

A SYMMETRY RESULT FOR ELLIPTIC SYSTEMS IN PUNCTURED DOMAINS

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ABSTRACT. We consider an elliptic system of equations in a punctured bounded domain. We prove that if the domain is convex in one direction and symmetric with respect to the reflections induced by the normal hyperplane to such a direction, then the solution is necessarily symmetric under this reflection and monotone in the corresponding direction. As a consequence, we prove symmetry results also for a related polyharmonic problem of any order with Navier boundary conditions.

1. Introduction. In this paper we will study general cooperative systems of second-order elliptic equations in the spirit of Troy [20]. The equations are set in a punctured domain, thus allowing possible singularities; nevertheless, the structure of the system being (partially) cooperative, we are allowed to use the maximum principle. Exploiting the moving plane method, we will prove that positive solutions in a domain which is symmetric under a given reflection also possess the same type of symmetry, and moreover they are monotone with respect to the direction of symmetry (in particular, if the domain is a punctured ball, the solution is necessarily radial and radially decreasing).

We now introduce the mathematical setting in which we work.

Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 2$) be a domain (i.e., open, bounded and connected set) satisfying the following *structural assumptions*, which we assume to be satisfied throughout the paper:

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- Ω is convex in the x_1 -direction, that is, if $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ belong to Ω and $p_j = q_j$ for any $j \in \{2, \dots, n\}$, then

$$(1-t)p + tq \text{ belongs to } \Omega \text{ for all } t \in [0, 1];$$

- Ω is symmetric with respect to the hyperplane $\{x_1 = 0\}$, that is,

$$\text{if } p = (p_1, p_2, \dots, p_n) \in \Omega \text{ then } p_0 := (-p_1, p_2, \dots, p_n) \in \Omega,$$

- 0 lies in Ω .

We point out that, since Ω is connected, the same is true of $\Omega \setminus \{0\}$.

Let now $m \in \mathbb{N}$ be fixed, we consider the following elliptic system

$$\begin{cases} -\Delta u_i = f_i(u_1, \dots, u_m), & \text{in } \Omega \setminus \{0\} \\ u_i = 0, & \text{on } \partial\Omega \\ u_i > 0, & \text{in } \Omega \setminus \{0\}. \end{cases} \quad (1.1)$$

We consider classical solutions of (1.1), namely $u_i \in C^2(\Omega \setminus \{0\}) \cap C(\bar{\Omega} \setminus \{0\})$ for every $i = 1, \dots, m$. We assume further that $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$ satisfy

$$f_i \in \text{Lip}(\mathbb{R}^m) \quad \text{and} \quad \frac{\partial f_i}{\partial u_k} \geq 0, \quad i \neq k, \quad 1 \leq i, k \leq m, \quad (f1)$$

where the sign assumption on the partial derivatives of f_i has to be intended in the L^∞ -sense. As anticipated, the main result of the present paper concerns symmetry properties of the positive solutions of (1.1) and is the following

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain satisfying the structural assumptions introduced above and let $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$ satisfy (f1). Moreover, let $U = (u_1, \dots, u_m) \in C^2(\Omega \setminus \{0\}; \mathbb{R}^m) \cap C(\bar{\Omega} \setminus \{0\}; \mathbb{R}^m)$ be a classical solution of the elliptic system (1.1).*

Then the following facts hold true: for every $i = 1, \dots, m$,

- (1) *u_i is symmetric in x_1 , i.e., $u_i(x_1, x_2, \dots, x_n) = u_i(-x_1, x_2, \dots, x_n)$ for every $x \in \Omega$;*
- (2) *u_i is decreasing with respect to x_1 on $\Omega \cap \{x_1 > 0\}$.*

Theorem 1.1 extends a classical result by Troy [20] to the case of punctured domains. In particular, when $m = 1$, our result recovers the classical result in [4]; on the other hand, when $m = 2$, the system (1.1) finds natural applications in engineering, for instance in the description of “hinged” rigid plates, see e.g. [12]. This leads to consider other examples of polyharmonic operators which naturally appear in the phase separation of a two component system, as described by the Cahn-Hilliard equation (see [5]), and when comparing the pointwise values of a function with its average (see [16]).

