Superoscillating Sequences Towards Approximation in $\mathscr{S}$ or $\mathscr{S}'$-Type Spaces and Extrapolation

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Abstract Aharonov–Berry superoscillations are band-limited sequences of functions that happen to oscillate asymptotically faster than their fastest Fourier component. In this paper we analyze in what sense functions in the Schwartz space $\mathscr{S}(\mathbb{R}, \mathbb{C})$ or in some of its subspaces, tempered distributions or also ultra-distributions, could be approximated over compact sets or relatively compact open sets (depending on the context) by such superoscillating sequences. We also show how one can profit of the existence of such sequences in order to extrapolate band-limited signals with finite energy from a given segment of the real line.

Keywords Approximation by superoscillations · Schwartz space · Tempered distributions · Band-limited signals

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1 Introduction

In a series of papers [1–3], Aharonov and his collaborators introduced the notion of weak measurements, based on the idea that one can pre- and post-select an ensemble of particles, and then calculate the so-called weak value of an observable. To be more

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specific, if $|\psi_{fin}>$ and $|\psi_{in}>$ are the final and initial state of an ensemble of particles, and if $\hat{A}$ is the operator representing an observable, then the *weak value* of $\hat{A}$ is defined to be

$$\hat{A}_w := \frac{\langle \psi_{fin}|\hat{A}|\psi_{in}\rangle}{\langle \psi_{fin}|\psi_{in}\rangle}.$$

It is clear that if $|\psi_{fin}>$ and $|\psi_{in}>$ are almost orthogonal, then the weak value of an operator can assume arbitrarily large values (this is dramatically demonstrated, for example, in [3]). It is in this context that the notion of superoscillations arose in a natural way. According to Aharonov and Berry (who identified this same phenomenon in more classical settings [13–17]), superoscillations are defined as band-limited functions that can oscillate faster than their fastest Fourier component. It turns out that there are many interesting questions regarding superoscillations, both from a mathematical as well as from a physical point of view. In the last few years, these functions have been given a rather thorough mathematical treatment, see [4,5,7–9,19] and the monograph [10], with a particular focus on the longevity of the superoscillatory phenomenon when such functions are evolved according to a wide class of differential equations of physical interest (in particular the Schrödinger equation). For further applications of superoscillations in physics see also [23,24,26–28]. For those who have never encountered superoscillations before, we might mention that the classical superoscillatory function that appears when considering weak measurements is of the type

$$F_n(x, \lambda_0) = \left(\cos\left(\frac{x}{n}\right) + i\lambda_0 \sin\left(\frac{x}{n}\right)\right)^n = \sum_{k=0}^{n} C_k(n, \lambda_0) e^{i(1-2k/n)x}$$

where $\lambda_0 > 1$ and

$$C_k(n, \lambda_0) = \binom{n}{k} \left(\frac{1+\lambda_0}{2}\right)^{n-k} \left(\frac{1-\lambda_0}{2}\right)^{k},$$

where $\binom{n}{k}$ denotes the binomial coefficient. If we fix $x \in \mathbb{R}$ and we let n go to infinity, we obtain that

$$\lim_{n\to+\infty} F_n(x, \lambda_0) = e^{i\lambda_0 x}.$$

The so-called superoscillatory behavior occurs because the terms $(1 - 2k/n)$ that appear in the Fourier representation of F_n are bounded in modulus by one, but the limit function $x \mapsto e^{i\lambda_0 x}$ oscillates with frequency λ_0 arbitrarily large. In this paper we will show that these functions can be utilized to approximate Schwartz functions and tempered distributions, as well as to extrapolate band-limited functions, in novel ways.

In the Banach space

$$(M_2(\mathbb{R},\mathbb{C}), \|f\|_{\mathrm{erg}}) := \frac{\left\{f\in L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{C})\ :\ \|f\|^2_{\mathrm{erg}} := \limsup\limits_{T\to+\infty}(1/T)\int_{-T/2}^{T/2}|f(x)|^2\,dx < +\infty\right\}}{\{f\in L^2_{\mathrm{loc}}(\mathbb{R},\mathbb{C}) :\ \|f\|^2_{\mathrm{erg}} = 0\}},$$

harmonic characters $x \mapsto e_\lambda(x) = e^{i\lambda x}$ (namely homomorphisms from $(\mathbb{R},+)$ to $(\mathbb{R}/(2\pi\mathbb{Z}),+)$) span as a Hilbertian basis the non-separable Hilbert *Besicovitch space* $(B_2(\mathbb{R},\mathbb{C}), \langle\ ,\ \rangle_{\mathrm{erg}})$ of *almost periodic functions* (one could say also *stationary signals from the determinist point of view*), where

$$\langle f_1, f_2\rangle_{\mathrm{erg}} := \lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} f_1(x)\,\overline{f_2(x)}\,dx.$$

Given $f\in B_2(\mathbb{R},\mathbb{C})$, we denote as $\Lambda[f] := \{\lambda\in\mathbb{R}\ :\ \langle f, e_\lambda\rangle \neq 0\}$ its *spectrum* (which may be discrete as well as continuous). Elements in the subspace

$$T(\mathbb{R},\mathbb{C}) := \{f\in B_2(\mathbb{R},\mathbb{C})\ :\ \#(\Lambda[f]) < +\infty\}$$

will be called *generalized trigonometric polynomial functions* ; such functions extend to $\mathbb{C}$ as entire functions belonging to the weighted algebra

$$A_1(\mathbb{C}) = \mathrm{Exp}(\mathbb{C}) := \{f\in H(\mathbb{C})\ :\ |F(z)| = O(e^{B|z|}) \text{ for some } B\geq 0\}.$$

Any $f\in B_2(\mathbb{R},\mathbb{C})$ such that $f(x+T_0) = f(x)$ a.e on $\mathbb{R}$ for some $T_0 > 0$ is such that

$$\Lambda[f]\subset(2\pi/T_0)\,\mathbb{Z}\quad\text{and}\quad \lim_{n\to+\infty}\Big\|f-\sum_{k=-n}^{n}\langle f, e_{2\pi k/T_0}\rangle\, e_{2\pi k/T_0}\Big\|_{L^2([x_0,x_0+T_0])} = 0,\quad \forall\, x_0\in\mathbb{R},$$

the convergence of the sequence of trigonometric polynomial functions (so-called *f-Fejér partial sums*)

$$\left(\sum_{k=-n}^{n}(1-|k|/n)\,\langle f, e_{2\pi k/T_0}\rangle\, e_{2\pi k/T_0}\right)_{n\geq 1} \tag{1}$$

towards f being uniform on any segment $[a,b]$ of $\mathbb{R}$ as soon as f admits a continuous T_0-periodic representative.

Definition 1.1 Given $f\in B_2(\mathbb{R},\mathbb{C})$, a sequence $(Y_n)_{n\geq 1}$ with entries in $T(\mathbb{R},\mathbb{C})$ is said to be f-superoscillating whenever there exists a frequential threshold $\lambda_f > 0$, together with a temporal segment $K=[a,b]$ with strictly positive diameter, such that

- $\Lambda[f] \cap (\mathbb{R} \setminus [-\lambda_f, \lambda_f]) \neq \emptyset$;
- $\Lambda[Y_n] \subset [-\lambda_f, \lambda_f]$ for any $n \in \mathbb{N}^*$;
- $f_{|K}$ admits a continuous representant which is achieved as the uniform limit over K of the sequence $(Y_n)_{n\geq 1}$.

The compact segment K is then called a superoscillation set for the f-superoscillating sequence $(Y_n)_{n\geq 1}$.

Remark 1.2 Note that whenever f admits a T_0-periodic continuous representant, the sequence (1) of the f-Fejér partial sums fails of course to be a f-superoscillating sequence since the spectrum of Y_n equals $\Lambda[f] \cap [-2\pi(n-1)/T_0, 2\pi(n-1)/T_0]$ for any $n \in \mathbb{N}^*$.

Remark 1.3 It is important to notice that whether Definition 1.1 is stated here with respect to an almost-periodic function f in the Besicovitch space $B_2(\mathbb{R}, \mathbb{C})$, it makes sense as well when f is an element of $L^2(\mathbb{R}, \mathbb{C})$. This will be important for us in the sequel since we will deal with functions $f \in L^2(\mathbb{R}, \mathbb{C})$ which therefore admit a spectrum $\widehat{f}$ in the L^2 sense (the spectrum $\Lambda[f]$ being here defined as the complement of the largest open set on which $\widehat{f} = 0$ almost everywhere) but do not fit with the frame of $B_2(\mathbb{R}, \mathbb{C})$ since they are such that $\|f\|_{\text{erg}} = 0$.

Mathematically speaking, the evidence for the existence of f-superoscillating sequences with arbitrary large superoscillation sets is provided (when f is an harmonic character e_{λ_0}, $\lambda_0 \in \mathbb{R}^*$) by the following simple observation that we will use extensively all through this paper. Suppose $|\lambda_0| > 1$ (which is possible up to rescaling eventually the real line). Since

$$e_{\lambda_0}(x) = e^{i\,\lambda_0\,x} = \lim_{n\to+\infty}\Big(1 + \lambda_0\,\frac{i\,x}{n}\Big)^n \quad \forall\, x \in \mathbb{R},$$

the convergence being uniform on any segment $[-T, T] \subset \mathbb{R}$ and

$$\Big|\cos\Big(\frac{x}{n}\Big) - 1\Big| \leq \frac{T^2}{2n^2}, \quad \Big|\sin\Big(\frac{x}{n}\Big) - \frac{x}{n}\Big| \leq \frac{T^3}{6\,n^3} \quad \forall\, x \in [-T, T]$$

thanks to Taylor–Lagrange inequality, one has also

$$\begin{aligned} e_{\lambda_0}(x) &= \lim_{n\to+\infty}\Big(\cos\Big(\frac{x}{n}\Big) + \lambda_0\,\sin\Big(\frac{x}{n}\Big)\Big)^n \\ &= \lim_{n\to+\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{1+\lambda_0}{2}\Big)^{n-k}\Big(\frac{1-\lambda_0}{2}\Big)^{k} e_{1-2k/n}(x)\Big), \end{aligned} \tag{2}$$

the convergence being again uniform on any segment $[-T, T]$, such convergence in x remaining uniform in λ_0 provided $\lambda_0 \in [-\Omega, \Omega]$ (see [10], Chapter 3, Theorem 3.1.8 and Remark 3.1.15). Note that this still holds when $x = z \in \overline{D(0, T)} \subset \mathbb{C}$

and λ_0 keeps values in the closed disk $\overline{D(0,\Omega)} \subset \mathbb{C}$. The sequence of generalized trigonometric polynomials

$$\Big(F_n(x,\lambda_0)\Big)_{n\geq 1} = \left(\sum_{k=0}^{n} \binom{n}{k} \Big(\frac{1+\lambda_0}{2}\Big)^{n-k} \Big(\frac{1-\lambda_0}{2}\Big)^{k} e_{1-2k/n}(x)\right)_{n\geq 1}$$

is clearly an e_{λ_0}-superoscillating sequence (with arbitrary large oscillation sets $[-T,T]$) since one has $|1-2k/n| \leq 1 < |a|$ for any $0 \leq k \leq n$. Observe that such construction relies deeply on the divisibility property of the abelian group $\mathbb{R}$ and that the price to pay in order to realize such an e_{λ_0}-superoscillating sequence $(Y_{\lambda_0,n})_{n\geq 1}$ is that the (real) coefficients

$$\big\langle Y_{\lambda_0,n}, e_{1-2k/n}\big\rangle = C_k(n,\lambda_0) := \binom{n}{k}\Big(\frac{1+\lambda_0}{2}\Big)^{n-k}\Big(\frac{1-\lambda_0}{2}\Big)^{k} \tag{3}$$

form a sequence $C(n,\lambda_0)$ in $\ell^1(\mathbb{N})$ such that

$$\lim_{n\to+\infty} \|C(n,\lambda_0)\|_1 = \lim_{n\to+\infty} \Big(\frac{|1+\lambda_0|+|1-\lambda_0|}{2}\Big)^n = +\infty.$$

Up to now, the study of such concept of superoscillation over the group $(\mathbb{R},+)$ from the mathematical point of view focused on two aspects.

