifference zone $\delta_k \subseteq \mathbb{R}^z$ such that $x^*_k \in \delta_k$. Denote $S_k = \{x \in X | x \in \delta_k\}$ as the preferred solution set. Solutions in S_k are considered indifferent to x^*_k for DM_k. DM_p and DM_q reach common solution(s) if $S_p \cap S_q \neq \emptyset$. A group common solution is a solution $x^* \in S_k, \forall k$. Due to the inconsistency in group opinion, it is normal that such common solution does not exist at the beginning. For this reason, consensus has to be made. Consensus refers to the unanimity of individuals, by which the option or course of action attained will be the best representative for the entire group [36]. To this end, a consensus model is used to evaluate the group opinion state, and a CRP is applied to revise expert preference and improve the group consensus level step by step, until the group common solution is reached.

In this research, we focus on developing a consensus model and the corresponding CRP for the group decision making problem when the preference information is provided in the format of $\delta_k, k = 1, \ldots, m$.

3. Consensus model

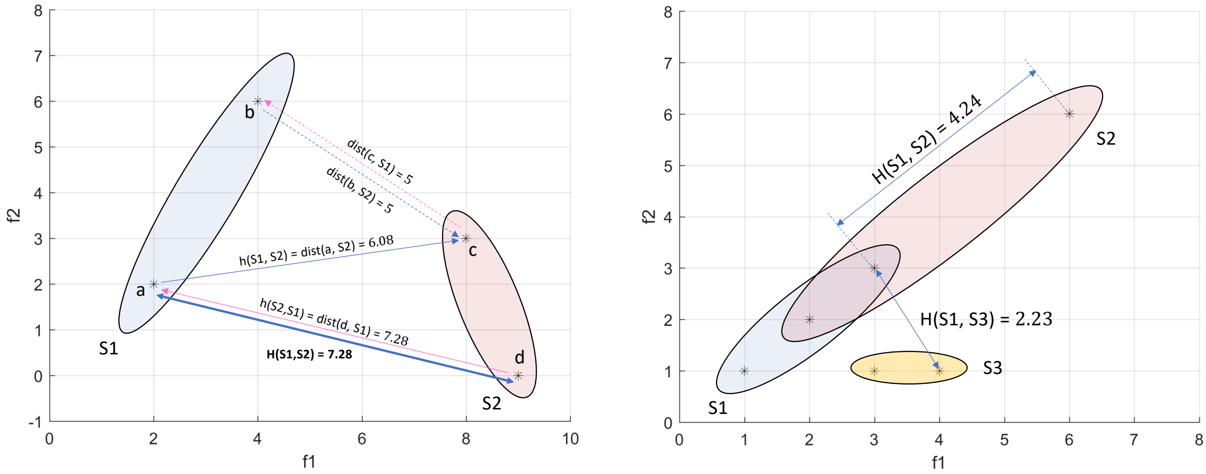
The consensus model aims at revealing the consensus status of the group and monitoring the agreement reaching process by means of several consensus indexes. Herrera et al. [19] proposed a paradigm for the consensus model with preference relation structure, which has been adopted in many other GDM researches [41, 39, 40, 30, 37, 53]. In this paradigm, two types of consensus indexes, consensus degrees and proximity measures, are calculated for helping the moderator to identify the critical preference values and the critical experts contributing to the group inconsistency. Guiding by these indexes, the CRP is performed.

By capturing the idea of this paradigm, in this section, we describe a consensus model much simpler to fulfill the basic requirements: evaluating group consensus level and guiding CRP.

3.1. Distance between opinions of decision makers

To derive the consensus model, one crucial task is to measure the closeness or degree of consensus between the opinions. In our case, the opinions of DMs are given as IZ. To simplify the problem, we use the solutions located within the IZ, i.e., S_k, to represent the spatial information. In this way, the closeness between two DMs' opinions can be considered as the distance between two sets of points.

One standard function for measuring point set distance is the Hausdorff distance [54], which is widely applied in research fields such as image matching, fractal geometry, among others. The Hausdorff distance is described as follows. Without loss of generality, in $\mathbb{R}^z$, the distance between two alternatives x and y is given by the p-norm $\|x-y\|_p$. The setting $p = 1$ corresponds to the Manhattan distance, whilst $p = 2$ refers to the Euclidean distance. In this work, we adopt the Euclidean distance.



(a) Example 1

(b) Example 2

Figure 2: Examples of distance calculation

The distance between a point x and a set Q is defined as the distance between x and the nearest point belonging to Q [54], which is given by:

$$dist(x, Q) = \min_{y \in Q} \|x - y\| \tag{1}$$

The directed Hausdorff distance between sets P and Q is defined as the maximum distance between each point $x \in P$ and the nearest point $y \in Q$, which is given by:

$$h(P, Q) = \max_{x \in P} dist(x, Q) \tag{2}$$

And finally, the Hausdorff distance between P and Q is calculated as

$$H(P, Q) = \max(h(P, Q), h(Q, P)) \tag{3}$$

An example of calculating the Hausdorff distance is given in Figure 2(a). Four points a,b,c and d have the coordinates (2,2), (4,6), (8,3) and (9,0), respectively. Let S1 = {a,b}, S2 = {c,d}. Point-set distances are marked by dashed lines; the directed Hausdorff distances are marked by solid lines; the Hausdorff distance is marked by thicker solid line.

It is observed that the Hausdorff distance is determined by the farthest point-set distance. This feature, however, may lead to unreasonable results in evaluating the closeness of expert opinions. For a better explanation, we consider the case of three DMs. The preferred solutions sets S_1, S_2 and S_3 are plotted in Figure 2(b). As shown, $S_1 \cap S_2 \neq \emptyset, S_1 \cap S_3 = \emptyset$. Intuitively, the opinion of DM1 is considered closer to DM2 than to DM3 because they share common solutions. But, when Hausdorff distance is used to measure such closeness, we obtain the opposite conclusion. More specifically, the Hausdorff distance $H(S_1, S_2)$ is larger than $H(S_1, S_3)$, this is because S_2 contains a solution which is quite distant from S_1. Indeed, the closeness measure should consider the contribution of preferred solutions in a more balanced way. To this end, we adopt the averaged Hausdorff distance.

The averaged Hausdorff distance proposed in [55] is denoted as Δ_p, where $p = 1, 2, \ldots, \infty$ is a parameter smoothing the contribution of the distant solutions. The larger p, the larger the contribution of distant solutions and when $p = \infty$, Δ_p coincides with the Hausdorff distance. In our case, to mitigate the contribution of distant solutions, we use Δ_1 to define the distance between two sets P and Q, which is given by:

$$\Delta_1(P,Q) = \max\{\frac{1}{|P|}\sum_{x\in P} dist(x,Q), \frac{1}{|Q|}\sum_{y\in Q} dist(y,P)\} \quad (4)$$

where $|\cdot|$ is the set cardinality. For example, in Figure 2(a), the averaged Hausdorff distance between S1 and S2 is calculated as

$$\Delta_1(S1,S2) = \max\{\frac{dist(a,S2)+dist(b,S2)}{2}, \frac{dist(c,S1)+dist(d,S1)}{2}\} = 6.14$$

Now we use the averaged Hausdorff distance to evaluate the opinion closeness of the three DMs (Figure 2(b)). We found that $\Delta_1(S1,S2) = 1.41$, and $\Delta_1(S1,S3) = 1.81$. The relationship $\Delta_1(S1,S2) < \Delta_1(S1,S3)$ coincides with the intuition that DM_1 and DM_2 have closer opinions.

3.2. Consensus indexes

The consensus indexes are used to serve the CRP. For a better explanation, we describe here briefly the idea of the adopted CRP whose details will be given in section 4. The CRP identifies the pair of DMs whose opinions are the most incompatible, then, opinion adjustments are proposed to them. Iteratively, the group consistency increases and finally the group consensus is obtained. To identify the critical DMs pair, we define the *individual consensus index* as in [33] but with a different closeness measure, which is given by:

Definition 1. *Let S_p and S_q be the preferred solution set of DM_p and DM_q. The individual consensus index (ICI) between DM_p and DM_q is calculated by:*

$$ICI_{pq} = \Delta_1(S_p, S_q), \forall p, q \in 1, \dots, m \quad (5)$$

ICI_{pq} inherits the semi-metric properties of Δ_p [55]: (1) (positive property) $ICI_{pq} \geq 0$ with equality if, and only if, $P = Q$, i.e., DM p and q hold identical opinions; (2) (symmetric property) $ICI_{pq} = ICI_{qp}$; (3) (relaxed triangle inequality) $ICI_{pq} \leq V(ICI_{pk} + ICI_{kq})$, where $|P|, |Q|, |K| \leq V$.

To evaluate the group consensus level, we propose two measures: Group decision maker consensus index (GDCI) and Group alternative consensus index (GACI).

Definition 2. *The group decision maker consensus index, denoted as $GDCI$, is defined as the maximum individual consensus index between any pair of decision makers within the group, given by:*

$$GDCI = \max_{p,q\in E, p\neq q}(ICI_{pq}) \quad (6)$$

$GDCI$ is used to evaluate the overall closeness of the opinions in the group. Obviously, the smaller $GDCI$ is, the closer the DMs' opinions are. When $GDCI = 0$, each pair of DMs is holding identical set of preferred solutions, the group reaches absolute consensus. Yet, absolute consensus is not a necessary condition for reaching a common solution. To verify how far is the group from reaching a common solution, the following measures are used.

Definition 3. *Let X be the set of alternatives and E be the set of DMs, $x_i \in X$ represents the i-th alternative in the objective space. $S_k, k \in 1, \dots, m$ is the preferred solution set of the k-th decision maker. The consensus* *index of the i-th alternative, denoted as ACI_i, is defined as the averaged distance of the alternative to the preferred solution sets of all decision makers, given by:*

$$ACI_i = \frac{1}{|E|} \sum_{k \in E} dist(x_i, S_k) \tag{7}$$

The ACI value evaluates the averaged distance of a specific alternative to the group opinions. A zero ACI_i value indicates that x_i is a common solution for the group.

Definition 4. *The group alternative consensus index, denoted as $GACI$, is defined as the minimum alternative consensus index among the alternative set, given by:*

$$GACI = \min_{i \in X}(ACI_i) \tag{8}$$

When $GACI = 0$, there exists at least one solution which falls simultaneously in the IZs of all DMs, i.e., there is at least one common solution. In summary, to monitor the group consensus, $GDCI$ measures how far the group is away from absolute consensus, $GACI$ measures how far the group is away from reaching a common solution.

4. Consensus reaching process

In the group CRP, it is necessary that the DMs adjust their preferences or opinions according to the suggestions of the moderator. Numerous CRPs have been proposed [40, 30, 37, 53, 27, 38], and most of them are based on the aforementioned paradigm [19]. The core functionality is the feedback mechanism, which generates the suggestions by tackling the following questions: (a) which DMs have to modify their preferences; (b) how to modify the DMs' preference information.