As is well-known, the literature concerning symmetry results for elliptic PDEs is extremely wide, and is far from our scopes to present here an exhaustive list of references. We must mention the seminal papers [3, 13, 18] for the use of the moving planes method in the PDEs setting; we also highlight [4, 10, 15, 17, 19] (for symmetry results for singular solutions of scalar semilinear equations in local and non-local setting) and [2, 6–9, 11, 20] (for symmetry results for semilinear polyharmonic problems and cooperative elliptic systems).

To continue in the direction traced in the aforementioned papers, it is then natural to apply Theorem 1.1 to a suitable class of polyharmonic problems with Navier boundary conditions, which comes from Pizzetti-type superpositions of polyharmonic operators with appropriate structural assumptions on the coefficients. More

precisely, given $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, we define $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and we consider the characteristic polynomial expansion

$$\det \begin{pmatrix} \alpha_1 + t & 0 & \dots & 0 \\ 0 & \alpha_2 + t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_m + t \end{pmatrix} = \sum_{k=0}^m s_k(\alpha) t^k, \quad (1.2)$$

where $s_m(\alpha) = 1$ (independently on α) and, for every $k = 0, \dots, m-1$, we have

$$s_k(\alpha) = \sum_{1 \leq i_1 < \dots < i_{m-k} \leq m} \alpha_{i_1} \dots \alpha_{i_{m-k}}. \quad (1.3)$$

We stress that, by the *Descartes rule of signs*, the positivity of $s_0(\alpha), \dots, s_{m-1}(\alpha)$ is equivalent to the positivity of all α_i ; see also Lemma A.1 for a self-contained proof.

Then, we consider the equation

$$\begin{cases} \sum_{k=0}^m s_k(\alpha) (-\Delta)^k u = f(u) & \text{in } \Omega \setminus \{0\}, \\ u > 0 & \text{in } \Omega \setminus \{0\}, \\ u = -\Delta u = \dots = (-\Delta)^{m-1} u = 0 & \text{on } \partial\Omega, \\ \inf_{\Omega \setminus \{0\}} (-\Delta)^j u > -\infty & j = 1, \dots, m-1. \end{cases} \quad (1.4)$$

In this context, we require f to satisfy the following assumptions:

$$f \in \text{Lip}(\mathbb{R}^+), \quad f(0) \geq 0 \quad \text{and} \quad f \text{ is non-decreasing.} \quad (\text{f2})$$

The symmetry and monotonicity result for (1.4) then goes as follows.

Theorem 1.2. *Assume that $s_0(\alpha), \dots, s_{k-1}(\alpha) \in [0, +\infty)$. Let $\Omega \subseteq \mathbb{R}^n$ be a domain satisfying the structural assumptions introduced above and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfy (f2). Moreover, let $u \in C^{2m}(\Omega \setminus \{0\}) \cap C^{2m-2}(\overline{\Omega} \setminus \{0\})$ be a classical solution of the $2m$ -th order boundary value problem (1.4).*

Then the following facts hold true:

- (1) *u is symmetric in x_1 , i.e., $u(x_1, x_2, \dots, x_n) = u(-x_1, x_2, \dots, x_n)$ for every $x \in \Omega$;*
- (2) *u is decreasing with respect to x_1 on $\Omega \cap \{x_1 > 0\}$.*

We notice that, if one aims to prove the *existence* of such a solution, some regularity on the boundary $\partial\Omega$ of Ω must be required see e.g., [12, Theorem 2.19]. On the other hand, we do not need to take any additional assumption here, since we are assuming a priori that a solution exists and we aim at proving its symmetry and monotonicity properties.