1. Show that the f-superoscillating behavior of a sequence $(Y_n)_{n\geq 1}$ persists when $(Y_n)_{n\geq 1}$ and f are considered as initial values $[\boldsymbol{f}(t,x)]_{t=0} = f(x)$, $[\boldsymbol{Y}_n(t,x)]_{t=0} = Y_n(x)$ of a Cauchy problem

$$\Big(\partial/\partial t + Q(x,\partial/\partial x)\Big)[\boldsymbol{\Phi}](t,x) \equiv 0 \quad (x \in \mathbb{R}, \quad t > 0) \tag{4}$$

   (mostly for fundamental differential operators in quantum physics such as the Schrödinger operator $\partial/\partial t - i\,\partial^2/\partial x^2$ or the harmonic oscillator $\partial/\partial t - i\,\big(\partial^2/\partial x^2 - x^2\big)/2$, see [4,6–8,10]).
2. Generate new and more examples of f-superoscillating sequences for various classes of complex functions f such as for example those represented as restrictions to the real line of sums of generalized Dirichlet series (see [10], Sect. 4.3).

As it turns out, these two aspects complement each other: solving a Cauchy-type problem such as (4) for a large class of differential equations (or even convolution equations when Q happens to be a convolution operator) leads to the realization of more general classes of f-superoscillating sequences for suitable functions f ; for example, given $p \in \mathbb{N}^*$, those realized as

$$\lim_{n\to+\infty} \left(\sum_{k=0}^{n} C_{p,k}(n,\lambda_0)\, e_{(1-2k/n)^p}\right)$$

for convenient coefficients $C_{p,k}(n,\lambda_0)$ $(n \in \mathbb{N}^*, k \in \mathbb{N})$.

Unfortunately, the Besicovitch space $B_2(\mathbb{R}, \mathbb{C})$, which realizes indeed a mathematical frame according the deterministic concept of stationarity, fails to integrate fundamental classes of complex functions f that essentially vanish at infinity. Among such classes, one of the most important ones is certainly the *Schwartz space* $\mathscr{S}(\mathbb{R}, \mathbb{C})$, since it contains in particular all Gabor atoms $x \mapsto e^{-(x-x_0)^2/(2\sigma)} e_{\lambda_0}(x)$ ($x_0, \lambda_0 \in \mathbb{R}$, $\sigma > 0$) as well as the modulated Gaussian chirps in which such Gabor atoms propagate along the action of the Schrödinger operator, and also the Hermite functions $(h_m)_{m \geq 0}$ which play (with respect to the uncertainty principle) the essential role of being eigenvectors of the Fourier transform $f \mapsto \widehat{f}$ from $L^2(\mathbb{R}, \mathbb{C})$ into itself. Since the concept of f-superoscillating sequence appeals only (see Definition 1.1) to compact subsets $K = [a, b]$ of $\mathbb{R}$ as superoscillations sets (namely subsets of $\mathbb{R}$ where the phenomenon of superoscillation concretely appears), it makes sense to speak about f-superoscillation sequences when $f \in \mathscr{S}(\mathbb{R}, \mathbb{C})$, more generally when $f \in L^2(\mathbb{R}, \mathbb{C})$ admits a continuous representative on the whole $\mathbb{R}$, and ask then whether there is a f-superoscillating sequence $(Y_n)_{n \geq 1}$ (depending only on f) for which any segment $[-T, T]$ is a superoscillation set. This is the question we address in Sect. 2. We will give a positive answer to this question (Theorem 2.1). The result will remain valid for a larger subclass of $L^2(\mathbb{R}, \mathbb{C})$ whose elements admit a continuous representative f on $\mathbb{R}$ such that $f(x)$ tends uniformly towards 0 when $|x|$ tends to $+\infty$ (Theorem 2.4).

In Sect. 3, we will revisit the so-called Hermite spectral decomposition of $L^2(\mathbb{R}, \mathbb{C})$ along which elements in $\mathscr{S}(\mathbb{R}, \mathbb{C})$ are developed in Sect. 2 (in order to be treated next) from a different point of view : precisely that of the persistence of superoscillations under the action of convolutor operators acting on weighted algebras of entire functions. Under stronger conditions on f that just being in the Schwartz space $\mathscr{S}(\mathbb{R}, \mathbb{C})$ as in the previous section, one will be able to state a persistence result which tells that the g-superoscillation phenomenon for a Gaussian g propagates precisely along the Hermite spectral decomposition to a subclass of $\mathscr{S}(\mathbb{R}, \mathbb{C})$ that contains all Gaussians (Theorem 3.3).

In Sect. 4, we will introduce the concept of $\boldsymbol{T}$-superoscillating sequence when $\boldsymbol{T} \in \mathscr{S}'(\mathbb{R}, \mathbb{C})$ (Definition 4.1) and show that, according to this concept, any tempered distribution $\boldsymbol{T}$ with support such that $\mathrm{Supp}\, \widehat{\boldsymbol{T}} \cap (\mathbb{R} \setminus [-1, 1]) \neq \emptyset$ admits a $\boldsymbol{T}$-superoscillating subsequence for which any relatively compact open subset U can be considered as a superoscillation set (Theorem 4.3). We introduce also in this section Gevrey-type subclasses $\mathscr{S}_{\mathrm{Gev}}$ of the Schwartz space which are essentially characterized by the ultra fast decrease of their coordinates along the orthonormal basis of Hermite functions. When $\boldsymbol{T}$ is an ultra-distribution belonging to a convenient subclass of $\mathscr{S}'_{\mathrm{Gev}}$, one will exhibit (in a sense to be defined) $\boldsymbol{T}$-superoscillating sequences.

In the final Sect. 5 of this paper, we exploit the existence of superoscillating-sequences towards extrapolation problems for some extremely rigid class of functions in $L^2(\mathbb{R}, \mathbb{C})$. Any element in the $\mathbb{C}$-vector space

$$\begin{aligned} \mathrm{BL}_2(\mathbb{R}, \mathbb{C}) &:= \bigcup_{\Omega > 0} \left\{ f \in L^2(\mathbb{R}, \mathbb{C})\, ;\ \widehat{f} = 0 \text{ a.e. on } \mathbb{R} \setminus [-\Omega, \Omega] \right\} \\ &= \bigcup_{\Omega > 0} H^{\Omega} \subset L^2(\mathbb{R}, \mathbb{C}) \end{aligned}$$

of *band-limited complex signals with finite energy* admits a continuous representant $f : \mathbb{R} \to \mathbb{C}$ that extends as an entire function which belongs to the weighted algebra $A_1(\mathbb{C})$. So does any generalized trigonometric polynomial function Y, which on the opposite fails to be in $L^2(\mathbb{R}, \mathbb{C})$ but nevertheless keeps finite spectrum, hence can be considered band-limited as well. The class $\mathrm{BL}_2(\mathbb{R}, \mathbb{C})$ stands then as a natural class of functions f to which the concept of f-superoscillating sequence can be carried. We will precisely profit from such a concept in Sect. 5 in order to extrapolate over the whole real line the continuous representant f of an element in $\mathrm{BL}_2(\mathbb{R}, \mathbb{C})$ from the values of this continuous representant taken over $[-1, 1]$, even under the single assumption that for example just the values of f over the grid of triadic points $\ell/3^j$, $\ell = -3^j, \ldots, 3^j$, $j \in \mathbb{N}$, are known with an arbitrary precision (Theorem 5.1).

2 Developments Along the Hermite's Orthonormal Basis and e_λ-Superoscillations

Let $(h_m)_{m\geq 0}$ be the orthonormal basis of $L^2(\mathbb{R}, \mathbb{C})$ defined by

$$h_m(x) = \frac{(-1)^m}{\pi^{1/4}\, 2^{m/2}\sqrt{m!}}\, e^{x^2/2} \Big(\frac{d}{dx}\Big)^m [e^{-x^2}] \qquad \forall\, m \in \mathbb{N}.$$

Each such normalized Hermite function h_m is related to the corresponding Hermite polynomial (here in the sense of physicists)

$$H_m(X) := (-1)^m e^{X^2} \Big(\frac{d}{dX}\Big)^m [e^{-X^2}] = m! \sum_{k=0}^{[m/2]} \frac{(-1)^k}{k!\,(m-2k)!}\,(2X)^{m-2k} \quad (m \in \mathbb{N}) \tag{5}$$

through the relation

$$h_m(x) = \frac{1}{\pi^{1/4}\, 2^{m/2}\, \sqrt{m!}}\, e^{-x^2/2}\, H_m(x) \quad \forall\, m \in \mathbb{N}.$$

It follows from the right-hand side equalities (5) that one has the formal identity

$$\sum_{m=0}^{\infty} \frac{H_m(X)}{m!}\, Y^m = \exp\big(2XY - X^2\big) \in \mathbb{C}[[X, Y]],$$

which implies, thanks to the Cauchy integral formulas for derivatives, that

$$\begin{aligned} \forall\, \lambda \in \mathbb{R}, \quad |h_m(\lambda)| &\leq \frac{\sqrt{m!}}{\pi^{1/4}\, 2^{m/2}}\, e^{-\lambda^2/2} \Big| \int_{|\zeta|=1} e^{2\lambda\zeta - \zeta^2}\, \frac{d\zeta}{\zeta^{m+1}} \Big| \\ &\leq \frac{\sqrt{m!}}{\pi^{1/4}\, 2^{m/2}}\, e^{-\lambda^2/2 + 2|\lambda| + 1}. \end{aligned} \tag{6}$$