In our case, DMs' opinions are represented by indifference zones distributed in the objective space. To make consensus, the process involves the shifting of these zones to reduce the dispersion of group opinion. To this end, we apply the peer-to-peer opinion adjustment framework proposed in [33], which is found quite appropriate for this purpose. The framework is shown in Figure 6. At each iteration, the DM pair that hold the most incompatible opinions are identified; and feedbacks are generated to them for preference modification. The DM can either accept or reject the request. If he/she rejects, the feedback strategy is adjusted to avoid further rejection. This cycle repeats until a group common solution is obtained or other termination conditions are met.

In the peer-to-peer strategy, improving consensus between a pair of DMs acts as a building block. For this reason, in this section, we first describe the feedback mechanism to improve consensus between two decision makers; then, the complete procedure for the group is presented.

4.1. Feedback mechanism

In Dong et al. [33], the feedback mechanism advises the DM pair to move their preference to certain intermediate points between them by applying a weighted product operator on the preference matrix. This, however, cannot be applied in our case due to first, the preference structure is different; second, unlike in the continuous space of preference matrix, we need to consider the feasibility in the objective space, i.e., the proposed region should contain feasible alternatives. For this reason, we propose an approach based on the opinion transition path to suggest intermediate points.

We firstly consider a special case where DM has only one preferred alternative in their IZs. Denote these two alternatives as x_p and x_q, respectively. The consensus reaching between DM_p and DM_q can be considered as a negotiation procedure in which they shift their preferred solutions towards each other. Each move is called a *transition*. We assume a transition could happen only between neighboring solutions, i.e., $x \to y, y \in \mathcal{N}(x)$. To define the neighbors of a query point, common methods are like the r-neighbors, which defines the points located within a radius r as neighbors; or the the k-nearest neighbors, which defines the first k nearest points as neighbors. Yet, both methods require a user-specific parameter, whose value is generally not easy to determine properly. To avoid this, a parameter-free neighborhood structure, known as the Voronoi neighbors [56], is used. The definition of Voronoi neighbors is given in Definition 5. To give an example, Figure 3 shows the Voronoi neighbors of a query point.

Definition 5. *Given a set of points N ($|N| \geq 3$) and a query point $x \in N$, the Voronoi neighbors of x, denoted as $\mathcal{N}(x)$, are the points whose Voronoi polygons share edges with the Voronoi polygon of x.*

Under the assumption of opinion transition, we can construct the opinion transition path along which the negotiators would shift their preferred solutions. To guarantee the efficiency of negotiation, the shortest path connecting x_p and x_q, denoted as $\Theta(x_p, x_q)$, is obtained as follows. Let $G = (N, \mathscr{E})$ be an undirected graph with nodes N and edges $\mathscr{E}$ connecting the neighboring nodes, i.e., $\mathscr{E} = \{(x, y)|x, y \in N, y \in \mathcal{N}(x)\}$. The length of edge (x, y) is given by $\|x - y\|$. Then, the shortest transition path linking x_p and x_q can be found by applying the Dijkstra algorithm [57].

To improve the consensus degree, negotiators are suggested to move their solutions to some intermediate alternatives on the opinion transition path, denoted as x'_p and x'_q, respectively. To select x'_p and x'_q, the following procedure is used. Let $\Theta(x_p, x_q) = \{\Theta(1), \ldots, \Theta(l)\}$ be the transition path between points x_p and x_q, where $\Theta(i)$ is the i-th node on the path, and $\Theta(1) = x_p$, $\Theta(l) = x_q$. Denote $cdist(i, j)$ as the cumulated distance between $\Theta(i)$ and $\Theta(j)$, which is given by $cdist(i, j) = \sum_{i \leq k < j} dist(\Theta(k), \Theta(k + 1))$. Let the total path length be $L = cdist(1, l)$. Then, the intermediate points x'_p and x'_q are located as

$$x'_p = \Theta(k^*), k^* = \underset{k=1,\ldots,l}{\arg\min} |cdist(1, k) - (1 - \alpha_p) * L| \tag{9}$$

$$x'_q = \Theta(k^*), k^* = \underset{k=1,\ldots,l}{\arg\min} |cdist(k, l) - (1 - \alpha_q) * L| \tag{10}$$

where $\alpha_k, k \in [p, q]$ is a user-defined parameter adjusting the transition distance, we name it the opinion preservation degree. In the formula, $(1 - \alpha_k) * L$ calculates the transition distance that DM_k is suggested to move for improving consensus. The proposed solution x'_k is the one closest to the suggested distance. It is obvious that the larger α_k is, the more the DM_k is able to stick to his/her original opinion. For example, when $\alpha_p = 1$ and $\alpha_q = 0$, DM_q is suggested to abandon his/her preferred solution and move to x_p whilst DM_p can retain his/her original opinion. However, to preserve the sovereignty of DMs, we use $\alpha_k \in [0.5, 1], k \in [p, q]$ so DMs are suggested to move to some intermediate alternatives in the transition path. If the special case $(x'_p = x_p \wedge x'_q = x_q)$ happens, we set $x'_p = \Theta(2)$ or $x'_q = \Theta(l - 1)$ to guarantee at least one transition is made.

It can be easily proved that consensus can be reached between two negotiators after a finite number of iterations of the above procedure. An example of the opinion transition path and the proposed intermediate

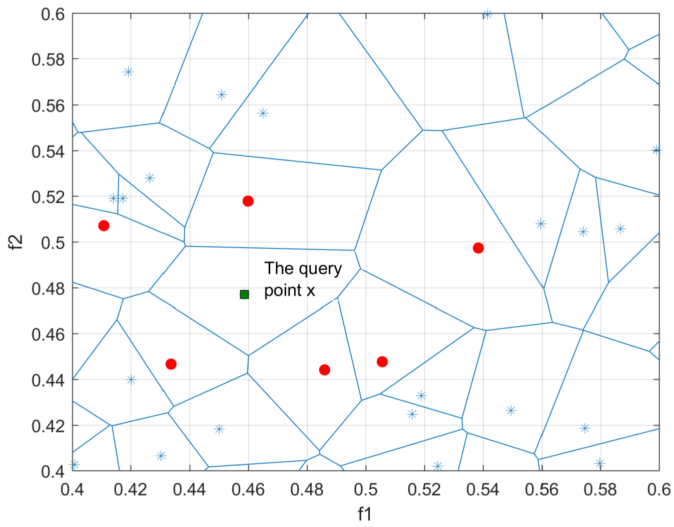


Figure 3: Example of the Voronoi neighbors of a query point

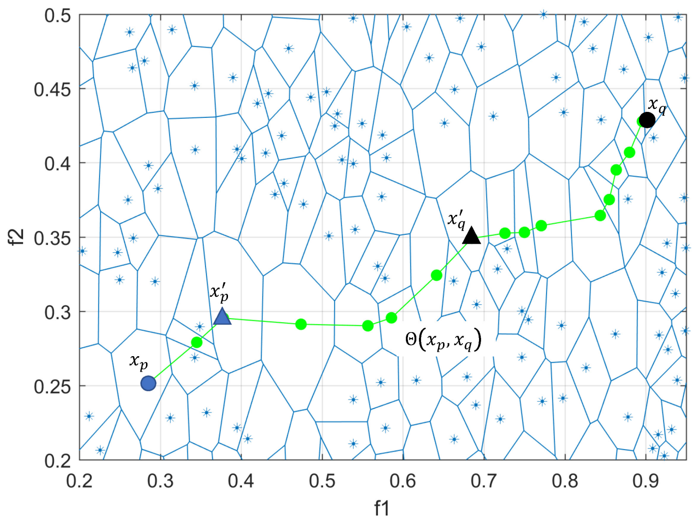


Figure 4: Example of generating suggestions for two decision makers with single preferred solution

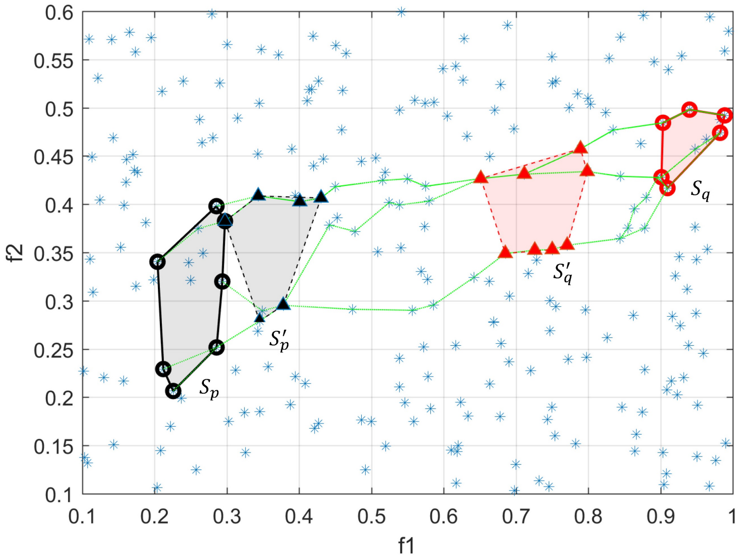


Figure 5: Example of generating suggestions for two decision makers with multiple preferred solutions

points is illustrated in Figure 4.

Now we consider the general case where both DM_p and DM_q have a set of preferred solutions, denoted as S_p and S_q, respectively. Since the alternatives in IZs are considered indifferent, the problem of finding intermediate solutions between S_p and S_q can be decomposed into a series of sub-problems of finding intermediate point between every possible pair of $x_p \in S_p$ and $x_q \in S_q$. This approach is simple but time-consuming because the number of possible paths linking S_p and S_q equals $|S_p| \cdot |S_q|$. To reduce the computational burden, we use only a subset of S_p and S_q to represent their spatial information. The procedure is described below.

Let V_p and V_q be the vertices of the convex hull of set S_p and S_q, respectively. More specifically, $V_k = \text{Vert}(S_k) = \{x \in S_k | x \notin \text{Conv}(S_k \backslash x)\}, k \in \{p, q\}$, where $\text{Conv}(\cdot)$ is the convex hull of a set. Denote $\mathbb{T} = \{\Theta_1, \Theta_2, \ldots\}$ as the set of all possible opinion transition paths linking $x_p \in V_p$ and $x_q \in V_q$. Initialize $V'_k = \emptyset, k \in \{p, q\}$. For each path $\Theta \in \mathbb{T}$, locate two intermediate points x'_p and x'_q according to Equation (9) and (10) , and update $V'_k = V_k + \{x'_k\}, k \in \{p, q\}$. Finally, we obtain the intermediate alternative set $S'_k = \{x | x \in \text{Conv}(V'_k)\}, k \in \{p, q\}$.