We also point out that, choosing $\alpha_1 = \dots = \alpha_m = 0$ (which gives that $s_i(\alpha) = 0$ for every $i = 0, \dots, k-1$), the above (1.4) becomes

$$\begin{cases} (-\Delta)^m u = f(u) & \text{in } \Omega \setminus \{0\}, \\ u > 0 & \text{in } \Omega \setminus \{0\}, \\ u = -\Delta u = \dots = (-\Delta)^{m-1} u = 0 & \text{on } \partial\Omega, \\ \inf_{\Omega \setminus \{0\}} (-\Delta)^j u > -\infty & j = 1, \dots, m-1. \end{cases} \quad (1.5)$$

which can be viewed as a natural extension of the second-order PDE studied in [4].

We now briefly describe how Theorem 1.1 can be used to prove Theorem 1.2. First of all, if $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, we set

$$u_1 := u, \quad u_{i+1} = (-\Delta + \alpha_i)u_i \quad (\text{for } i = 1, \dots, m-1); \quad (1.6)$$

with this notation, the $2m$ -th order boundary value problem (1.4) is equivalent to the following system

$$\begin{cases} (-\Delta + \alpha_i)u_i = u_{i+1} & \text{in } \Omega \setminus \{0\}, \quad i = 1, \dots, m-1, \\ (-\Delta + \alpha_m)u_m = f(u_1) & \text{in } \Omega \setminus \{0\}, \\ u_1 > 0 & \text{in } \Omega \setminus \{0\}. \end{cases} \quad (1.7)$$

We stress that (1.7) is a particular case of (1.1) with

$$f_i(u_1, \dots, u_m) := u_{i+1} \quad (\text{for } i = 1, \dots, m-1) \quad \text{and} \quad f_m(u_1, \dots, u_m) := f(u_1).$$

In view of this, we spend few words on the relation between (f1) and (f2). The request $f(0) \geq 0$ is closely related to the fact that in (1.4) we asked only $u_1 > 0$ and so the positivity of all the other components has to be proved. Indeed, condition (f2) together with the weak maximum principle in punctured domains (see [4, Lemma 2.1]), the lower-boundedness of the u_i 's and the standard strong maximum principle, yields the positivity of the components u_1, \dots, u_m of U in $\Omega \setminus \{0\}$. If we ask immediately for $u_i > 0$ for every $i = 1, \dots, m$, we can then avoid the extra assumption $f(0) \geq 0$, and then (f2) becomes a particular instance of (f1).

Finally, a couple of comments on the regularity assumption of f in (f2). When $m = 1$, in [4] the analogue of Theorem 1.2 (in the case $\alpha_1 = \dots = \alpha_m = 0$) is proved under the weaker assumption that f is only *locally* Lipschitz-continuous (and possibly depending on the spatial variable x). In our case, we instead assume a *global* Lipschitz assumption in (f1), since boundedness issues become more involved when $m \geq 2$ (roughly speaking, for the case of systems, the positivity of one component in a subdomain does not imply the positivity of the other components). For a similar reason, we also assume the bound

$$\inf_{\Omega \setminus \{0\}} u_i > -\infty. \quad (1.8)$$

Indeed, when $m = 1$ dealing with positive solutions implies immediately the former bound. On the other hand, for $m \geq 2$, while this is still true for u_1 , this is not automatically inherited by the other components of the system. We think that it is an interesting *open problem* to further investigate whether either the global Lipschitz regularity assumption or the bound from below of the $u_j = (-\Delta)^j u$ can be relaxed as in [4].

The paper is organized as follows. After stating some notation, in Section 2 we present the main technical results, related to suitable versions of the maximum principle for cooperative systems; then, Theorem 1.1 will be proved in Section 3 obtaining the symmetry result by the moving plane and reflection methods.