Given $T > 0$, we need to recall also how the sequence $((h_m)_{|[-T,T]})_{m \geq 0}$ behaves asymptotically as the index m goes to $+\infty$. As soon as $\sqrt{2m} > T$, the whole segment $[-T, T]$ lies entirely in the so-called *oscillatory domain* of the Hermite polynomial function H_m. In order to quantify this heuristic assertion, we refer for example to Theorem 5 in [21]. Since $T^2 > T^2/2$, one deduces from such result that there exist two sequences $(\varepsilon_{T,m})_{m \geq T^2}$ and $(\eta_{T,m})_{m \geq T^2}$ of continuous real functions on $[-T, T]$ which both converge uniformly towards 0 on this segment and are related to the sequence $\big((h_m)_{[-T,T]}\big)_{m \geq T^2}$ as follows :

$$\forall x \in [-T, T], \quad \forall m \geq T^2, \quad h_m(x) = \big(1 + \varepsilon_{T,m}(x)\big) \sqrt{\frac{2}{\pi}} \, \frac{\cos\big(m\,\lambda(x, m)\big)}{(2m)^{1/4}},$$

where

$$\begin{aligned} \lambda(x, m) &= \frac{x}{\sqrt{2m}} \sqrt{1 - \frac{x^2}{2m}} + \Big(1 + \frac{1}{2m}\Big) \operatorname{Arcsin}\Big(\frac{x}{\sqrt{2m}}\Big) - \frac{\pi}{2} \\ &= \sqrt{\frac{2}{m}}\, x - \frac{\pi}{2} + \frac{\eta_{T,m}(x)}{m}. \end{aligned}$$

In particular

$$\forall m \geq T^2, \quad \sup_{[-T,T]} |h_m(x)| \leq \frac{k_T}{m^{1/4}}, \quad \text{where} \quad k_T := \frac{2^{1/4}}{\sqrt{\pi}} \Big(1 + \sup_{m \geq T^2} \sup_{[-T,T]} |\eta_{T,m}|\Big). \tag{7}$$

Theorem 2.1 *Let $c_m = \pi^{-1/4} 2^{-m/2} (m!)^{-1/2}$ for any integer $m \geq 0$ and for any $n \in \mathbb{N}^*$, $0 \leq k \leq n$, let*

$$\begin{aligned} J_{m,n,k} : &= \frac{1}{2^n} \sum_{\kappa_0=0}^{[m/2]} \sum_{\kappa_1=0}^{n-k} \sum_{\kappa_2=0}^{k} (-1)^{\kappa_0+\kappa_2}\, 2^{m-2\kappa_0} \frac{m!}{\kappa_0!(m-2\kappa_0)!} \frac{n!}{\kappa_1!(n-k-\kappa_1)!\,\kappa_2!(k-\kappa_2)!} \\ &\quad \times 2^{\frac{m-1}{2} + \frac{\kappa_1+\kappa_2}{2} - \kappa_0}\, \Gamma\Big(\frac{m+1}{2} + \frac{\kappa_1+\kappa_2}{2} - \kappa_0\Big). \end{aligned} \tag{8}$$

Let $f \in \mathscr{S}(\mathbb{R}, \mathbb{C})$ such that $\operatorname{Supp} \widehat{f} \cap (\mathbb{R} \setminus [-1, 1]) \neq \emptyset$. *One can find two sequences $(M_n)_{n \geq 1}$, $(N_n)_{n \geq 1}$ of strictly positive integers such that the sequence $(Y_n^f)_{n \geq 0}$, where*

$$Y_n^f = \frac{1}{\sqrt{2\pi}} \left(\sum_{k=0}^{N_n} \left(\sum_{m=0}^{M_n} (-i)^m c_m \, \langle f, h_m \rangle \, J_{m,N_n,k} \right) e_{1-2k/N_n} \right)$$

is a f-superoscillating sequence for which any segment $[-T, T]$ is a superoscillation set.

Proof Any $f \in L^2(\mathbb{R}, \mathbb{C})$ expands in $L^2(\mathbb{R}, \mathbb{C})$ as

$$f = \sum_{m \geq 0} \langle f, h_m \rangle \, h_m \tag{9}$$

since $(h_m)_{m\geq 0}$ is an orthonormal basis of $L^2(\mathbb{R}, \mathbb{C})$. The fact that f represents an element in $\mathscr{S}(\mathbb{R}, \mathbb{C})$ is characterized by the conditions

$$\forall\, p \in \mathbb{N}, \quad \sum_{m\geq 0} (1+m)^p \, |\langle f, h_m\rangle| < +\infty \tag{10}$$

(see [31, Lemma 3]). It follows from Fourier inversion formula, together with the fact that for any $m \in \mathbb{N}$, the Hermite function h_m is an eigenvector of the Fourier transform (with corresponding eigenvalue $(-i)^n\sqrt{2\pi}$), that

$$\begin{aligned}
\forall\, x \in \mathbb{R}, \quad f(x) &= \sum_{m=0}^{\infty} \langle f, h_m\rangle \, h_m(x) = \sum_{m=0}^{\infty} \frac{\langle f, h_m\rangle}{2\pi} \int_{\mathbb{R}} \widehat{h_m}(\lambda)\, e^{i\lambda x}\, d\lambda \\
&= \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} (-i)^m \, \langle f, h_m\rangle \int_{\mathbb{R}} h_m(\lambda)\, e_\lambda(x)\, d\lambda,
\end{aligned}$$

the series of functions of x in the right-hand side of this equality being normally convergent on any compact set $[-T, T]$ of $\mathbb{R}$ because of uniform estimates (7) and the rapid decrease (10) of the sequence $|\langle f, h_m\rangle|$. Fix now $m \in \mathbb{N}$. For any $\lambda \in \mathbb{R}$, one has (see (2))

$$e_\lambda(x) = \lim_{n\to+\infty} \Big(\cos\Big(\frac{x}{n}\Big) + \lambda\, \sin\Big(\frac{x}{n}\Big)\Big)^n.$$

For any $\varepsilon > 0$, one can find $\eta_\varepsilon > 0$ such that $|t| \leq \eta_\varepsilon \Longrightarrow |\tan(t)| \leq (1+\varepsilon)|t|$. Therefore

$$\begin{aligned}
\forall\, x \in [-T, T],\ \forall\, n \geq \frac{T}{\sqrt{\eta_\epsilon}}, \quad &\Big| \cos\Big(\frac{x}{n}\Big) + \lambda\, \sin\Big(\frac{x}{n}\Big)\Big|^n \\
&= \Big(\cos^2\Big(\frac{x}{n}\Big) + \lambda^2\, \sin^2\Big(\frac{x}{n}\Big)\Big)^{n/2} \\
&\leq \Big[\Big(1 + (1+\varepsilon)\, \frac{\lambda^2}{n^2}\Big)^{n^2} \Big]^{1/2n} \leq \exp\Big(\Big(\frac{1+\varepsilon}{2}\Big) \frac{\lambda^2}{2n}\Big).
\end{aligned} \tag{11}$$

It then follows from Lebesgue's domination theorem and upper estimate (6) that

$$\begin{aligned}
&\forall\, x \in [-T, T], \\
&\int_{\mathbb{R}} h_m(\lambda)\, e_\lambda(x)\, d\lambda \\
&= \lim_{n\to+\infty} \frac{c_m}{2^n} \sum_{k=0}^{n} \binom{n}{k} \Big(\int_{\mathbb{R}} (1+\lambda)^{n-k}\, (1-\lambda)^k\, e^{-\lambda^2/2}\, H_m(\lambda)\, d\lambda \Big)\, e_{1-2k/n}(x)
\end{aligned} \tag{12}$$

Moreover, splitting $\mathbb{R}$ as $[-\Omega, \Omega] \cup \{|\lambda| > \Omega\}$ with $\Omega >> 1$ and using the results quoted in the introduction about the uniformity of the convergence (2) with respect to

$x \in [-T, T]$ and $\lambda_0 \in [-\Omega, \Omega]$, one gets also that the convergence in (12) is uniform for $x \in [-T, T]$. Observe now that for any $n \geq 1$, for any $k = 0, ..., n$,

$$(\lambda + 1)^{n-k}(1-\lambda)^k = \sum_{\kappa_1=0}^{n-k} \sum_{\kappa_2=0}^{k} \binom{n-k}{\kappa_1}\binom{k}{\kappa_2}(-1)^{\kappa_2}\, \lambda^{\kappa_1+\kappa_2}.$$

Then, using the development of the Hermite polynomial H_m such as given in (5), one has

$$\begin{aligned}
&\frac{1}{2^n}\binom{n}{k}\int_{\mathbb{R}} (1+\lambda)^{n-k}\,(1-\lambda)^k\, e^{-\lambda^2/2}\, H_m(\lambda)\, d\lambda \\
&= \frac{1}{2^n}\sum_{\kappa_0=0}^{[m/2]}\sum_{\kappa_1=0}^{n-k}\sum_{\kappa_2=0}^{k}(-1)^{\kappa_0+\kappa_2}\, 2^{m-2\kappa_0}\,\frac{m!}{\kappa_0!(m-2\kappa_0)!}\,\frac{n!}{\kappa_1!(n-k-\kappa_1)!\,\kappa_2!(k-\kappa_2)!} \\
&\quad\times \int_{\mathbb{R}} e^{-\lambda^2/2}\lambda^{m+\kappa_1+\kappa_2-2\kappa_0}\, d\lambda \\
&= \frac{1}{2^n}\sum_{\kappa_0=0}^{[m/2]}\sum_{\kappa_1=0}^{n-k}\sum_{\kappa_2=0}^{k}(-1)^{\kappa_0+\kappa_2}\, 2^{m-2\kappa_0}\,\frac{m!}{\kappa_0!(m-2\kappa_0)!}\,\frac{n!}{\kappa_1!(n-k-\kappa_1)!\,\kappa_2!(k-\kappa_2)!} \\
&\quad\times 2^{\frac{m-1}{2}+\frac{\kappa_1+\kappa_2}{2}-\kappa_0}\int_{\mathbb{R}} e^{-t}\, t^{\frac{m-1}{2}+\frac{\kappa_1+\kappa_2}{2}-\kappa_0}\, dt \\
&= \frac{1}{2^n}\sum_{\kappa_0=0}^{[m/2]}\sum_{\kappa_1=0}^{n-k}\sum_{\kappa_2=0}^{k}(-1)^{\kappa_0+\kappa_2}\, 2^{m-2\kappa_0}\,\frac{m!}{\kappa_0!(m-2\kappa_0)!}\,\frac{n!}{\kappa_1!(n-k-\kappa_1)!\,\kappa_2!(k-\kappa_2)!} \\
&\quad\times 2^{\frac{m-1}{2}+\frac{\kappa_1+\kappa_2}{2}-\kappa_0}\,\Gamma\Big(\frac{m+1}{2}+\frac{\kappa_1+\kappa_2}{2}-\kappa_0\Big) = J_{m,n,k}.
\end{aligned}$$