In summary, the feedback mechanism can be denoted as a function $[S'_p, S'_q] = \text{feedback}(S_p, S_q, \alpha_p, \alpha_q)$. An example of applying the feedback function between two negotiators is plotted in Figure 5, where $\alpha_p = 0.842$ and $\alpha_q = 0.676$. As shown, DM_p makes a smaller concession due to a larger α value.

4.2. The complete process

In the complete CRP, at each iteration, say t, the critical DM pair is identified and the feedback mechanism is used to generate suggestions for opinion adjustment. One problem here is how to preserve the original judgment of DMs, or saying, how to decide the amount of concession between two negotiators. As suggested by [33], the judgment preservation should be related to the proximity of his/her opinion to the rest of the group. Generally, the higher the proximity, the more reliable his/her opinion is and therefore the less concession he/she should make. For this reason, the opinion preservation degree α in Equation (9) and (10) are calculated as below:

$$\alpha_p = 1 - \frac{\sum\limits_{k=1,k\neq p,q}^{m} ICI_{kp}}{2(\sum\limits_{k=1,k\neq p,q}^{m} ICI_{kp} + \sum\limits_{k=1,k\neq p,q}^{m} ICI_{kq})} \tag{11}$$

$$\alpha_q = 1 - \frac{\sum\limits_{k=1,k\neq p,q}^{m} ICI_{kq}}{2(\sum\limits_{k=1,k\neq p,q}^{m} ICI_{kp} + \sum\limits_{k=1,k\neq p,q}^{m} ICI_{kq})} \tag{12}$$

In these formula, the numerator of the second term measures the incompatibility between the DM and the rest of the group, whereas the denominator is the summation of group incompatibility of the DM and his/her negotiator. It is observed that $\alpha_k, k \in \{p,q\}$ is bounded to $[0.5, 1]$, this guarantees that at each turn the suggested transition distance will not exceed $L/2$ in order to preserve the sovereignty of the DMs. Parameter α acts as an indicator measuring the relative opinion proximity to the group. Therefore, if the special case $(x'_p = x_p \wedge x'_q = x_q)$ occurs, it is reasonable to suggest the DM with the smaller α to do the minimum transition.

Another problem is how to adjust the feedback strategy if a decision maker rejects the provided suggestion. Apparently, if DM_p rejects the suggestion to move towards DM_q, it implies that there may exist large disagreement in their opinions, and he/she would probably not revise his/her opinion according to DM_q in the remaining process. To better manage this phenomenon, we introduce the definition of ordered pair from [33]:

Definition 6. *Let $E = \{DM_1, \ldots, DM_m\}$ be a set of m decision makers. An ordered pair of decision makers in the group is defined as $(DM_p, DM_q), DM_p, DM_q \in E$, where (DM_p, DM_q) indicates that DM_p may revise his/her judgments according to DM_q.*

From definition 6 it is clear that $(DM_p, DM_q) \neq (DM_q, DM_p)$. We use set $\mathfrak{D}$ to represent all active ordered pairs in the group, which is given by:

$$\mathfrak{D} = \{(DM_1, DM_2), \ldots, (DM_1, DM_m), \ldots, (DM_m, DM_1), \ldots, (DM_m, DM_{m-1})\} \tag{13}$$

Whenever DM_p rejects to revise his/her opinion according to another DM, say DM_q, we remove the corresponding ordered pair from $\mathfrak{D}$ and avoid to generate suggestion for DM_p based on the opinion of DM_q.

Given that the alternative set could be quite large, some alternatives distant from the group preference can be pruned to reduce the computational burden. For this end, we use the following pruning procedure. Let $S_E = \bigcup_{k\in 1,\ldots,m} S_k$ be the union of DMs' preferred solution sets, which represents the group opinion. Let d^* be the diameter of set S_E, i.e., $d^* = \max\limits_{x,y\in S_E} \|x-y\|$. Then, any alternative whose distance to S_E is larger than d^*, i.e., $\{x \in X | dist(x, S_E) > d^*\}$, is pruned because it is not interested by the group.

By gathering all building blocks, the CRP is described as follows.

Inputs: Alternative set X; initial indifference zone $\delta_k, \forall k$; the maximum number of iterations T.

Outputs: Final preferred solution sets of DMs $S_k^*, \forall k$; group common solution(s); and the iteration counter t^*, $0 \leq t^* \leq T$.

Step 1. Derive the preferred solution sets $S_k, \forall k$ from $\delta_k, \forall k$. Initialize the active ordered pair set $\mathfrak{D}^0$ by Equation (13), set $t \leftarrow 0, S_k^0 \leftarrow S_k, \forall k \in 1, \ldots, m$.

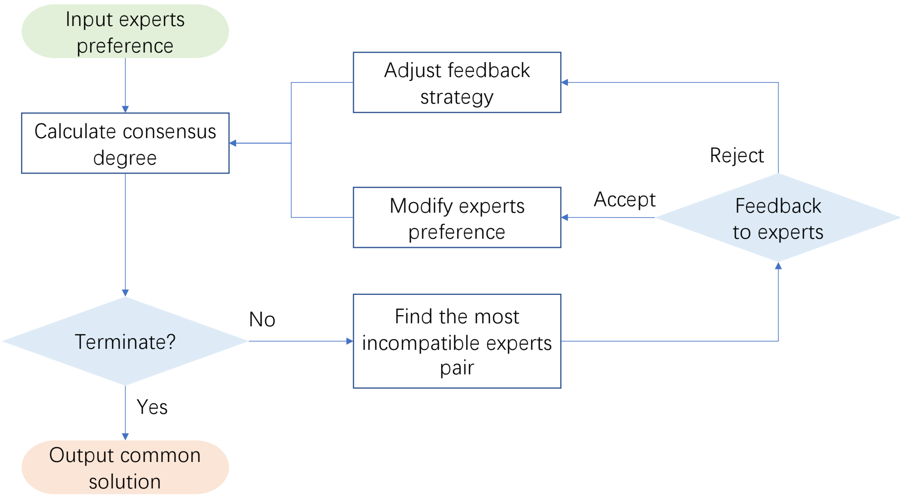


Figure 6: Flow chart of the peer-to-peer opinion adjustment strategy

Step 2. Prune the alternative set X by removing alternatives distant from the group opinion.

Step 3. Calculate the individual consensus index $ICI^t_{p,q}$ with Equation (5) for all $(DM_p, DM_q) \in \mathfrak{D}^t$, $GDCI^t$ with Equation (6), and $GACI^t$ with Equation (8). If one of the following termination conditions (1) $t \geq T$, (2) $\mathfrak{D}^t = \emptyset$, (3) $GACI^t = 0$ meets, go to Step 5; otherwise continue.

Step 4. Identify the most incompatible decision makers DM_a and DM_b, with $ICI^t_{ab} = \max_{\mathfrak{D}^t} ICI^t_{pq}$. At least one of the ordered pair (DM_a, DM_b) or (DM_b, DM_a) should be in $\mathfrak{D}^t$. Calculate the opinion preservation degree $\alpha_k, k \in [a, b]$ with Equation (11) and (12). Generate suggestions using the feedback mechanism $[S^{t+1}_{sug,a}, S^{t+1}_{sug,b}] =$ feedback $(S^t_a, S^t_b, \alpha_a, \alpha_b)$, and feedback to the DMs according to the following cases:

1. Both (DM_a, DM_b) and (DM_b, DM_a) are in $\mathfrak{D}^t$: Suggestions $S^{t+1}_{sug,a}$ and $S^{t+1}_{sug,b}$ are given to DM_a and DM_b, respectively.
2. Only (DM_a, DM_b) ((DM_b, DM_a)) is in $\mathfrak{D}^t$: Suggestion $S^{t+1}_{sug,a}$ ($S^{t+1}_{sug,b}$) is given to DM_a (DM_b).

Upon receiving the suggestion, the DM, e.g., DM_a, has two options:

1. Accept the suggestion. In this case, we set $S^{t+1}_a \leftarrow S^{t+1}_{sug,a}$.
2. Reject the suggestion and refuse to revise opinion according to his/her negotiator, in this case, DM_b, any longer. We set $S^{t+1}_a = S^t_a$, and update $\mathfrak{D}^{t+1} \leftarrow \mathfrak{D}^t - (DM_a, DM_b)$.

For any other decision makers $DM_k \in E, k \neq a, b$, set $S^{t+1}_k \leftarrow S^t_k$. Update $t \leftarrow t + 1$, and return to Step 2.

Step 5. Terminate the procedure. Output the obtained group common solution(s) , $t^* = t$, $S^*_k = S^t_k, \forall k$.

When the procedure is terminated, there could be two situations: (1) One or more common solutions are obtained. In this case, any of them can be chosen as the final solution because they are equivalent for the group; (2) No common solution is obtained. This may happen if there exist big disagreements among the group, or DMs refuse to revise their opinions during the consensus reaching procedure.

The proposed CRP has been proved to be convergent in the case of infinite number of alternatives and homogeneous distribution, i.e., the solution space is continuous, details are in Appendix A. This implies that the CRP can always lead to a common solution if the group follows the suggestions. We also provide a bound for the required iterations for reaching convergence in Appendix B, using the same assumption of Appendix A. This analysis shows a time complexity of $O(m^2)$, where m is the number of DMs. This implies that the efficiency of the CRP is neither affected by the number of alternatives nor by the solution space dimension, yet the group size may be a critical factor. It should be noted that this time complexity of $O(m^2)$ is for the rounds of discussion to reach consensus. Since we use the Dijkstra algorithm to define the path between two DMs, the nested complexity is higher because the Dijkstra algorithm has a time complexity that depends on the number of solutions. However, in real-world cases, group discussions are generally taking much longer time than running the computer algorithm. For this reason, we consider $O(m^2)$ as a proper description for the method efficiency.

5. Numerical results

5.1. Illustrative example

In this section, we give an illustrative example of the proposed consensus model and CRP. A group of three DMs chooses a final solution from a large set of 500 alternatives. Without loss of generality, we randomly sample the alternatives in 2-dimensional objective space with $f_1, f_2 \in [0, 1]$. The IZs of the DMs are given in Figure 7(a). More specifically, $R_1 = \text{Square}[(0.1, 0.7), (0.2, 0.8)]$, which means for DM1, alternatives with $f_1 \in [0.1, 0.2] \wedge f_2 \in [0.7, 0.8]$ are preferred and indifferent. $R_2 = \text{Square}[(0.2, 0.2), (0.3, 0.4)]$, and $R_3 = \text{Square}[(0.9, 0.4), (1.0, 0.5)]$.

Since no common solution exists for the group, the proposed CRP is applied. The maximum iteration T is set as 12. The ordered pair of DMs is initialized as

$$\mathfrak{D}^0 = \{(DM_1, DM_2), (DM_1, DM_3), (DM_2, DM_1), (DM_2, DM_3), (DM_3, DM_1), (DM_3, DM_2)\}$$

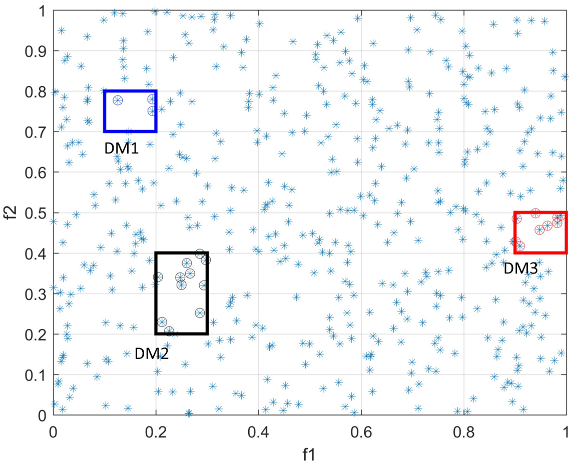
Iteration 0: The individual consensus index matrix is calculated using Equation (5):

$$(ICI_{pq}^0)_{m\times m} = \begin{bmatrix} 0 & 0.4374 & \mathbf{0.8054} \\ 0.4374 & 0 & 0.6558 \\ 0.8054 & 0.6558 & 0 \end{bmatrix}$$

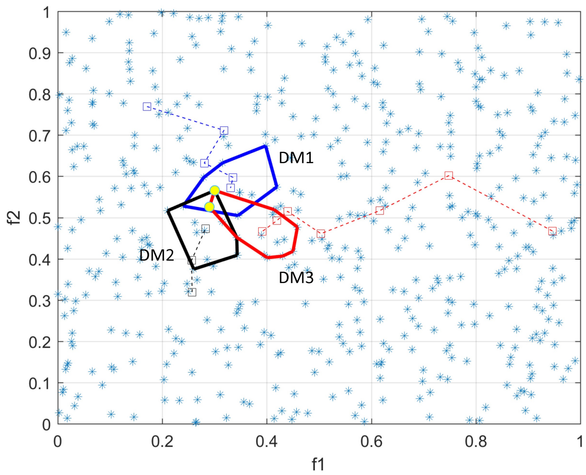
By Equation (6) and (8), we obtain $GDCI^0 = 0.8054$ and $GACI^0 = 0.3233$. The most incompatible decision maker pair is DM_1 and DM_3. To increase their consensus level, the feedback mechanism is used to generate new preferred solutions. Using Equation (11) and (12) we obtain $\alpha_1^0 = 0.7999, \alpha_3^0 = 0.7001$. For DM_1, four solutions, denoted as $S_{sug,1}^1$, are proposed, whose f_1 varies from 0.248 to 0.380, f_2 varies from 0.675 to 0.759; For DM_3, $S_{sug,3}^1$ contains four solutions as well, with $f_1 \in [0.688, 0.802]$ and $f_2 \in [0.553, 0.648]$. Both decision makers accept the suggestions, then we update $S_1^1 = S_{sug,1}^1, S_3^1 = S_{sug,3}^1$, and $S_2^1 = S_2^0$.

Iteration 1 - 7: The information of iteration 1 to 7 is summarized in Table 1. During the CRP, DM_2 rejects to revise according to DM_3 at iteration 1, while in other cases, the decision makers accept the suggestions given by the feedback algorithm. At iteration 7, the individual consensus index matrix is as below:

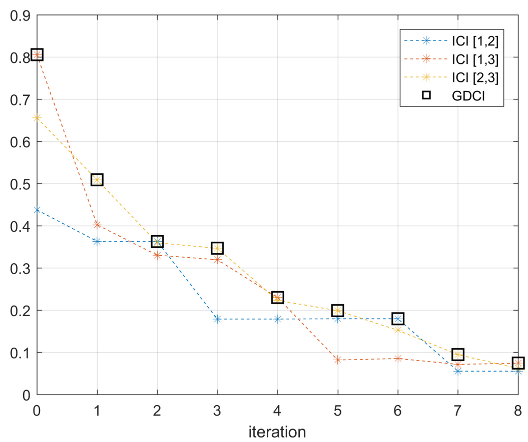
$$(ICI_{pq}^7)_{m\times m} = \begin{bmatrix} 0 & 0.0551 & 0.0713 \\ 0.0551 & 0 & \mathbf{0.0943} \\ 0.0713 & 0.0943 & 0 \end{bmatrix}$$



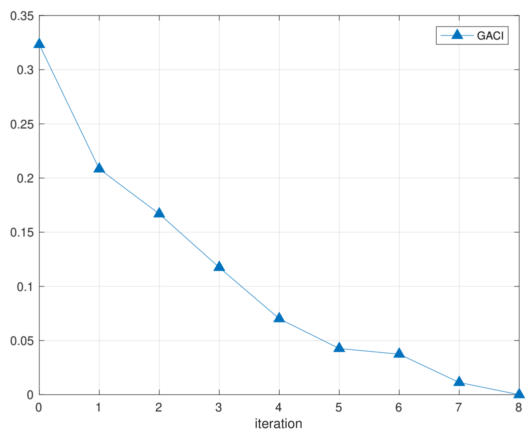
(a) Initial state



(b) Final state



(c) GDCI



(d) GACI

Figure 7: Illustrative example

Table 1: Consensus reaching process of the illustrative example: Iteration 1 to 7

Iteration	ICI matrix	DM pair	α value	$\lvert S_{sug}\rvert$	Range of proposed solutions	Accept ?
1	$\begin{bmatrix} 0 & 0.3630 & 0.4022 \\ 0.3630 & 0 & \mathbf{0.5088} \\ 0.4022 & 0.5088 & 0 \end{bmatrix}$	2	0.7628	9	f1: [0.343, 0.442] f2: [0.278, 0.439]	NO
		3	0.7372	11	f1: [0.559, 0.679] f2: [0.471, 0.572]	YES
2	$\begin{bmatrix} 0 & \mathbf{0.3630} & 0.3301 \\ 0.3630 & 0 & 0.3594 \\ 0.3301 & 0.3594 & 0 \end{bmatrix}$	1	0.7606	9	f1: [0.225, 0.327] f2: [0.566, 0.696]	YES
		2	0.7394	9	f1: [0.169, 0.297] f2: [0.321, 0.488]	YES
3	$\begin{bmatrix} 0 & 0.1787 & 0.3193 \\ 0.1787 & 0 & \mathbf{0.3465} \\ 0.3193 & 0.3465 & 0 \end{bmatrix}$	3	0.6794	9	f1: [0.411, 0.575] f2: [0.566, 0.696]	YES
		2	0.8206	-	- -	-
4	$\begin{bmatrix} 0 & 0.1787 & \mathbf{0.2298} \\ 0.1787 & 0 & 0.2231 \\ 0.2298 & 0.2231 & 0 \end{bmatrix}$	1	0.7776	6	f1: [0.279, 0.398] f2: [0.555, 0.674]	YES
		3	0.7224	8	f1: [0.371, 0.506] f2: [0.448, 0.556]	YES
5	$\begin{bmatrix} 0 & 0.1794 & 0.0817 \\ 0.1794 & 0 & \mathbf{0.1987} \\ 0.0817 & 0.1987 & 0 \end{bmatrix}$	3	0.8436	10	f1: [0.354, 0.486] f2: [0.418, 0.561]	YES
		2	0.6564	-	- -	-
6	$\begin{bmatrix} 0 & \mathbf{0.1794} & 0.0851 \\ 0.1794 & 0 & 0.1522 \\ 0.0851 & 0.1522 & 0 \end{bmatrix}$	1	0.8206	10	f1: [0.239, 0.419] f2: [0.505, 0.674]	YES
		2	0.6794	12	f1: [0.210, 0.343] f1: [0.375, 0.566]	YES
7	$\begin{bmatrix} 0 & 0.0551 & 0.0713 \\ 0.0551 & 0 & \mathbf{0.0943} \\ 0.0713 & 0.0943 & 0 \end{bmatrix}$	3	0.718	14	f1: [0.290, 0.459] f2: [0.402, 0.566]	YES
		2	0.782	-	- -	-

The procedure stops because two common solutions are found for the group, as shown in Figure 7(b). In Figure 7(c) and (d) are illustrated the trend of GDCI and GACI during the consensus reaching procedure. As shown, these two indexes continue to decrease as the procedure goes, which brings the group to a high consensus level and, finally, to common solution(s). It should be noted that common solution can be reached despite some rejections from the DMs.

5.2. Simulation campaign

In this section, we launch a simulation campaign to study the properties of the proposed method in terms of effectiveness and efficiency. We aim at answering the following questions by numerical experiments:

1. Could we always obtain common solution(s) if the group follows all suggestions?
2. What is the contribution of the indifference zone in group consensus making?
3. Is the efficiency of the proposed approach sensitive to the cardinality of the alternative set or the group size?

We perform a full factorial design of experiments (DOE) with the following factors and levels:

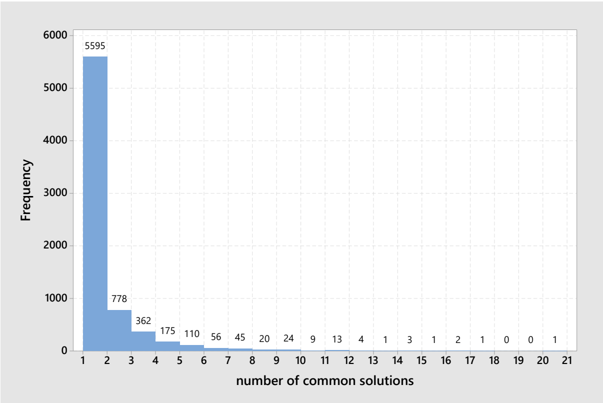
- Number of solutions, n: two levels (500, 750)
- Number of decision makers, m: three levels (3, 5 and 7)
- Indifference zone size, $Size$: three levels (0, 0.1 and 0.2)
- Number of objective space dimensions, z: two levels (2, 3)

n alternatives are randomly sampled in the z-dimensional objective space $[0,1]^z$. m decision makers are involved, whose initial IZs are randomly specified as a hypercube with edge length $Size$. Note that $Size = 0$ represents the special case in which DM holds only one preferred solution. All these factors make $2 \cdot 3 \cdot 3 \cdot 2 = 36$ different problem settings. For each setting, 200 instances are generated and tested, resulting in 7200 GDM problems. The proposed approach is applied with a maximum iteration $T = 150$. We record in each experiment the number of iterations required for reaching common solution(s), and the number of common solutions obtained.