It then follows from (12) that for any $M \in \mathbb{N}$ one has uniformly on $[-T, T]$,

$$\sum_{m=0}^{M}\langle f, h_m\rangle\, h_m = \frac{1}{\sqrt{2\pi}}\lim_{n\to+\infty}\left(\sum_{k=0}^{n}\left(\sum_{m=0}^{M}(-i)^m c_m\ \langle f, h_m\rangle\, J_{m,n,k}\right) e_{1-2k/n}\right). \tag{13}$$

Let us proceed now as follows in order to construct the two sequences $(M_n)_{n\geq 1}$ and $(N_n)_{n\geq 1}$. Fix $n \in \mathbb{N}^*$. It follows from estimates (7) relative to the uniform asymptotic behaviour of the sequence $\big((h_m)_{[-n,n]}\big)$ when $m \geq n^2$ tends to $+\infty$, together with estimates (10) ensuring the rapid decrease of the sequence $(\langle f, h_m\rangle)_{m\geq 0}$, that there exists $M_m \geq m^2$ such that

$$\sup_{[-n,n]}\left| f - \sum_{m=0}^{M_n}\langle f, h_m\rangle\, h_m\right| \leq \frac{1}{n}. \tag{14}$$

For such $M_n >> n^2$ fixed, (13) implies that one can find $N_n >> 1$ such that

$$\sup_{[-n,n]}\left|\sum_{m=0}^{M_n}\langle f, h_m\rangle\, h_m - \frac{1}{\sqrt{2\pi}}\sum_{k=0}^{N_n}\Big(\sum_{m=0}^{M_n}(-i)^m c_m\ \langle f, h_m\rangle\, J_{m,N_n,k}\Big)\, e_{1-2k/N_n}\right| \leq \frac{1}{n}. \tag{15}$$

Combining (14) with (15), one gets

$$\sup_{[-n,n]}\left|f-\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{N_n}\Big(\sum_{m=0}^{M_n}(-i)^m c_m\ \langle f,h_m\rangle\ J_{m,N_n,k}\Big)\,e_{1-2k/N_n}\right|\leq\frac{2}{n}=o(1).$$

Since the sequence $([-n,n])_{n\geq 1}$ exhausts $\mathbb{R}$ as an increasing sequence with respect to inclusion and $(2/n)_{n\geq 1}$ decreases towards 0, the conclusion of Theorem 2.1 follows.

Remark 2.2 One can express also the conclusion of Theorem 2.1 in two steps as follows: given any segment $[-T,T]\subset\mathbb{R}$ and any $M\in\mathbb{N}$ such that there exists at least one $m\leq M$ with $\langle f,h_m\rangle\neq 0$, the sequence $(Y^f_{M,n})_{n\geq 1}$, where

$$Y^f_{M,n}:=\frac{1}{\sqrt{2\pi}}\left(\sum_{k=0}^{n}\Big(\sum_{m=0}^{M}(-i)^m c_m\ \langle f,h_m\rangle\ J_{m,n,k}\Big)\,e_{1-2k/n}\right)\quad\forall n\geq 1$$

is $\sum_0^M\langle f,h_m\rangle\,h_m$-superoscillating (since $\sum_0^M\langle f,h_m\rangle\,h_m$ has an unbounded spectrum as all h_m have) and admits $[-T,T]$ as a superoscillation set (step 1). Then (step 2) the sequence

$$(Y^f_{M,\infty})_{M\geq 0}=\Big(\sum_{k=0}^{M}\langle f,h_m\rangle\,h_m\Big)_{M\geq 0}=\Big(\mathrm{Proj}^{\perp}_{\mathrm{vec}(h_0,\ldots,h_M)}[f]\Big)_{M\geq 0}$$

converges uniformly towards f on $[-T,T]$.

It could be worthwhile to point out the following consequence of Theorem 2.1, which clearly emphasizes the fact testing the superoscillation phenomenon makes sense only on compact segments of the real temporal line.

Corollary 2.3 *Let $-\infty<a<b<+\infty$ and $\varphi\ :[a,b]\to\mathbb{C}$ be a C^∞ complex valued function. There exists a sequence of generalized trigonometric polynomial functions $(Y_{[a,b],n}[\varphi])_{n\geq 1}$ with spectrum in $[-1,1]\cap\mathbb{Q}$ such that $(Y_{[a,b],n}[\varphi])_{n\geq 1}$ converges uniformly towards φ on the segment $[a,b]$.*

Proof Thanks to Borel's theorem, one can extend φ to a smooth function $\boldsymbol{\varphi}\ :\mathbb{R}\to\mathbb{C}$. Multiplying $\boldsymbol{\varphi}$ by a test-function $\psi_{[a,b]}\in\mathscr{D}(\mathbb{R},[0,1])$ which identically equals 1 in an open neighborhood of $[a,b]$, one may assume that $\boldsymbol{\varphi}\in\mathscr{D}(\mathbb{R},\mathbb{C})\subset\mathscr{S}(\mathbb{R},\mathbb{C})$. Then the conclusion follows from Theorem 2.1. □

In fact, the conclusion of Theorem 2.1 stands for a much wider class of continuous complex-valued functions from $\mathbb{R}$ to $\mathbb{C}$ than just the Schwartz class $\mathscr{S}(\mathbb{R},\mathbb{C})$.

Theorem 2.4 *Let $f\in L^2(\mathbb{R},\mathbb{C})$ such that the sequence $(\langle f,h_m\rangle)_{m\geq 0}$ of its coordinates in the orthonormal basis of Hermite functions is in $\ell^q(\mathbb{N})$ with $1\leq q<4/3$. Then f admits a continuous representant for which the assertions in Theorem 2.1 and Remark 2.2 remains valid.*

Proof For any $T > 0$ and for any $k \geq 0$, one has combining asymptotic estimates (7) with Hölder's inequality

$$\begin{aligned}
&\sum_{m>[T^2]+1+k} |\langle f, h_m\rangle| \sup_{[-T,T]} |h_m| \\
&\leq k_T \left(\sum_{m>[T^2]+1+k} |\langle f, h_m\rangle|^q\right)^{1/q} \left(\sum_{m>[T^2]+1+k} \frac{1}{m^{q'/4}}\right)^{1/q'} \\
&= o_k(1) \qquad (16)
\end{aligned}$$

since $q' = q/(q-1) > 4$. Therefore f admits a continuous representative in $\mathbb{R}$ defined on each $[-T, T]$ as the sum of the normally convergent series $\sum_m \langle f, h_m\rangle h_m$ in $C([-T, T], \mathbb{C})$. Since f develops in $L^2(\mathbb{R}, \mathbb{C})$ as (9), one can express such continuous representant f on $[-T, T]$ as follows : given any $k \geq 0$ (for the moment kept fixed), it follows from (13) that

$$\begin{aligned}
&\forall x \in [-T, T], \quad f(x) = \sum_{m=0}^{[T^2]+1+k} \langle f, h_m\rangle\, h_m(x) \\
&\quad + \sum_{m>[T^2]+1+k} \langle f, h_m\rangle\, (1 + \varepsilon_{T,m}(x)) \sqrt{\frac{2}{\pi}}\, \frac{\cos\left(\sqrt{2m}\, x - m\pi/2 + \eta_{T,m}(x)\right)}{(2m)^{1/4}} \\
&= \lim_{n\to+\infty} \frac{1}{\sqrt{2\pi}} \left(\sum_{k=0}^{n} \left(\sum_{m=0}^{[T^2]+1+k} (-i)^m c_m\, \langle f, h_m\rangle\, J_{m,n,k}\right) e_{1-2k/n}(x)\right) \\
&\quad + \sum_{m>[T^2]+1+k} \langle f, h_m\rangle\, (1 + \varepsilon_{T,m}(x)) \sqrt{\frac{2}{\pi}}\, \frac{\cos\left(\sqrt{2m}\, x - m\pi/2 + \eta_{T,m}(x)\right)}{(2m)^{1/4}},
\end{aligned} \qquad (17)$$

where the combinatorial coefficients $J_{m,n,k}$ have been introduced in (8). One just need to repeat word for word at this point the final arguments in the proof of Theorem 2.1, starting from (13). □

Remark 2.5 Observe here that for any $m \geq [T^2] + 1$, the signal

$$\varphi_m\ : x \mapsto \cos(\sqrt{2m}\, x - m\pi/2) = \begin{cases} (-1)^k\, \frac{e_{\sqrt{2m}}(x) + e_{-\sqrt{2m}}(x)}{2} & \text{if} \quad m = 2k \\ (-1)^k\, \frac{e_{\sqrt{2m}}(x) - e_{-\sqrt{2m}}(x)}{2i} & \text{if} \quad m = 2k+1 \end{cases}$$

can be approximated uniformly on $[-T, T]$ in a φ_m-superoscillating way as follows

$$e_{\pm\sqrt{2m}} = \lim_{n\to+\infty} \left(\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(1 + (\pm\sqrt{2m})\right)^{n-k} \left(1 - (\pm\sqrt{2m})\right)^k e_{1-2k/n}\right),$$

the limits along n being uniform on $[-T, T]$. The tail function defined as the restriction over $[-T, T]$ of the orthogonal projection of f on $[\mathrm{vec}(h_0, \dots, h_{[T^2]+k})]^\perp$ for $k >> 1$ (depending on T) can be considered as a quasi almost periodic function

$$\sum_{m \geq [T^2]+k+1} (\alpha_m \, e_{\sqrt{m}} + \beta_m \, e_{-\sqrt{m}}),$$

that is the sequences $m \mapsto \alpha_m$ and $m \mapsto \beta_m$ $(m \geq [T^2] + k + 1)$ are almost constant complex sequences : the modulus of α_m (respectively of β_m) stands for the amplitude of the monic function $\alpha_m e_{\sqrt{2m}}$ (respectively $\beta_m \, e_{-\sqrt{2m}}$) it affects, while its argument stands for the its phase shift. This provides an alternative way to realize a superoscillating sequence, provided one enlarges slightly the constraints in Definition 1.1 in order to tolerate infinitesimal variations of amplitudes and phase shifts of the monic trigonometric polynomial functions involved in the decomposition of the entries $(Y_n)_{n \geq 1}$, which seems reasonable from the physicists's point of view.

3 Persistence of g-Superoscillations when $g : x \mapsto e^{-\varepsilon x^2}$ is a Gaussian

Though the explicitation of the superoscillation sequence $(Y_n^f)_{n \geq 0}$ in Theorem 2.1 relies on the coordinates of $f \in \mathscr{S}(\mathbb{R}, \mathbb{C})$ in the orthonormal basis of Hermite functions $(h_m)_{m \geq 1}$, combined with the absolute combinatoric constants $J_{m,n,k}$ defined as (8), the fact that the approximation procedure needs to be conducted in two steps (see Remark 2.2) makes the superoscillating approximation sequence $(Y_n^f)_{n \geq 0}$ indeed hard to handle. Let us therefore state the following Proposition, which shows that under stronger conditions on f than just being in $\mathscr{S}(\mathbb{R}, \mathbb{C})$, one has indeed a much more clear reformulation of Theorem 2.1.