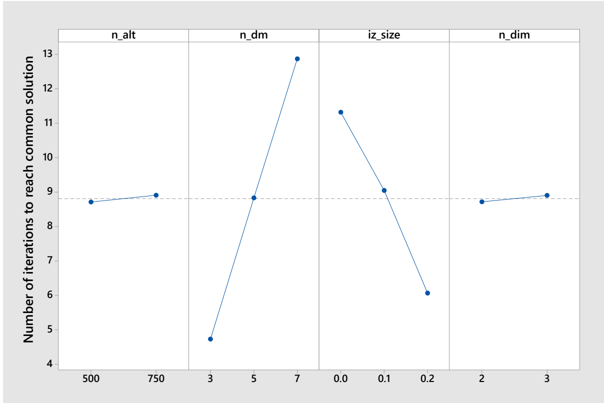
The number of common solutions found in each experiment is summarized in Figure 8(a). In all experiments, at least one common solution is obtained. In most cases, one to five common solutions are obtained. The cases with a large number of common solutions are due to the large indifferent zones. This result answers the first question previously raised and validates the convergence proof provided in Appendix A.

The number of required iterations is analyzed with ANOVA. As show in Figure 8(c), all main factors, as well as some interactions between them, are significant. The most influential factors are m (or n_dm in the table) and $Size$. From Figure 8(b) we have some observations:

- The larger the IZs, the less iterations are required to reach common solution(s). The IZ indicates not only the uncertainty but also the willingness of DM to reach consensus with others. When DMs hold larger IZs, it becomes easier for the group to find overlapping zones in the objective space, which facilitates the consensus reaching of the group.



(a) Histogram of found common solutions in 7200 experiments



(b) Main effect plot of iteration number

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
n_alt	1	70	69.6	26.64	0.000
n_dm	2	79497	39748.4	15210.16	0.000
iz_size	2	33329	16664.3	6376.79	0.000
n_dim	1	62	62.0	23.72	0.000
n_alt*n_dm	2	9	4.3	1.66	0.190
n_alt*iz_size	2	228	114.2	43.70	0.000
n_alt*n_dim	1	1	1.0	0.38	0.540
n_dm*iz_size	4	3931	982.7	376.03	0.000
n_dm*n_dim	2	121	60.6	23.18	0.000
iz_size*n_dim	2	4169	2084.6	797.68	0.000
Error	7180	18763	2.6		
Lack-of-Fit	16	519	32.4	12.73	0.000
Pure Error	7164	18244	2.5		
Total	7199	140180			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
1.61656	86.61%	86.58%	86.54%

(c) ANOVA table of iteration number

Figure 8: Results of numerical experiments

- The required number of iterations increases with the group size m. As the number of DM pairs increases, the peer-to-peer preference adjustment scheme requires more iterations to reduce the group inconsistency. This observation matches the time complexity $O(m^2)$ given in Appendix B.

- The effect of n (or n_alt) is not obvious. This shows that the efficiency of the method is not dependent on the size of the alternative set. Indeed, by making use of the spatial preference information, it avoids the tedious modifications of the preference relation matrix whose size depends on the alternative size .This concusion matches the time complexity analysis given in Appendix B. This property reveals the potential of the method for handling larger problems.

- The effect of z (or n_dim) is not obvious. This implies that the method is able to solve problem with more criteria without losing efficiency. This matches the time complexity analysis given in Appendix B.

In summary, the numerical results on the convergence property as well as the influence of m, n and z on the method efficiency coincide with the analytical analysis provided in the appendix.

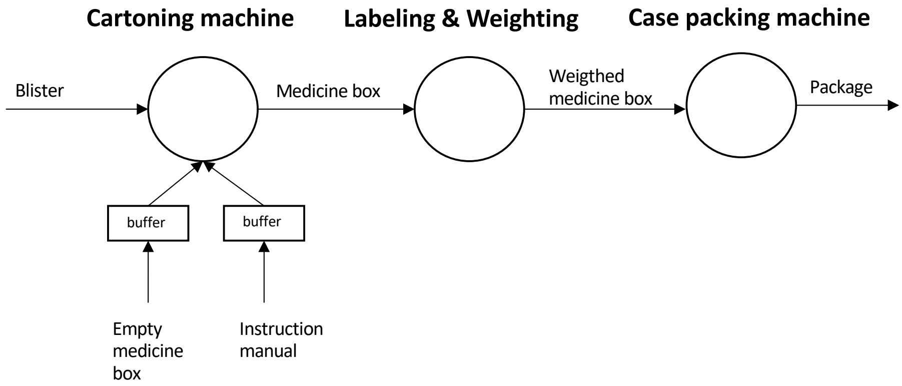


Figure 9: The packaging line

Table 2: Activities for improving the line productivity

Improvement activity	Levels
(a) Set higher alarm levels for empty medicine box buffer	[50, 200, 400]
(b) Set higher alarm levels for instruction manuals	[100, 300, 600]
(c) Increase the buffer capacity of the empty medicine box	[0, 200, 450]
(d) Increase the capacity of big package box	[0, 108, 258]
(e) Reduce the transportation time between the line and the warehouse	[0%, 25%, 50%]
(f) Increase the number of operators	[0, 1, 2]

6. A case study

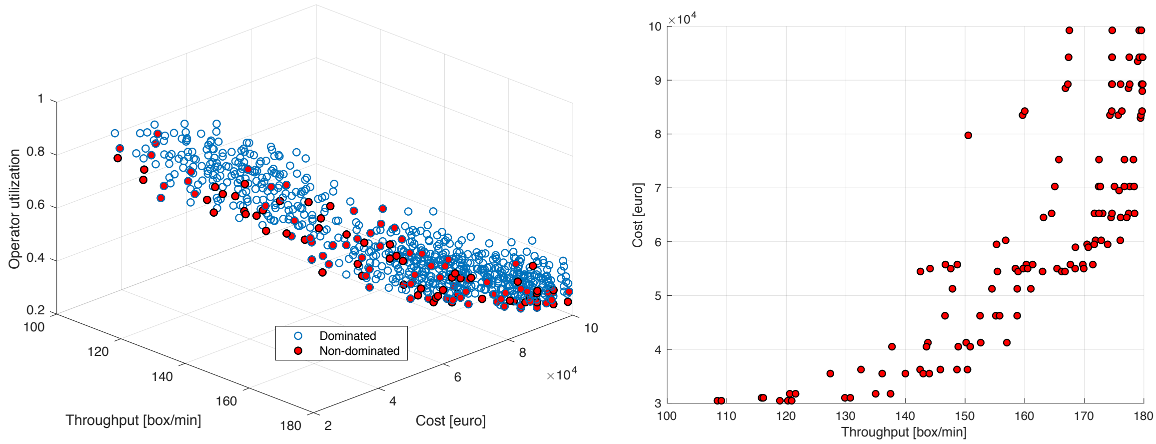
To verify the usefulness of the proposed approach, in this section, we present a case study on a real-world decision making problem. The aim is to select a production line configuration from a set of alternatives for a pharmaceutical manufacturing company. To emulate the decision making procedure, we invited Ph.D and master of science students to play the roles of DMs.

6.1. The decision making problem

The packaging line in a pharmaceutical company is used for the packaging of a medicine product. The line is depicted in Figure 9. Pill blisters enter the line as raw material. In the first station, a cartoning machine is used to insert two blisters together with an instruction manual into a small medicine box. Then, these boxes enter the second station for labeling and weighting. Finally, in the packing machine, the medicine boxes are packed into a big package box, then exit the line as products.

The line is synchronized and automatic, yet some operations are manual. These include replenishing the medicine boxes buffer and the instruction manual buffer of stage 1, loading and unloading the big package box in stage 3, retrieving from the warehouse the medicine boxes and instruction manuals when necessary, etc. Note that when the medicine boxes buffer or instruction manual buffer becomes empty, the line will be forced to stop. To prevent such stoppage, alarms are set to alert the operator for replenishing activities.

Currently, the line has a throughput of 108 [box/min] and runs with an annual cost of about 30000 [euros]. For the next production period, to fulfill the increasing market, the company has decided to improve the



(a) All possible solutions (b) Alternatives

Figure 10: Case study alternatives

production capacity of the line through the activities indicated in Table 2. Considering the realistic features of the line, we chose to use a simple and intuitive design of experiment to generate feasible solutions for the company. Note that by coupling the simulation model of the line to a multi-objective optimization algorithm, one could result in more alternatives in a larger space. Based on the information provided by the company, we set the corresponding levels for the improvement activities as in Table 2. More specifically, improving the alarm level (a)(b) triggers more frequently the buffer replenishment to avoid line stoppage but also occupies more time of the operator. These alarm levels are chosen respecting the specification of the machines, e.g., the level could not be set higher than the current buffer capacity. Activity (c) has the same purpose for avoiding line stoppage. Activity (d) aims at reducing the frequency of loading/unloading the final package. The package capacity levels (d) are set based on the query of the company, e.g., the company is highly interested in the simulation results of increasing the package box capacity by 258. Activities (e) saves the time of operator in transportation. Activity (f) helps to mitigate the workload of operators so as to reduce the waiting time of triggered tasks. The levels are set considering the maximum number of operators allowed in the working area of the line. Among these improvements, modifying the alarm level (a)(b) can be realized by adjusting the control codes in the machines; other activities incur a certain amount of costs, like purchasing new machine modules or equipment, hiring employees, etc. A full factorial experiment with different levels of activities is performed, generating a total of $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$ configurations.

The performance of different configurations is evaluated using discrete-event-simulation by considering three criteria: line throughput, cost and operator utilization. Results are in Figure 10(a). Basically, higher investment yields greater throughput and less operator occupation. By filtering the dominated solutions, the analysts provide 126 alternatives for the decision making group (as in Figure 10(b)). Given that the operator utilization is less important than the other two criteria, for simplicity, the group focuses only on the throughput and the cost.

6.2. The decision making group

The decision making group consists of four experts from different departments, they hold distinct opinions as shown in Table 3. In the case study, each experiment was performed by four volunteers with the randomly assigned roles.

Table 3: The opinion of decision makers from different departments

DM	Department	Opinion
DM1	Finance	Our company has better projects to inverst. We propose a low investment in this project.
DM2	Marketing	According to our market occupation, a throughput of about 160 would satisfy the orders from different channels.
DM3	Production	We should guarantee the customer satisfaction with a high throughput to earn company reputation.
DM4	Stakeholder	This project might be promising, yet there is certain risk for expanding too much the productivity. The opinion of the Board is to limit the cost at about 40000 euros.