Proposition 3.1 *Let* $f \in \mathscr{S}(\mathbb{R}, \mathbb{C})$ *such that* $\mathrm{Supp}\, \widehat{f} \cap (\mathbb{R} \setminus [-1, 1]) \neq \emptyset$ *and*

$$\sum_{m \geq 0} \frac{|\langle f, h_m \rangle|}{(m!)^{-1/2}} \Big(\frac{1}{\sqrt{2}}\Big)^m < +\infty. \tag{18}$$

Let $\boldsymbol{\mu} = (\mu_n)_{n \geq 0}$ *and* $\boldsymbol{\nu} = (\nu_n)_{n \geq 1}$ *two arbitrary sequences of integers tending to* $+\infty$. *The sequence* $(\boldsymbol{Y}_n^{\boldsymbol{\mu},\boldsymbol{\nu}})_{n \geq 1}$, *where*

$$\boldsymbol{Y}_n^{\boldsymbol{\mu},\boldsymbol{\nu}} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\nu_n} \Big(\sum_{m=0}^{\mu_n} (-i)^m c_m \, \langle f, h_m \rangle \, J_{m,\nu_n,k} \Big) e_{1-2k/\nu_n}$$

is a f*-superoscillating sequence which admits any compact set* $[-T, T]$ *as a superoscillating set.*

Proof Let $\varepsilon > 0$. It follows from condition (18) and estimates (6) that for any $n \geq 1+\varepsilon$,

$$\int_{\mathbb{R}} \Big(\sum_{m=0}^{\infty} |\langle f, h_m \rangle| \, |h_m(\lambda)| \Big) \exp\Big(\Big(\frac{1+\varepsilon}{2}\Big) \frac{\lambda^2}{2n} \Big) \, d\lambda < +\infty. \tag{19}$$

Proposition 3.1 then follows from the approximations (13), together with domination estimates (11) and finally (19) which validates the application of Fubini's theorem. □

As we have seen in Sect. 2, any Gaussian function $g : x \mapsto e^{-(x-x_0)^2/(2\sigma)}$, which can be transformed up to translation and rescaling and multiplication by a positive constant into the standard Gaussian $x \mapsto e^{-x^2}$ or better the Hermite function h_0, can be uniformly approximated on any compact set $[-T, T]$ by the g-superoscillating sequence $(Y_n[g])_{n\geq 1}$ where

$$Y_n[g] = \frac{1}{2\pi}\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\Big(\int_{\mathbb{R}}\widehat{g}(\lambda)\,(1+\lambda)^{n-k}(1-\lambda)^k\,d\lambda\Big)\,e_{1-2k/n} \quad \forall n\geq 1,$$

any compact segment $[-T, T]$ being then a superoscillation set. There could be indeed other candidates for such a g-superoscillating sequence. In order to state a result about the persistence of such an g-superoscillating phenomenon along the expansion of any $f \in \mathscr{S}(\mathbb{R}, \mathbb{C})$ (or in a convenient subspace of $\mathscr{S}(\mathbb{R}, \mathbb{C})$ along the development (9)), one will appeal to the theory of convolutor operators in weighted algebras of entire functions.

We recall first the following classical notion in differential equations (Malgrange, Ramis, etc.) as well as in number theory :

Definition 3.2 A power series $\sum_{m\geq 0} a_m z^m$ is called Gevrey with order $s \in \mathbb{R}$ if and only if the power series $\sum_{m\geq 0} a_m/(m!)^s\, z^m$ has a strictly positive radius of convergence. When $s < 0$, this is equivalent to say that the series $\sum_{m\geq 0} a_m z^m$ defines an element in the weighted algebra

$$A_{|\ |^{-1/s}}(\mathbb{C}) := \Big\{F \in H(\mathbb{C})\,;\ |F(z)| = O\big(e^{B|z|^{-1/s}}\big)\ for\ some\ B \geq 0\Big\}.$$

We can now state the following *persistence result* about g-superoscillating phenomenon when g is a Gaussian function.

Theorem 3.3 *Let $f \in \mathscr{S}(\mathbb{R}, \mathbb{C})$ be of the form $f(x) = e^{-x^2/2}\Phi(x)$ where $\sum_{m\geq 0}\langle \Phi, h_m\rangle\, z^m$ is a $(-1/2)$-Gevrey series. Let $(y_n)_{n\geq 1}$ be any $(g : x \mapsto e^{-x^2})$-superoscillating sequence which admits $[-T, T]$ as superoscillating set and moreover is such that the entire analytic continuations $(z \mapsto \boldsymbol{y}_n(z))_{n\geq 1}$ of the $(x \mapsto y_n(x))_{n\geq 1}$ define a bounded set in $A_1(\mathbb{C})$. Then the sequence of functions $(\boldsymbol{Y}_n)_{n\geq 1}$, where*

$$\boldsymbol{Y}_n(x) = \sum_{m=0}^{n}\langle \Phi, h_m\rangle\left(\frac{d}{dx}\right)^m [y_n]$$

is a f-superoscillating sequence which admits also any $[-T, T]$ as a superoscillating set.

Proof The fact that $\sum_{m\geq 0}\langle \Phi, h_m\rangle z^m$ is a Gevrey series of order $-1/2$ is equivalent to the fact that the entire function $z \mapsto \sum_{m\geq 0}\langle \Phi, h_m\rangle\, z^m$ belongs to the weighted algebra

$$A_2(\mathbb{C}) = \left\{F \in H(\mathbb{C})\,;\ F(z) = O(e^{B|z|^2}) \text{ for some } B \geq 0\right\},$$

which implies that the symbol $\sum_{m\geq 0}\langle \Phi, h_m\rangle\, z^m$ of the operator

$$\mathbb{D}_\Phi := \sum_{m\geq 0} \langle \Phi, h_m\rangle \left(\frac{d}{dx}\right)^m$$

induces a continuous multiplication operator of $A_2(\mathbb{C})$ into itself. Hence the operator $\mathbb{D}_\Phi$ acts as a continuous convolution operator from the dual algebra (via the Fourier-Borel transform, see [10], Chapter 4, also [12,34])

$$A_{2,0} := \{F \in H(\mathbb{C})\,;\ \forall\, \varepsilon > 0\,,\ F(z) = O(e^{\varepsilon|z|^2})\}$$

into itself. It follows from the fact that the $(h_m)_{m\geq 0}$ form an orthonormal system in $L^2(\mathbb{R}, \mathbb{C})$ that one has in $L^2(\mathbb{R}, \mathbb{C})$ (also uniformly on any compact set if one takes continuous representants), see (9),

$$\Phi(x) = \sum_{m=0}^{\infty} \langle \Phi, h_m\rangle\, h_m(x) = e^{x^2/2} \sum_{m=0}^{\infty} \langle \Phi, h_m\rangle \left(\frac{d}{dx}\right)[e^{-x^2}].$$

From the hypothesis on the g-superoscillating sequence $(y_n)_{n\geq 0}$, it follows from Montel's theorem that one can extract from the sequence $(y_n)_{n\geq 1}$ a subsequence $(y_{n_k})_{k\geq 0}$ such that the entire functions $z \mapsto \boldsymbol{y}_{n_k}(z)$ form a convergent sequence in $\text{Exp}(\mathbb{C})$, hence in $A_{2,0}(\mathbb{C})$, with limit $\boldsymbol{y}_\infty$ in $A_{2,0}(\mathbb{C})$. Due to the continuity of the convolutor operator $\mathbb{D}_\Phi$, the sequence $(\boldsymbol{Y}_{n_k})_{k\geq 0}$, where

$$\boldsymbol{Y}_{n_k} = \sum_{m=0}^{n_k} \langle \Phi, h_m\rangle \left(\frac{d}{dx}\right)^m [\boldsymbol{y}_{n_k}]$$

converges in $\text{Exp}(\mathbb{C})$ towards the element $\mathbb{D}_\Phi[\boldsymbol{y}_\infty] \in \text{Exp}(\mathbb{C})$. Hence the sequence $(e^{x^2/2}\boldsymbol{Y}_{n_k}(x))_{k\geq 0}$ converges uniformly on any compact of K towards $x \mapsto e^{x^2/2} y_\infty(x)$. Since the sequence $(y_{n_k})_{k\geq 0}$ converges uniformly to $x \mapsto e^{-x^2}$ and

$$f(x) = e^{-x^2/2}\Big(e^{x^2/2} \sum_{m\geq 0} \langle \Phi, h_m\rangle \left(\frac{d}{dx}\right)^m [e^{-x^2}]\Big) \quad \forall\, x \in K,$$

one has $y_\infty(x) = f(x)$ on $\mathbb{R}$. Since the result does not depend on the choice of the subsequence $(y_{n_k})_{k\geq 0}$, the conclusion of the Theorem follows. □

Remark 3.4 The sequence $(y_n)_{n\geq 1}$ realized as in (12) by averaging conveniently e_λ-superoscillating sequences according to Fourier inversion formula

$$e^{-x^2} = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-\lambda^2/4}\, e_\lambda(x)\, d\lambda$$

satisfies the required conditions in Theorem 3.3.

Remark 3.5 Up to a rescaling, the Gaussian g can be replaced by $x \mapsto e^{-x^2/(2\sigma)}$ for any $\sigma > 0$; the hypothesis needed then on f becomes $f(x) = e^{-x^2/(4\sigma)}\, \Phi(x)$, where $\sum_{m\geq 0} \langle \Phi, h_m\rangle\, z^m$ is a $(-1/2)$-Gevrey series.