Table 4: Summary of case study experiment results

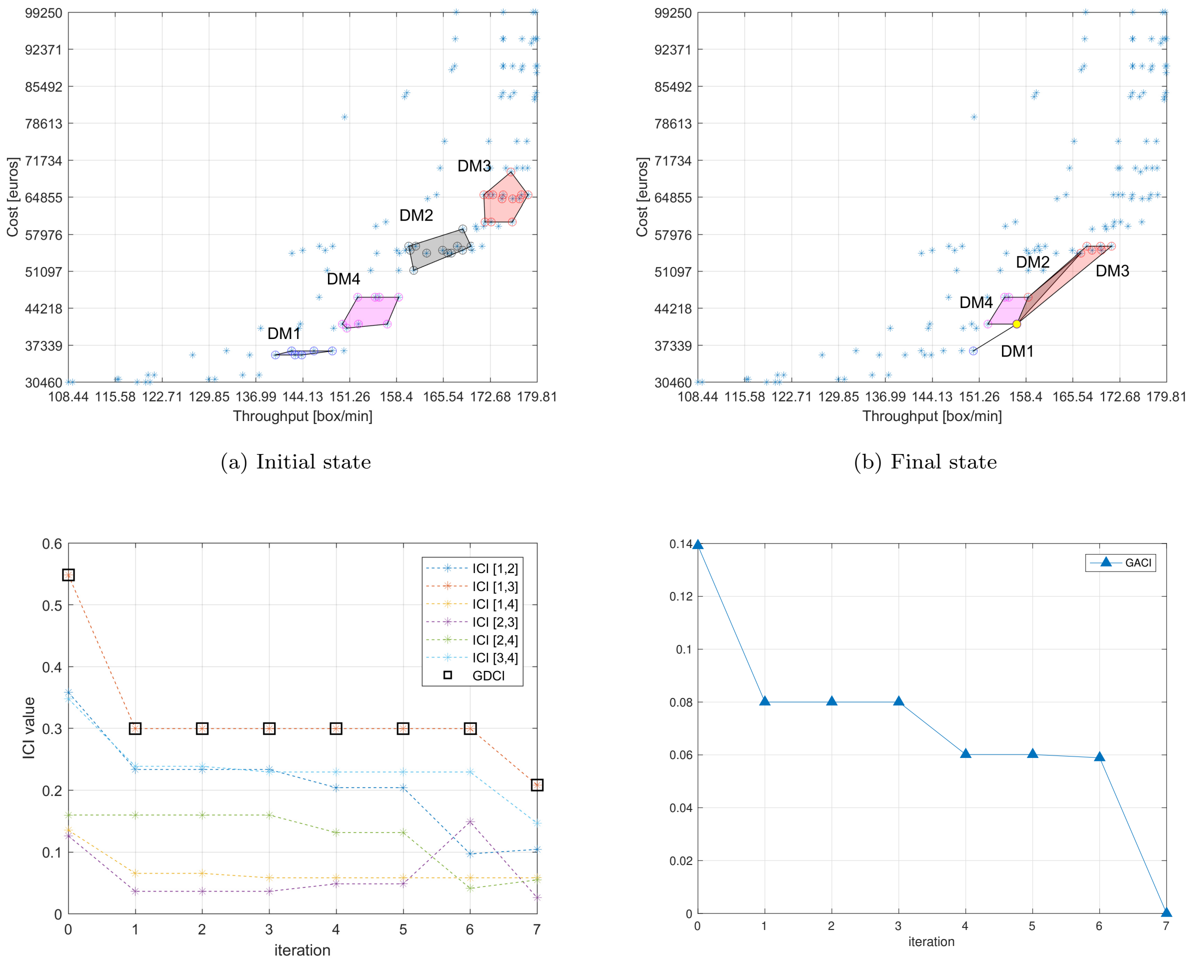
Experiment ID	Group ID	# common solutions	Common solution	# Iterations	# DM rejections
1	1	1	[157.01, 41250]	7	5
2	2	1	[158.74, 46250]	4	2
3	3	0	-	20	8
4	1	1	[158.74, 46250]	5	0

6.3. Experiment results

We have invited three groups of volunteers for the experiments. Before running the approach, the coordinates of the alternatives are normalized to [0,1] by the utopia-nadir point method [58]. Note that if the spans of alternatives are highly heterogeneous in different dimensions, the adopted normalization method may introduce scale bias to the normalized space. To mitigate such bias, one may divide each criterion by a corresponding scaling parameter, which should be determined by experts. See [59] for more details on normalization techniques. For each group, the volunteers first specify their IZs based on the opinion of their roles, then the CRP is applied. The maximum number of iterations is set as 20. The results are summarized in Table 4.

For Group 1, the initial preference information is shown in Figure 11 (a). During the procedure, we observed in total five rejections. These were mainly due to DM_1 and DM_3 whose opinions were far away from the rests. Even though, the common solution was obtained at the 7-th iteration. Compared to Group 1, the DMs in Group 2 showed more willingness to reach consensus because only two rejections were made during the process. As a result, common solution was obtained with fewer iterations.

The result of Group 3 is shown in Figure 12. The initial opinion inconsistency in this group is much higher than Group 1 due to a larger GDCI value. In the process we observed 8 rejections. The CRP stopped at maximum iteration without any common solution. As shown in Figure 11 (b), the group opinions finally divided into two subgroups: DM_1 and DM_4, DM_2 and DM_3. As we observed, the DM has rejected almost any suggestion of moving towards to the other subgroup. Figure 11 (c) shows that the DMs in the same subgroup reached absolute consensus at iteration 13 with $ICI_{14} = 0$ and $ICI_{23} = 0$. After that, no further improvement



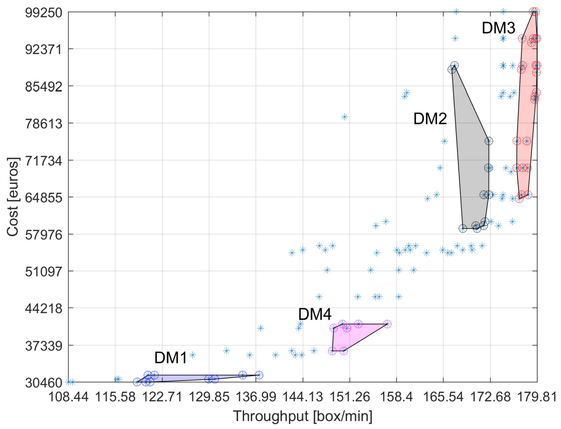
(a) Initial state

(b) Final state

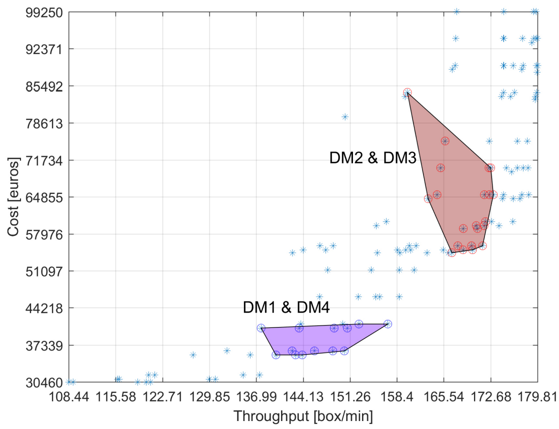
(c) GDCI

(d) GACI

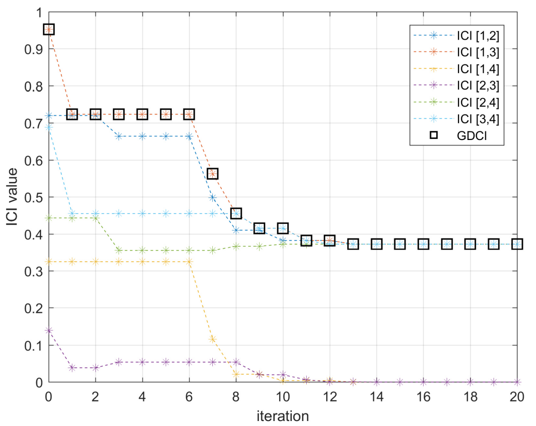
Figure 11: GDM case study: decision making group 1



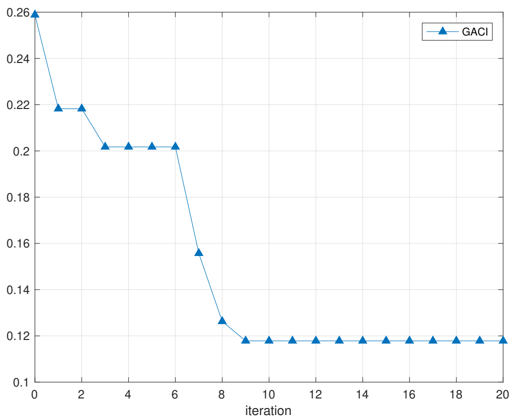
(a) Initial state



(b) Final state

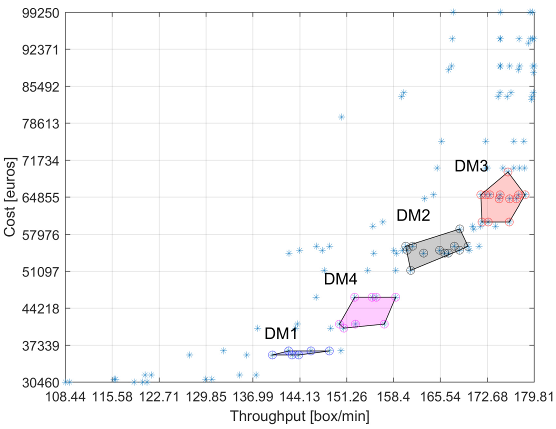


(c) GDCI

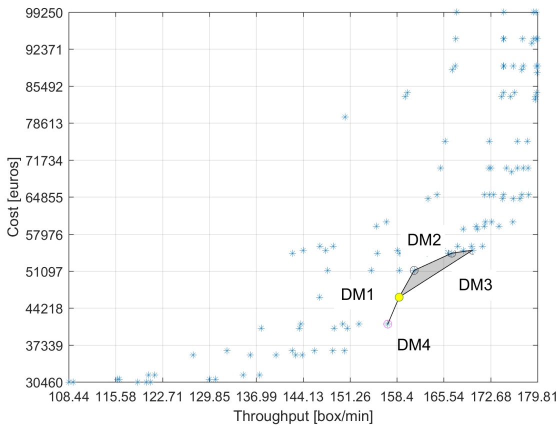


(d) GACI

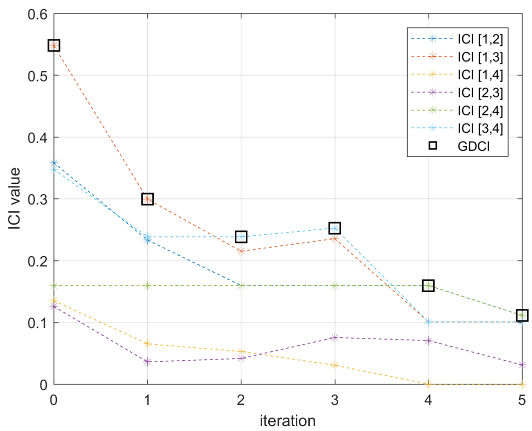
Figure 12: GDM case study: decision making group 3



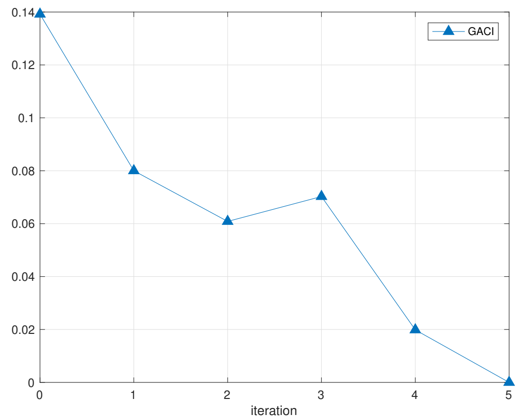
(a) Initial state



(b) Final state



(c) GDCI



(d) GACI

Figure 13: GDM case study: decision making group 3 (allowing partial acceptance)

Table 5: Comparison between the CRP solution and two benchmarks

Experiment ID	Group ID	CRP solution	Center solution	Equilibrium solution
1	1	[157.01, 41250]	[163.00, 54460]	[158.74, 46250]
2	2	[158.74, 46250]	[154.51, 51250]	[158.74, 46250]
3	3	-	[163.00, 54460]	[165.43, 55000]

was made before the termination. Although common solution was not obtained, the proposed model has helped to reduce the group incompatibility level as indicated by GDCI and GACI.