4 $\boldsymbol{T}$-Superoscillating Sequences ($\boldsymbol{T} \in \mathscr{S}'(\mathbb{R}, \mathbb{C})$)

The space $\mathscr{S}'(\mathbb{R}, \mathbb{C})$ of complex tempered distributions in $\mathbb{R}$, namely complex distributions in $\mathbb{R}$ which are restrictions to the real line of complex valued distributions on its compactification $\mathbb{S}^1 = \mathbb{P}^1(\mathbb{R})$, can be characterized as well in terms of the asymptotic behavior of the coefficients in the spectral Hermite developments of its elements. More precisely, any element in $\boldsymbol{T} \in \mathscr{S}'(\mathbb{R}, \mathbb{C})$ can be expressed as the limit (in $\mathscr{S}'(\mathbb{R}, \mathbb{C})$)

$$\boldsymbol{T} = \lim_{M\to+\infty} \left(\sum_{m=0}^{M} \langle \boldsymbol{T}, h_m\rangle\, [h_m] \right), \tag{20}$$

where $|\langle \boldsymbol{T}, h_m\rangle| = O(m^p)$ for some $p \in \mathbb{N}$ and $[h_m]$ denotes the distribution-function induced by the Hermite function h_m. The action of T is therefore described as

$$\forall \varphi \in \mathscr{S}(\mathbb{R}, \mathbb{C}), \quad \langle \boldsymbol{T}, \varphi\rangle = \sum_{m=0}^{\infty} \langle \boldsymbol{T}, h_m\rangle\, \overline{\langle \varphi, h_m\rangle}$$

(see [22], Theorem 2.2). For example, the Dirac measure $\delta_0 \; : \varphi \mapsto \varphi(0) = (1/2\pi) \int_{\mathbb{R}} \widehat{\varphi}(\lambda)\, d\lambda$ can be approximated in two ways :

- either as the limit in $\mathscr{S}'(\mathbb{R}, \mathbb{C})$ of the sequence of distributions-functions $\big([\varepsilon_m^{-1}\, g(x/\varepsilon_m)]\big)_{m\geq 0}$, where $(\varepsilon_m)_{m\geq 0}$ denotes an arbitrary sequence of strictly positive numbers that converges towards 0, which is the most classical way to modelize the impulsion at the origin ;
- either as $\delta_0 = \sum_{m=0}^{\infty} h_m(0)\, [h_m]$, the convergence of the series being understood in $\mathscr{S}'(\mathbb{R}, \mathbb{C})$, that is $\varphi(0) = \sum_{m\geq 0} h_m(0)\, \overline{\langle h_m, \varphi\rangle}$ for any $\varphi \in \mathscr{S}(\mathbb{R}, \mathbb{C})$.

It does not make sense to restrict a distribution to a segment $[a, b]$ of $\mathbb{R}$; in order to adapt the notion of f-oscillating sequence to the frame of tempered complex distributions on the real line, one needs then to modify slightly Definition 1.1.

Definition 4.1 Given $\boldsymbol{T} \in \mathscr{S}'(\mathbb{R}, \mathbb{C})$, a sequence $([Y_n])_{n\geq 1}$ with entries distributions-functions such that $Y_n \in T(\mathbb{R}, \mathbb{C})$ for any $n \in \mathbb{N}^*$ is said to be $\boldsymbol{T}$-superoscillating whenever there exists a frequential threshold $\lambda_f > 0$, together with a relatively compact open set U, such that

- $\operatorname{Supp} \widehat{\boldsymbol{T}} \cap (\mathbb{R} \setminus [-\lambda_{\boldsymbol{T}}, \lambda_{\boldsymbol{T}}]) \neq \emptyset$;
- $\Lambda[Y_n] \subset [-\lambda_{\boldsymbol{T}}, \lambda_{\boldsymbol{T}}]$ for any $n \in \mathbb{N}^*$;
- one has $\lim_{n\to+\infty}[(Y_n)_{|U}] = \boldsymbol{T}_{|U}$ in $\mathscr{D}'(U, \mathbb{C})$.

The relatively compact open set U is then called a superoscillation set for the $\boldsymbol{T}$-superoscillating sequence $([Y_n])_{n\geq 1}$.

Remark 4.2 Since the concept of superoscillating sequence involves by itself necessarily the notion of spectrum, it is not possible to extend Definition 4.1 to the case where $\boldsymbol{T}$ would belong just to $\mathscr{D}'(\mathbb{R}, \mathbb{C})$. One needs indeed the spectrum $\Lambda[\mathbf{T}] = \operatorname{Supp} \widehat{\boldsymbol{T}}$ to be well defined, thus in principle $\boldsymbol{T}$ to be tempered.

Theorem 4.3 *Let the constants c_m ($m \in \mathbb{N}$) and $J_{m,n,k}$ ($m \in \mathbb{N}$, $n \in \mathbb{N}^*$, $k \in [[0, n]]$) be defined as in Theorem 2.1. Let $\boldsymbol{T} \in \mathscr{S}'(\mathbb{R}, \mathbb{C})$ such that $\operatorname{Supp} \widehat{\boldsymbol{T}} \cap (\mathbb{R} \setminus [-1, 1]) \neq \emptyset$. One can find a sequence $(N_n)_{n\geq 1}$ of strictly positive integers such that the sequence $([Y_n^{\boldsymbol{T}}])_{n\geq 0}$, where*

$$Y_n^{\boldsymbol{T}} = \frac{1}{\sqrt{2\pi}} \left(\sum_{k=0}^{N_n} \left(\sum_{m=0}^{n} (-i)^m c_m \, \langle \boldsymbol{T}, h_m \rangle \, J_{m,N_n,k} \right) e_{1-2k/N_n} \right),$$

is a $\boldsymbol{T}$-superoscillating for which any bounded open set $]-T, T[$ is a superoscillation set.

Proof The sequence of distributions-functions $\left(\sum_{m=0}^{M} \langle \boldsymbol{T}, h_m \rangle \, [h_m] \right)_{M\geq 0}$ converges to the distribution $\boldsymbol{T}$ in $\mathscr{S}'(\mathbb{R}, \mathbb{C})$ (see [22], Theorem 2.2). Hence, for any $M \in \mathbb{N}$, the sequence of restrictions $\left(\sum_{m=0}^{M} \langle \boldsymbol{T}, h_m \rangle \, [h_m]_{|]-T,T[} \right)_{M\geq 0}$ converges to $\boldsymbol{T}_{]-T,T[}$ in $\mathscr{D}'(]-T, T[, \mathbb{C})$. For any $m \in \mathbb{N}$, one can find, repeating the arguments leading to (12), then (13) in the proof of Theorem 2.1, a strictly increasing sequence of strictly positive integers $(N_{M,n})_{n\geq 1}$ such the sequence of functions $(Y_{M,n})_{n\geq 1}$ in $T(\mathbb{R}, \mathbb{C})$ defined as

$$Y_{M,n}^{\boldsymbol{T}} = \frac{1}{\sqrt{2\pi}} \left(\sum_{k=0}^{N_{M,n}} \left(\sum_{m=0}^{M} (-i)^m c_m \, \langle \boldsymbol{T}, h_m \rangle \, J_{m,N_{M,n},k} \right) e_{1-2k/N_{M,n}} \right) \quad \forall n \geq 1$$

converges uniformly on $[-M, M]$ towards $\sum_{m=0}^{M} \langle \boldsymbol{T}, h_m \rangle \, h_m$, hence is such the sequence of distributions-functions $([Y_{M,n}]_{|]-M,M[})_{n\geq 1}$ (as a sequence with entries in $\mathscr{D}'(]-M, M[, \mathbb{C})$) converges in $\mathscr{D}'(]-M, M[, \mathbb{C})$ towards the distribution-function $\sum_{m=0}^{M} \langle \boldsymbol{T}, h_m \rangle \, [h_m]_{|]-M,M[}$. One can then use the Cantor diagonal process and take for example $M_n = n$, $N_n = M_{n,n}$ for any $n \geq 1$ in order to conclude.

If one wishes to reformulate within the frame of distributions Proposition 3.1, keeping track the method (relying on Fubini's theorem) used in Sect. 3 in order to formulate and prove Proposition 3.1 within the frame of the Schwartz space, one needs first to introduce for any $s \geq 0$ the following adhoc Gelfand-Shilov-Gevrey subspace of $\mathscr{S}(\mathbb{R}, \mathbb{C})$ through the Hermite development of its elements:

$$\begin{aligned}\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C}) := \Big\{\Phi \in \mathscr{S}(\mathbb{R}, \mathbb{C})\,;\ &\sum_{m=0}^{\infty} |\langle \Phi, h_m\rangle|^2 [(m!)^{1+s}]^2 r^m \\ &< +\infty \quad \forall\, r > 0\Big\} \\ = \big\{\Phi \in \mathscr{S}(\mathbb{R}, \mathbb{C})\,;\ &\sup_{m\in\mathbb{N}} |\langle \Phi, h_m\rangle|\,|(m!)^{1+s}|\, r^m \\ &< +\infty \quad \forall\, r > 0\big\} \\ = \Big\{\Phi \in \mathscr{S}(\mathbb{R}, \mathbb{C})\,;\ &\Big(z \in \mathbb{C} \mapsto \sum_{m=0}^{\infty} \langle \Phi, h_m\rangle\, z^m\Big) \\ &= O(e^{\varepsilon |z|^{1/(1+s)}}) \quad \forall\, \varepsilon > 0\Big\}\end{aligned}$$

(note that $1/(1+s) \in]0, 1]$). The $\mathbb{C}$-vector space $\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})$ is equipped here with its topology of projective limit of weighted $\ell^2_{\mathbb{C}}(\mathbb{N})$-spaces. The Fourier transform $f \mapsto \widehat{f}$ realizes a continuous automorphism of $\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})$ since it admits each Hermite function h_m as an eigenvector (with corresponding eigenvalue $(-i)^m\sqrt{2\pi}$).

Similarly, for each $s' \geq 0$, let us define the following $\mathbb{C}$-vector space of *Gevrey-ultra-distributions* $\mathscr{S}'_{\mathrm{Gev},s'}(\mathbb{R}, \mathbb{C})$ which elements are the formal series

$$\boldsymbol{T} = \sum_{m\geq 0} \boldsymbol{t}_m\, [h_m] \quad (\boldsymbol{t}_m \in \mathbb{C}),$$

where $\sum_{m\geq 0} \boldsymbol{t}_m\, z^m$ is a Gevrey series with order $s' \geq 0$. Note that

$$\mathscr{S}'_{\mathrm{Gev},s'}(\mathbb{R}, \mathbb{C}) = \Big\{\boldsymbol{T} = \sum_{m\geq 0} \boldsymbol{t}_m\, [h_m]\,;\ \sum_{m=0}^{\infty} \frac{|\boldsymbol{t}_m|^2}{[(m!)^{s'}]^2}\, r^m < +\infty \text{ for some } r > 0\Big\}.$$

The vector space $\mathscr{S}'_{\mathrm{Gev},s'}(\mathbb{R}, \mathbb{C})$ is equipped here with its inductive limit of weighted $\ell^2_{\mathbb{C}}(\mathbb{N})$-vector spaces. When $s' = 1 + s$, the space $\mathscr{S}'_{\mathrm{Gev},1+s}(\mathbb{R}, \mathbb{C})$ stands for the dual of $\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})$, the duality being realized as

$$\Big\langle \sum_{m=0}^{\infty} \boldsymbol{t}_m\, [h_m]\,,\ \sum_{m=0}^{\infty} \langle \Phi, h_m\rangle\, h_m \Big\rangle = \sum_{m\geq 0} \boldsymbol{t}_m\, \overline{\langle \Phi, h_m\rangle} \langle [h_m], h_m\rangle = \sum_{m\geq 0} \boldsymbol{t}_m\, \overline{\langle \Phi, h_m\rangle}, \tag{21}$$

according to the fact that the system $\{h_m\,;\ m \geq 0\}$ is an Hilbertian basis of $L^2(\mathbb{R}, \mathbb{C})$. The space $\mathscr{S}'_{\mathrm{Gev},s}(\mathbb{R}, \mathbb{C})$ embeds continuously in $\mathscr{S}'_{\mathrm{Gev},1+s}(\mathbb{R}, \mathbb{C}) = [\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})]'$.

When $\boldsymbol{T} \in \mathscr{S}'_{\mathrm{Gev},s}(\mathbb{R}, \mathbb{C})$, it follows from Cauchy-Schwarz inequality together with the estimates (6) that for any $\varepsilon > 0$ and $n \geq 1 + \varepsilon$, for any $\Phi \in \mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})$,

$$\sum_{m\geq 0} |\boldsymbol{t}_m|\, |\langle \Phi, h_m\rangle| \iint_{\mathbb{R}^2} \exp\Big(\Big(\frac{1+\varepsilon}{2}\Big)\frac{\lambda^2}{2n}\Big)\, |h_m(\lambda)|\, |h_m(x)|\, d\lambda\, dx$$
$$\leq C_\Phi \sqrt{\sum_{m\geq 0} \frac{|\boldsymbol{t}_m|^2}{(m!)^{2s}}\, r^m} \sqrt{\sum_{m\geq 0} |\langle \Phi, h_m\rangle|^2\, [(m!)^{1+s}]^2\, (4r)^{-m}}$$
$$\times \iint_{\mathbb{R}^2} e^{-\lambda^2/4 - x^2/2 + 2(|\lambda|+|x|)}\, d\lambda\, dx < +\infty. \tag{22}$$

One can now state here the pendant of Proposition 3.1.

Proposition 4.4 *Let $s \geq 0$ and $\boldsymbol{T} = \sum_{m\geq 0} \boldsymbol{t}_m\, [h_m] \in \mathscr{S}'_{\mathrm{Gev},s}(\mathbb{R}, \mathbb{C}) \hookrightarrow [\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})]'$. Then for any $\Phi \in \mathscr{S}_{\mathrm{Gev},-s-1}$, one has*

$$\langle \boldsymbol{T}, \Phi\rangle$$
$$= \frac{1}{\sqrt{2\pi}} \lim_{T\to+\infty} \Big[\lim_{n\to+\infty} \Big(\sum_{m=0}^{\infty} (-i)^m\, c_m \boldsymbol{t}_m\, \overline{\langle \Phi, h_m\rangle} \int_{-T}^{T} \Big(\sum_{k=0}^{n} J_{m,n,k}\, e_{1-2k/n}(x)\Big)\, h_m(x)\, dx\Big)\Big]. \tag{23}$$

Proof Taking into account (22), it follows from Fourier inversion formula and Lebesgue's theorem that for any $\Phi \in \mathscr{S}'_{\mathrm{Gev},-s-1}(\mathbb{R}, \mathbb{C})$, one has

$$\langle \boldsymbol{T}, \Phi\rangle = \frac{1}{\sqrt{2\pi}} \lim_{T\to+\infty} \Big(\sum_{m=0}^{\infty} (-i)^m \boldsymbol{t}_m\, \overline{\langle \Phi, h_m\rangle} \int_{-T}^{T} \Big(\int_{\mathbb{R}} e^{ix\lambda}\, h_m(\lambda)\, d\lambda\Big)\, h_m(x)\, dx\Big).$$

The result follows then from (2), together with a second application of Lebesgue's theorem, which can be used here since (22) and estimates (11) are valid for n large (provided $x \in [-T, T]$ with $T > 0$ fixed). □

Remark 4.5 Since the Fourier transform realizes an automorphism of $\mathscr{S}_{\mathrm{Gev},-1-s}(\mathbb{R}, \mathbb{C})$, one can define naturally the Fourier transform $\widehat{\boldsymbol{T}}$ of $\boldsymbol{T}$ thanks to the usual rule $\langle \widehat{\boldsymbol{T}}, \Phi\rangle = \langle \boldsymbol{T}, \widehat{\Phi}\rangle$. What is unclear is the notion of spectrum $\Lambda[\boldsymbol{T}]$ for such an ultra-distribution, that is the support of $\widehat{\boldsymbol{T}}$. What Proposition 4.4 says is that given any $\boldsymbol{T}$ in $\mathscr{S}'_{\mathrm{Gev},s}(\mathbb{R}, \mathbb{C})$ ($s \geq 0$), it is in some sense possible to approximate it with distribution-functions associated to generalized trigonometric polynomial functions with spectrum in $[-1, 1]$.

As for the "persistence" Theorem 3.3, we do not know for the moment how to reformulate it in the setting of tempered distributions or (most probably) ultra-distributions of the Gevrey type introduced previously in this section. Given an ultra-distribution $\boldsymbol{T} = \sum_{m\geq 0} \boldsymbol{t}_m\, [h_m]$ in $\mathscr{S}'_{\mathrm{Gev},s}(\mathbb{R}, \mathbb{C})$ with $s > 0$, a natural suggestion in order to exploit the formal relation

$$\boldsymbol{T} = \sum_{m\geq 0} \boldsymbol{t}_m\, [h_m] = e^{x^2/2}\, \mathbb{D}_{\boldsymbol{T}}\big[[e^{-x^2}]\big],$$

where $\mathbb{D}_{\boldsymbol{T}} = \sum_{m\geq 0} \boldsymbol{t}_m\,(d/dx)^m$ (formally) as we did in Sect. 3 is to appeal now ($s > 0$ instead of $s = -1/2 < 0$, for example $s = 1/2$) to the concept of *Borel resummation*. The formal Borel transform $z \mapsto \widehat{\mathscr{B}}_{1/s}(z) := \sum_{m\geq 0} \big(\boldsymbol{t}_{m+1}/\,\Gamma(1+ms)\big)\,z^m$ of the symbol of $\mathbb{D}_{\boldsymbol{T}}$ defines an holomorphic function about the origin and Borel type resommation methods rely on the additional hypothesis of $1/s$- *summability along a specific direction* [32] : namely that for such direction $\theta \in \mathbb{S}^1$, the sum of the Borel transform $\widehat{\mathscr{B}}_{1/s}$ about the origin extends as an holomorphic function with growth in $O(e^{|z|^{1/s}})$ to a sector Γ_θ with aperture in $]0,\pi[$ which is bisected by θ, see [12] or [10] for related functionals and the dualizing role of the Laplace transform. If ρ_s denotes the ramification operator which consists in replacing z by z^s on the Riemann surface of the logarithm, then

$$\rho_{1/s} \circ \Big[\frac{1}{s}\int_{\theta\mathbb{R}^+} \widehat{\mathscr{B}}_{1/s}(u)\,e^{-u^{1/s}/z}\,u^{1/s-1}\,du\Big](z)$$

stands for a "resummation" $\mathrm{resumm}_\theta[f]$ of the symbol $f = \sum_{m\geq 0} \boldsymbol{t}_m\,z^m$, namely

$$\boldsymbol{t}_0 + \mathrm{resumm}_\theta[f](z) = \boldsymbol{t}_0 + \sum_{m\geq 0} \boldsymbol{t}_{m+1}\,z^{m+1}.$$

This leads to an asymptotic development of f in $1/z$ approaching the origin precisely along the direction θ. We plan to come back to these questions (e.g. clarify $\mathrm{resumm}_\theta(\mathbb{D}_{\boldsymbol{T}})$ and its action on the entries $(y_n)_{n\geq 1}$ of a $[g]$-superoscillating sequence) for a future project.

5 Extrapolation of Band-Limited Signals with Finite Energy by Means of e_λ-Superoscillating Sequences

Let T, Ω two strictly positive constants and H_T, H^Ω be the closed subspaces of the Hilbert space $L^2 = L^2(\mathbb{R},\mathbb{C})$ defined as

$$\begin{aligned} H_T &= \{f \in L^2\,;\ f = 0\ dx - \text{almost everywhere on } \mathbb{R}_x \setminus [-T,T]\} \\ H^\Omega &= \{f \in L^2\,;\ \widehat{f} = 0\ d\lambda - \text{almost everywhere on } \mathbb{R}_\lambda \setminus [-\Omega,\Omega]\}, \end{aligned}$$

where

$$\widehat{f} = \lim_{\substack{L^2 \\ T\to+\infty}} \Big(\int_{-T/2}^{T/2} f(x)\,e^{i(\cdot)x}\,dx\Big)$$

denotes the L^2-spectrum of f. The union of subspaces H^Ω for $\Omega > 0$ defines the (non-closed in L^2, though each H^Ω is closed) subspace $\mathrm{BL}_2(\mathbb{R},\mathbb{C})$ of band-limited signals. It is well known that if sinc denotes the sinus cardinal function

$$\mathrm{sinc}\ : x \in \mathbb{R} \longmapsto \frac{\sin(\pi x)}{\pi x},$$

one has the following identity

$$\sum_{\ell\in\mathbb{Z}}\Big[\mathrm{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\frac{\pi}{\Omega}\Big)\Big)\Big]^2 = 1 \quad \forall x\in\mathbb{R}. \tag{24}$$

Moreover the Nyquist–Shannon theorem (see for example [30]) ensures that any element in H^Ω admits a continuous representative $x\in\mathbb{R}\mapsto f(x)$ which extends as an entire function of $z\in\mathbb{C}$ and is such that for any $K\in\mathbb{N}^*$

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\Big|f(x)-\sum_{\ell=-K}^{K} f(\ell\pi/\Omega)\,\mathrm{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\frac{\pi}{\Omega}\Big)\Big)\Big|\\
&\quad\le\Big(\sum_{|\ell|>K}|f(\ell\pi/\Omega)|^2\Big)^{1/2}\sup_{x\in\mathbb{R}}\Big(\sum_{|\ell|>K}\Big[\mathrm{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\frac{\pi}{\Omega}\Big)\Big)\Big]^2\Big)^{1/2}\\
&\quad\le\Big(\sum_{|\ell|>K}|f(\ell\pi/\Omega)|^2\Big)^{1/2}\sup_{x\in\mathbb{R}}\Big(\sum_{k\in\mathbb{Z}}\Big[\mathrm{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\frac{\pi}{\Omega}\Big)\Big)\Big]^2\Big)^{1/2}\\
&\quad\le\Big(\sum_{|\ell|>K}|f(\ell\pi/\Omega)|^2\Big)^{1/2}.
\end{aligned} \tag{25}$$

We have in addition the following tolerance with respect to the fact that the band-limited condition fails to be exactly fulfilled : for any $f\in L^2$ such that $\widehat{f}\in L^1$ (but f is not assumed anymore to belong to H^Ω), f still admits a continuous representative $f:\mathbb{R}\to\mathbb{C}$ (which fails now in general to extend as an entire function of $t\in\mathbb{C}$) such that, for any $K\in\mathbb{N}^*$,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\Big|f(x)-\sum_{\ell=-K}^{K} f(\ell\pi/\Omega)\,\mathrm{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\frac{\pi}{\Omega}\Big)\Big)\Big|\\
&\qquad\le\Big(\sum_{|\ell|>K}|f(\ell\pi/\Omega)|^2\Big)^{1/2}+\frac{1}{\pi}\int_{|\lambda|>\Omega}|\widehat{f}(\lambda)|\,d\lambda.
\end{aligned} \tag{26}$$

This is true, in particular, when f is still band-limited, but such that $\mathrm{Supp}(\widehat{f})\subset[-\widetilde{\Omega},\widetilde{\Omega}]$ with $\widetilde{\Omega}>\Omega$, which may happen in practical situations since the threshold Ω cannot in general be precisely localized, $\widetilde{\Omega}$ standing in this case as a rough upper estimate of the true Ω.