Generally, rejections from DMs increase the difficulty for the group to reach common solution, because each rejection means the deletion of an ordered pair for opinion revision. According to the volunteers, they rejected a suggestion when the zone contained certain extreme point far away from his/her current IZ. Yet, the rests were still acceptable. For this reason, it would be reasonable to allow accepting only a subset of the proposed alternatives to avoid rejection. For validating this idea, we invited the decision making group 1 again and carried out the 4-th experiment . They started with the same initial preferences as in experiment 1. Result is given in Figure 13. As we observed, no rejection was made during the process. The previously rejections were substituted by partial acceptance. For this reason, all ordered pairs remained active during the CRP. This contributes to a faster convergence to the common solution compared to experiment 1.

In summary, supported by the proposed method, the group found common solutions in three out of four experiments. These solutions are closely located. More specifically, in experiment 1, the common solution was [157.01, 41250]; in experiment 2 and 4, the same solution [158.74, 46250] was obtained. These common solutions locate in the intermediate region of the DMs' initial IZs, which reveals the result of a compromising procedure.

Finally, in Table 5 we compare the solution obtained by using the CRP, i.e., with rounds of discussions, to two benchmarks obtained by aggregating directly the initial preference of the group. More specifically, the center solution is the closest solution to the centroid of the group initial preferred solutions; whilst the equilibrium solution is defined as the solution $x_{i^*} \in X$ with the shortest averaged distance to all initial indifference zones of DMs, where $i^* = \arg\min_{i \in 1,\ldots,n} ACI_i$. Both benchmarks reveal the group collective opinion in some way. However, since they are derived from DMs' initial opinions, they cannot capture the interactions and changes of mind emerging from the group consensus meeting. It is observed that the CRP solution deviates from the center solution. Surprisingly, for Group 1 and 2, the results of CRP are very similar to the equilibrium solution. Such consistency, on the one hand, results from a successful consensus meeting which leads to the group collective opinion, and validates the rationality of the CRP; on the other hand, however, it does not suggest skipping the CRP and selecting x_{i^*} directly. Intuitively, x_{i^*} is the most potential common solution because reaching it requires the minimum total transition distance for the group. Yet, reaching x_{i^*} is not always true but depends on the DMs' dynamic behaviors during the CRP. For example, a highly qualified expert would easily pull the others around his/her position and introduce impact on the group opinion transition pattern. Also, skipping CRP bears the risk of selecting a solution beyond the acceptance zone of all DMs, which is observed in the case of Group 3. This violates the purpose of GDM.

7. Conclusions

This research proposes an approach for supporting the group decision making in manufacturing systems. Firstly, we develop a consensus model using spatial information in the objective space as preference inputs. It avoids the tedious pairwise comparison and allows the use of state-of-art posterior preference articulation techniques to ease the opinion expression. As a result, the proposed model is able to tackle a much larger alternative set than common group decision making approaches, meanwhile it avoids the preference inconsistency issue incurred by pairwise comparison. Secondly, the indifference zone is incorporated into the consensus model. This not only allows the decision makers to express the preference vagueness, and more importantly, it has been shown beneficial for the consensus reaching process. Finally, we adopt a recent proposed peer-to-peer consensus reaching process [33] for improving group consensus. Several modifications are made on the feedback mechanism to allow its implementation in the discrete objective space with finite alternative points.

The properties of the proposed method are first investigated by analytical analysis. In the case of continuous solution space, the convergence is proved; the bound of the required iterations is given as well, which implies a time complexity of $O(m^2)$. Then, a full factorial design of experiments is carried out for the discrete space case. In all tested GDM problems, the proposed approach yields common solution(s) as long as the decision makers follow the given suggestions. The efficiency of the method is not sensitive to the size of alternative set and the objective space dimension, which shows its potential of tackling large decision making problems. Finally, the usefulness of the proposed method has been shown by a case study, where the method is applied to a packaging line configuration problem.

There are also some limitations of the proposed approach. Firstly, the proposed method requires rounds of group discussions to obtain a common solution. It is indeed taking more efforts (e.g. organizing meeting, evaluating alternatives, discussions) than direct preference aggregation, such as picking the 'center solution' of the initial group opinion without performing a consensus reaching process. In spite of the similarity of results in some cases, the proposed method is able to capture the interactions and changes of mind during the negotiations and facilitate a decision supported by the group, which is not guaranteed by direct preference aggregation. Due to the cost of consensus, the proposed method is suitable for problems where the decision has long-term effect such as production line configuration, product design and cyclic staff scheduling, but not so cost-effective for short-term managerial decisions. Secondly, as shown in one of our real-world experiments, group consensus meeting reduces the group disagreement to some extent, but may not necessary result in a common solution. This indeed depends on the degree of opinion discrepancy as well as decision makers' willingness of making consensus. For future study, it is valuable to investigate the use of soft consensus or fuzzy majority concept to yield a group decision in case no common solution is obtained from the meeting. Thirdly, due to the peer-to-peer feedback mechanism, the required number of iterations to reach common solution increases quickly with the number of decision makers in the group. This no doubt lowers the decision efficiency for large expert groups. For future work, it is interesting to study the effects of different consensus reaching mechanisms, for example, using group collective opinion to generate suggestion feedback. Lastly, in the model, it is assumed that the opinions from experts are equally important, which may not coincide with the scenarios where the group is formed by experts with different levels of qualifications. It is necessary to consider a more general model in which decision makers are of different importance as a future work.

Appendix A. Proof of convergence

When the number of alternatives $n \to \infty$ and the alternatives are homogeneously distributed in the objective space $\mathbb{R}^z$, i.e., the solution space is continuous, the feedback mechanism proposed in section 4.1, denoted here as $[S'_p, S'_q] = \Gamma(S_p, S_q, \alpha_p, \alpha_q)$, transits the convex hulls S_p, S_q to S'_p, S'_q as:

$$S'_p = \text{Conv}(\{x + \alpha_p(y - x), \forall x \in \text{Vert}(S_p), y \in \text{Vert}(S_q)\}) \tag{A.1}$$

$$S'_q = \text{Conv}(\{y + \alpha_q(x - y), \forall x \in \text{Vert}(S_p), y \in \text{Vert}(S_q)\}) \tag{A.2}$$

where $\text{Conv}(\cdot)$ represents the convex hull and $\text{Vert}(\cdot)$ is the set of strictly convex vertexes, and $\alpha_p, \alpha_q \in (0, 0.5)$. For simplification, we use $\Gamma(S_p, S_q)$ to represent this transition in the following context. The distance between two decision makers, say p and q, is given by the averaged Hausdorff distance, which is

$$\Delta_1(S_p, S_q) = \max\{\frac{1}{|S_p|}\int_{S_p} \min_{y \in S_q} \|x - y\|dx, \frac{1}{|S_q|}\int_{S_q} \min_{x \in S_p} \|y - x\|dy\} \tag{A.3}$$

in continuous solution space. The proposed algorithm, denoted as $\mathscr{A}$ here, iterates as follows. At each iteration t, the DMs with the largest Δ_1 among the group $E = \{1, \ldots, m\}$, say p and q, move their indifference zones S_p and S_q to the S'_p and S'_q, respectively. Here, we provide the demonstration that as the algorithm iteration $t \to \infty$, the maximum Δ_1 between any pair of DMs in the group, denoted as Δ_1^*, goes to zero.

We consider the special case where $S_i, \forall i \in E$ contains only one point x_i. In this case, $\Delta_1(S_i, S_j) = \|x_i - x_j\|$. Using the triangle property, we can show that the diameter of the set (the maximum distance between any of the m points) shrinks after each transition. This implies that as $t \to \infty$, the group opinion convergences to zero distance. Details are as follows. At t-th iteration, let (p, q) be the DMs having the largest distance Δ_1^* among the group, this means the set diameter equals to $\|x_p - x_q\|$. The transition $(x'_p, x'_q) = \Gamma(x_p, x_q)$ reduces the distance between DM p and q, i.e., $\|x'_p - x'_q\| < \Delta_1^*$. For any other DMs, say i, we have $\|x_i - x'_p\| < \max\{\|x_i - x_p\|, \|x_i - x_q\|\}$ and $\|x_i - x'_q\| < \max\{\|x_i - x_p\|, \|x_i - x_q\|\}$ due to the triangle property. Also, since $\|x_p - x_q\|$ is the diameter of the set, we have $\|x_i - x_p\| < \|x_p - x_q\|$ and $\|x_i - x_q\| < \|x_p - x_q\|$. As a consequence, we obtain that $\|x'_p - x_i\| < \Delta_1^*$ and $\|x'_q - x_i\| < \Delta_1^*$. That is, after the transition, the distance between x_p or x_q and any other point in the set is less than the diameter of the set before the transition. In summary, the transition $\Gamma(x_p, x_q)$ strictly decreases the Δ_1^* at each iteration, and the algorithm converges for $t \to \infty$.

For the general case in which DMs hold convex hulls as indifference zones, the triangle property between DMs vanishes because Δ_1 is a semi-metric. In order to show the convergence, we make use of the maximum distance between DMs as a bound for their Δ_1. Similar with the above special case, using some properties of the convex hull, we can see that the maximum pairwise distance between DMs strictly decreases after each transition, and the algorithm is convergent. Details are as follows. Firstly, a theorem is given to describe the property of convex hull. Based on this theorem, two corollaries are provided to show the relationship of the convex hulls after the transition is performed.

Theorem 1. *Let x and y be two points in a set of points S. If the distance between x and y is equal to the diameter of S, i.e., $\max_{x,y \in S} \|x - y\|$, then x and y are strictly convex points, i.e., $x, y \in Vert(S)$.*

Proof This theorem describes the basic property of a convex hull, which is essentially identical to the lemma proved in Bhattacharya and Toussaint [60] It states that the maximum distance between two convex hulls are due to their vertexes, as:

$$dmax(S_p, S_q) = dmax(\text{Vert}(S_p), \text{Vert}(S_q)) \tag{A.4}$$

where $dmax(S_p, S_q) = \max_{x \in S_p, y \in S_q} \|x - y\|$ returns the maximum distance between sets S_p and S_q. By setting $S_p = S_q$, it shows that the diameter of S_p is due to its vertexes, which is Theorem 1.