Take here $T=1$ (up to a dilation of f by $1/T$ which forces to change Ω into $T\Omega$). Given $f\in H^\Omega$ such that $f_0=\mathrm{Proj}^{\perp}_{H_1}[f]$ is known, the inductive procedure initiated at f_0 and ruled by

$$f_{k+1}=f_0+(1-\chi_{[-1,1]})\times\text{Fourier inverse}\Big[\chi_{[-\Omega,\Omega]}\,\widehat{f_k}\Big]\quad\forall k\in\mathbb{N}$$

inspired the extrapolation method developed by R.W. Gerchberg and A. Papoulis [25,29]. The singular value decomposition of the compact normal operator S_1^Ω from $L^2([-1,1],\mathbb{C})$ into itself

$$S_1^\Omega \;:\varphi \longmapsto \int_{-1}^{1} \varphi(x)\,\mathrm{sinc}\Big(\frac{\Omega}{\pi}((\cdot)-x)\Big)\,dx,$$

in particular the behavior of the decreasing sequence

$$1 = \|S_1^\Omega\| = \lambda_{1,0}^\Omega > \lambda_{1,1}^\Omega \geq \lambda_{1,2}^\Omega \cdots \ldots > 0$$

of its eigenvalues (more specifically the careful analysis of the slope of the discrete decreasing function $n \mapsto \lambda_{1,n}^\Omega$ starting from $n = 0$), together with the so-called prolate eigenfunctions (see [33]) plays a central role in the stability of an Hilbertian algorithmic approach. Other ways in order to face such extrapolation problem (which is not well-posed, but conditionally stable only, see in particular [11], VIII.31 for appropriate references) are based for example on the use of Carleman's interpolation technics as in [11] or either the Cauchy transform as in [20].

We propose here an alternative approach inspired by the concept of superoscillating approximations.

Theorem 5.1 *Let $\Omega > 0$, $f \in H^\Omega$ with continuous representant still denoted as f, and $(\nu_j)_{j\geq 0}$, $(\mu_j)_{j\geq 0}$ be two strictly increasing sequences of strictly positive integers such that $\mu_j^3 = o(\nu_j^2)$ when j tends to $+\infty$. For any $j \in \mathbb{N}$, let*

$$\boldsymbol{f}_j \;: x \longmapsto \frac{1}{2^{\nu_j}} \sum_{k=0}^{\nu_j} \binom{\nu_j}{k} \Big(\sum_{\ell=-\mu_j}^{\mu_j} \Big(\frac{\ell\,\Omega}{\pi}+1\Big)^{\nu_j-k} \Big(1-\frac{\ell\,\Omega}{\pi}\Big)^{k} \mathrm{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\,\frac{\pi}{\Omega}\Big)\Big)\Big)\, f(1-2k/\nu_j).$$

Then, for any j sufficiently large (depending on Ω and on the growth of the sequence $(\nu_j)_{j\geq 0}$),

$$\begin{aligned}\sup_{\mathbb{R}} \|f-\boldsymbol{f}_j\| &\leq \sqrt{\sum_{|\ell|>\mu_j} |f(\ell\pi/\Omega)|^2} + 2\pi\,\frac{\mu_j\sqrt{2\mu_j+1}}{\nu_j} \int_{[-\Omega,\Omega]} |\widehat{f}(\lambda)|\,d\lambda \\ &= o_j(1) \quad (as\ j \to +\infty).\end{aligned} \tag{27}$$

Moreover, if $f \in H^{\widetilde{\Omega}}$ with $\Omega < \widetilde{\Omega} < +\infty$, one has

$$\begin{aligned}\sup_{\mathbb{R}} \|f-\boldsymbol{f}_j\| \leq & \sqrt{\sum_{|\ell|>\mu_j} |f(\ell\pi/\Omega)|^2} + 2\pi\,\frac{\mu_j\sqrt{2\mu_j+1}}{\nu_j} \int_{[-\Omega,\Omega]} |\widehat{f}(\lambda)|\,d\lambda \\ & + \frac{1}{\pi}\int_{\Omega<|\lambda|\leq\widetilde{\Omega}} |\widehat{f}(\lambda)|\,d\lambda = o_j(1) \\ & + \frac{1}{\pi}|\widehat{f}(\lambda)|\,d\lambda \quad (as\ j \to +\infty).\end{aligned} \tag{28}$$

Proof Suppose first that $f \in H^\Omega$. Let also f denote the continuous representant of the class in $L^2(\mathbb{R},\mathbb{C})$ (f is known to be in fact the restriction to $\mathbb{R}$ of an entire function in Exp). Fourier inversion formula implies

$$f(\ell\pi/\Omega) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\lambda)\, e^{i\lambda\,\ell\pi/\Omega}\, d\lambda = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \widehat{f}(\lambda)\, e^{i\lambda(\ell\pi/\Omega)}\, d\lambda \quad \forall\,\ell \in \mathbb{Z}.$$

For any $\ell \in \mathbb{Z}$, for any $\lambda \in [-\Omega, \Omega]$, one has (see (2))

$$e^{i\lambda(\ell\pi/\Omega)} = \lim_{j\to+\infty} \frac{1}{2^{\nu_j}} \Big(\sum_{k=0}^{\nu_j} \binom{\nu_j}{k} \Big(\frac{\ell\pi}{\Omega}+1\Big)^{\nu_j-k}\Big(1-\frac{\ell\pi}{\Omega}\Big)^{k} e^{i(1-2k/\nu_j)\lambda}\Big). \quad (29)$$

Moreover, as seen in [10], Chapter 3, Theorem 3.1.8 and Remark 3.1.15, the estimation of the error in (29) is, for j sufficiently large (depending on Ω and on the growth of the sequence $(\nu_j)_{j\geq 0}$)

$$\begin{aligned} &\Big| e^{i\lambda(\ell\pi/\Omega)} - \frac{1}{2^{\nu_j}} \sum_{k=0}^{\nu_j} \binom{\nu_j}{k} \Big(\frac{\ell\pi}{\Omega}+1\Big)^{\nu_j-k}\Big(1-\frac{\ell\pi}{\Omega}\Big)^{k} e^{i(1-2k/\nu_j)\ell}\Big| \\ &\quad \leq 2\,\frac{\Omega}{\nu_j}\,\frac{|\ell|\pi}{\Omega} = 2\frac{|\ell|\pi}{\nu_j}. \end{aligned} \quad (30)$$

This implies for any $\ell \in \mathbb{Z}$, for any such $j \in \mathbb{N}$ sufficiently large (depending on Ω and on the growth of $(\nu_j)_{j\geq 0}$)

$$\begin{aligned} &\Big| f(\ell\pi/\Omega) - \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \frac{\widehat{f}(\lambda)}{2^{\nu_j}} \Big(\sum_{k=0}^{\nu_j} \binom{\nu_j}{k} \Big(\frac{\ell\pi}{\Omega}+1\Big)^{\nu_j-k}\Big(1-\frac{\ell\pi}{\Omega}\Big)^{k} e^{i(1-2k/\nu_j)\lambda}\Big)\, d\lambda\Big| \\ &= \Big| f(\ell\pi/\Omega) - \frac{1}{2^{\nu_j}} \sum_{k=0}^{\nu_j} \binom{\nu_j}{k} \Big(\frac{\ell\pi}{\Omega}+1\Big)^{\nu_j-k}\Big(1-\frac{\ell\pi}{\Omega}\Big)^{k} f\Big(1-\frac{2k}{\nu_j}\Big)\Big| \\ &\leq \frac{2|\ell|\pi}{\nu_j}\int_{[-\Omega,\Omega]} |\widehat{f}(\lambda)|\, d\lambda. \end{aligned} \quad (31)$$

For any such j sufficiently large, it follows from inequalities (31) for $-\mu_j \leq \ell \leq \mu_j$, together with inequality (25), Cauchy-Schwarz inequality and formula (24), that

$$\begin{aligned} &\sup_{\mathbb{R}} \Big| f(x) - \frac{1}{2^{\nu_j}} \sum_{k=0}^{\nu_j} \binom{\nu_j}{k} \Big(\sum_{\ell=-\mu_j}^{\mu_j} \Big(\frac{\ell\pi}{\Omega}+1\Big)^{\nu_j-k}\Big(1-\frac{\ell\pi}{\Omega}\Big)^{k} \operatorname{sinc}\Big(\frac{\Omega}{\pi}\Big(x-\ell\,\frac{\pi}{\Omega}\Big)\Big)\Big) f\Big(1-\frac{2k}{\nu_j}\Big)\Big| \\ &\leq \sqrt{\sum_{|\ell|>K} |f(\ell\pi/\Omega)|^2} + 2\pi\,\frac{\mu_j\sqrt{2\mu_j+1}}{\nu_j}\int_{[-\Omega,\Omega]} |\widehat{f}(\lambda)|\, d\lambda. \end{aligned}$$

This is the required inequality (27). When $f \in H^{\widetilde{\Omega}}$ with $\Omega < \widetilde{\Omega}$, one repeats the argument starting from inequality (26) instead of (25). Note that it remains essential that f is known for sure to belong to $\mathrm{BL}_2(\mathbb{R}, \mathbb{C})$, which is required each time one claims that j can be chosen large enough (namely to ensure (30), hence (31)), depending now on $\widetilde{\Omega}$ instead of Ω, and still on the behavior of the sequence $(\nu_j)_{j\geq 0}$. □

Example 5.2 If one choses, for any $j \geq 0$, $\mu_j := 2^j$ and $\nu_j := 3^j$, the required condition $\mu_j^{3/2} = o(\nu_j)$ is fulfilled since $\log 3/\log 2 \geq 1.58 > 3/2$. One needs (in order theoretically to extrapolate f uniformly on $\mathbb{R}$) to be able to evaluate exactly (that is up to an arbitrary precision !) f at all triadic points $\ell/3^j$, $\ell = -3^j, ..., 3^j$. Chosing the sequence $(\nu_j)_{j\geq 0}$ as the sequences of powers such as $(3^j)_{j\geq 0}$ allows indeed the memorization of the values of f at triadic points in $[-1, 1]$ from the instant where they have been selected for the first time.

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