Corollary 1. *$dmax(S'_p, S'_q) < dmax(S_p, S_q)$, where $[S'_p, S'_q] = \Gamma(S_p, S_q)$.*

Proof This corollary shows the relationship of two convex hulls after the transition acting on them. The proof is based on the fact that the transition moves their vertexes into the interior of the convex hull of their union set; then, it is obvious that the maximum distance between these two convex hulls shrinks according to Theorem 1. Details are as follows. Given that S'_p and S'_q are obtained by the convex combination of sets $\text{Vert}(S_p)$ and $\text{Vert}(S_q)$, it is obvious that $\text{Vert}(S'_p) \subseteq \text{int}(\text{Conv}(S_p \cup S_q))$ and $\text{Vert}(S'_q) \subseteq \text{int}(\text{Conv}(S_p \cup S_q))$, where $\text{int}(\cdot)$ represents the interior. Also, we have $\text{Vert}(S'_p \cup S'_q) \subseteq \text{int}(\text{Conv}(S_p \cup S_q))$, which means the vertexes of the union of S'_p and S'_q lie in the interior of $\text{Conv}(S_p \cup S_q)$. According to Theorem 1, we know that $dmax(S'_p, S'_q)$ are due to $\text{Vert}(S'_p)$ and $\text{Vert}(S'_q)$, which are two internal points of $\text{Conv}(S_p \cup S_q)$, so we have $dmax(S'_p, S'_q) < dmax(S_p, S_q)$.

Corollary 2. *After the transition $[S'_p, S'_q] = \Gamma(S_p, S_q)$, for any other decision maker $i \in E, i \neq p, q$, the following relationship holds:*

$$max\{dmax(S_i, S'_p), dmax(S_i, S'_q)\} < max\{dmax(S_i, S_p), dmax(S_i, S_q)\} \tag{A.5}$$

Proof This corollary shows the relationship between a convex hull moved by the transition and another convex hull that remains still. It provides a bound for their updated distance. The proof is made by treating the two convex hulls moved by the transition as an union. Since the transition moves the vertexes of this union to the interior of its convex hull, and we know that the maximum distance between sets are due to their vertexes, we can see that the maximum distance between this union and another convex hull shrinks after the transition. Details are as follows. Let $S_p \cup S_q$ be the union of the convex hulls moved by the transition. The maximum distance to another convex hull, say S_i, is given by $dmax(S_i, S_p \cup S_q)$. According to the lemma given in Equation (A.4), we know that $dmax(S_i, S_p \cup S_q) = dmax(\text{Vert}(S_i), \text{Vert}(S_p \cup S_q))$. This reveals that for any set, say S_k, if it satisfies $S_k \subseteq \text{int}(\text{Conv}(S_p \cup S_q))$, then $dmax(S_i, S_k) < dmax(S_i, S_p \cup S_q)$. Since the transition $\Gamma(S_p, S_q)$ makes $\{S'_p \cup S'_q\} \subseteq \text{int}(\text{Conv}(S_p \cup S_q))$, so we have $dmax(S_i, S'_p \cup S'_q) < dmax(S_i, S_p \cup S_q)$. Finally, it is obvious that $\max\{dmax(S_i, S'_p), dmax(S_i, S'_q)\} = dmax(S_i, S'_p \cup S'_q)$ and $\max\{dmax(S_i, S_p), dmax(S_i, S_q)\} = dmax(S_i, S_p \cup S_q)$. Then, Corollary 2 is obtained.

Based on these building blocks, we provide the following theorem indicating the decreasing property of the maximum pairwise distance between DMs in the group. For a better explanation, we define the pairwise maximum distance matrix as $D_{max}(i, j) = dmax(S_i, S_j)$. Note that $D_{max}(i, j) = D_{max}(j, i)$.

Theorem 2. *Assume the decision maker pair with the largest maximum distance leads to the greatest averaged Hausdorff distance in the group, i.e., $\arg\max_{i,j \in E, i \neq j} dmax(S_i, S_j) = \arg\max_{i,j \in E, i \neq j} \Delta_1(S_i, S_j)$. As the algorithm $\mathscr{A}$ iterates, the maximum element in the matrix D_{max} is strictly decreasing.*

Proof The proof is by analyzing the transition effect on the pairwise distance of DMs. The DM pairs can be divided into three categories: (a) both DMs in the pair are moved; (b) only one DM in the pair is moved; (c) none of DMs are moved. According to the previous corollaries, we can see that the pairwise distance in (a) is decreased; those in (b) are bounded; those in (c) are unchanged. So we obtain the conclusion. Details are as follows. Let t be the current iteration, $(p,q) = \underset{i,j\in E, i\neq j}{\arg\max}\, dmax(S_i,S_j) = \underset{i,j\in E, i\neq j}{\arg\max}\, \Delta_1(S_i,S_j)$, and D^t_{max} be the maximum distance matrix at iteration t.

For category (a): After the transition $\Gamma(S_p,S_q)$, according to Corollary 1, we have $D^{t+1}_{max}(p,q) < D^t_{max}(p,q)$.

For category (b): According to Corollary 2, for any $i \notin \{p,q\}$, we have

$$D^{t+1}_{max}(p,i) < \max\{D^t_{max}(p,i), D^t_{max}(q,i)\} < D^t_{max}(p,q),$$

$$D^{t+1}_{max}(q,i) < \max\{D^t_{max}(p,i), D^t_{max}(q,i)\} < D^t_{max}(p,q).$$

For category (c): for any $i,j \notin \{p,q\}$, the relative positions of S_i and S_j remain the same, then we have

$$D^{t+1}_{max}(i,j) = D^t_{max}(i,j) < D^t_{max}(p,q).$$

In summary, $\max_{i,j\in E, i\neq j}\{D^{t+1}_{max}(i,j)\} < \max_{i,j\in E, i\neq j}\{D^t_{max}(i,j)\}$.

As a result, for any DM pair (i,j), as the iteration $t\to\infty$, $D_{max}(i,j)\to 0$ according to Theorem 2. Given that the relationship $\Delta_1(S_i,S_j) < \Delta_\infty(S_i,S_j) < D_{max}(i,j)$ holds, we have $\Delta_1(S_i,S_j)\to 0, \forall i,j\in E, i\neq j$ as well, and the algorithm $\mathscr{A}$ converges.

Appendix B. Time complexity

Based on the assumptions and theorems we made in Appendix A, here we provide a bound of iterations required by the algorithm $\mathscr{A}$ to reach common solution for the group. Again, this analysis is based on the pairwise maximum distance matrix D_{max} defined in Appendix A. The basic idea is to quantify the decrease of the element's bound in D_{max} after each transition. By a simple deduction, we can obtain the number of iterations required to suppress the bounds to a sufficient small value.

Theorem 2 provides the bound on the maximum element of D_{max} after the transition. Based on the same assumption, the bound on any other elements of D_{max} is given by the following corollary.

Corollary 3. *At t-th iteration, after the transition of the most distant decision maker pair, i.e.,* $\Gamma(S_p,S_q)$*, the maximum distance between any decision maker pair other than* (p,q)*, denoted as* $D^{t+1}_{max}(i,j)$*, is bounded by the second largest maximum distance in the group before the transition, denoted as* $D^t_{max}(a,b)$ *here.*

Proof The proof is similar to that for Theorem 2, the only difference is that we compare the updated elements to $D^t_{max}(a,b)$. Details are as follows. We use the same notions as in the proof of Theorem 2. For category (b), for any $i\notin\{p,q\}$, we have $D^{t+1}_{max}(p,i) < \max\{D^t_{max}(p,i), D^t_{max}(q,i)\}$. Since (p,i) and (q,i) are different from (p,q), it is sure that $\max\{D^t_{max}(p,i), D^t_{max}(q,i)\} \leq D^t_{max}(a,b)$, so we have $D^{t+1}_{max}(p,i) < D^t_{max}(a,b)$. Similarly, we have $D^{t+1}_{max}(i,q) < D^t_{max}(a,b)$. For category (c), for any $i,j\notin\{p,q\}$, it is easy to see that $D^{t+1}_{max}(i,j) = D^t_{max}(i,j) \leq D^t_{max}(a,b)$. In summary, the corollary is proved.

Here we calculate the required iterations. Denote $\hat{d}^t$ as the second largest value in D^t_{max}. Let $\mathscr{D}$ be all possible undirected pairs of DMs in the group, and $|\mathscr{D}| = C^2_m$. Let DM pair (p,q) be the one having the largest maximum distance at iteration 0. Assume the transition Γ reduces the maximum distance between any DM pair at a constant rate, i.e., $D_{max}^{t+1}(i,j) = \alpha \cdot D_{max}^{t}(i,j), \forall (i,j) \in \mathscr{D}$, where $0 < \alpha < 1$. Now let us start at iteration $t = 0$. After a finite number of iterations, say c, the maximum element in D_{max} falls below $\hat{d}^0$. At iteration c, $D_{max}^{c}(p,q) = \alpha^c D_{max}^{0}(p,q)$; for any other DM pair, say (i,j), the bound is $D_{max}^{c}(i,j) \leq \hat{d}^0$ according to Corollary 3. Although it is not feasible that all these pairs hit the bound, but if it does happen, it requires $C_m^2 - 1$ transitions acting on all these pairs to realize the reduction on the maximum value in D_{max}. Let this happen and we move to iteration $t = c + C_m^2 - 1$, $D_{max}^{t}(p,q)$ is the largest element and is bounded by $\alpha^c D_{max}^{0}(p,q)$; and $D_{max}^{t}(i,j), \forall (i,j) \in \mathscr{D}, (i,j) \neq (p,q)$ are bounded by $\alpha \cdot \hat{d}^0$. Assume all of them hit the bounds, then another round of C_m^2 transitions are performed (one for (p,q) and the rest for the others), after which the bounds become $\alpha^{c+1} D_{max}^{0}(p,q)$ for (p,q) and $\alpha^2 \cdot \hat{d}^0$ for the others. This procedure repeats until $D_{max}^{t}(p,q)$ is less than a small enough quantity, say, ϵ, and the group obtains common solution.

Denote $d^* = \alpha^{c-1} D_{max}^{0}(p,q)$, and η as the number of rounds. To obtain $\alpha^{\eta} d^* \leq \epsilon$, the required number of rounds is given by

$$\eta^* = \frac{\log \epsilon - \log d^*}{\log \alpha}.$$

The maximum required iterations is given by $T = c - 1 + \eta^* C_m^2$, which implies a time complexity of $O(m^2)$.

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