2. Notation, assumptions and preliminary results. We introduce some notation and the standing assumptions used along the paper. For a function $U : \Omega \rightarrow \mathbb{R}^m$, $U = (u_1, \dots, u_m)$, we say that $U \geq 0$ if

$$u_i \geq 0 \quad \text{for every } i = 1, \dots, m.$$

The notation for the moving plane technique is as in the paper of Serrin [18], and it goes as follows. Given a point $x \in \mathbb{R}^n$, we denote by (x_1, \dots, x_n) its components;

moreover, when more practical, we equivalently write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. For a given unit vector $e \in \mathbb{R}^n$ and for $\lambda \in \mathbb{R}$, we define the hyperplane

$$T_\lambda := \{x \in \mathbb{R}^n : \langle e, x \rangle = \lambda\}.$$

From now on, without loss of generality we assume that $e = e_1$, i.e. the normal to T_λ is parallel to the x_1 -direction. To simplify the readability, we further assume that

$$\sup_{\Omega} x_1 = 1. \quad (\text{A1})$$

Now, for every $\lambda \in (0, 1)$ we define

$$R_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad R_\lambda(x) = x_\lambda := (2\lambda - x_1, x'). \quad (2.1)$$

We stress that (2.1) may lead to points that *do not belong* to $\overline{\Omega}$: for example, we have

$$0_\lambda = (2\lambda, 0, \dots, 0) \notin \overline{\Omega}$$

for $\lambda > \frac{1}{2}$, in view of (A1). Proceeding further with the notation, given any $\lambda \in \mathbb{R}$, we introduce the possibly empty set

$$\Sigma_\lambda := \{x \in \Omega : x_1 > \lambda\}$$

and its reflection with respect to T_λ ,

$$R_\lambda(\Sigma_\lambda) = \Sigma'_\lambda := \{x_\lambda \in \mathbb{R}^n : x \in \Sigma_\lambda\}.$$

Since $\Omega \subset \mathbb{R}^n$ is bounded, by (A1) we have that T_λ does not touch $\overline{\Omega}$ for $\lambda > 1$; moreover, T_1 touches $\overline{\Omega}$ and, for every $\lambda \in (0, 1)$, the hyperplane T_λ cuts off from Ω the portion Σ_λ . At the beginning of this process, the reflection Σ'_λ of Σ_λ will be contained in Ω .

Let now $U = (u_1, \dots, u_m) \in C^2(\Omega \setminus \{0\}; \mathbb{R}^m) \cap C(\overline{\Omega} \setminus \{0\}; \mathbb{R}^m)$ be a classical solution of the second order elliptic system (1.1). For every $i = 1, \dots, m$ and for every fixed $\lambda \in (0, 1)$, we introduce the functions on $\Sigma_\lambda \setminus \{0_\lambda\}$ defined as follows:

$$u_i^{(\lambda)}(x) := u_i(x_\lambda) \quad \text{and} \quad v_i^{(\lambda)}(x) := u_i^{(\lambda)}(x) - u_i(x).$$

Finally, to simplify the notation, we also define

$$U_\lambda(x) := \left(u_1^{(\lambda)}, u_2^{(\lambda)}, \dots, u_m^{(\lambda)}\right) \quad x \text{ in } \Sigma_\lambda \setminus \{0_\lambda\}, \quad (2.2)$$

and

$$V_\lambda(x) := \left(v_1^{(\lambda)}, v_2^{(\lambda)}, \dots, v_m^{(\lambda)}\right) \quad x \text{ in } \Sigma_\lambda \setminus \{0_\lambda\}, \quad (2.3)$$

We observe that, since Ω is symmetric and convex with respect to $T_0 = \{x_1 = 0\}$, the above definitions (2.2) and (2.3) are well-posed for every $\lambda \in [0, 1]$.

We now recall the maximum principle in small domains for cooperative systems proved by de Figueiredo in [9].

Lemma 2.1 (Proposition 1.1 of [9]). *Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected subset of \mathbb{R}^n and let $A(x) = (a_{ij}(x))$ be a $m \times m$ -valued map such that $a_{ij} \in L^\infty(\Omega)$ and*

$$a_{ij} \geq 0 \text{ on } \Omega \quad \text{for every } i, j \in \{1, \dots, m\} \text{ with } i \neq j.$$

Suppose that $U := (u_1, \dots, u_m)$ is a vector-valued function in $C^2(\Omega)$ such that

$$\begin{cases} -(\Delta + A(x))U \geq 0 & \text{in } \Omega, \\ \liminf_{x \rightarrow y} U(x) \geq 0 & \text{for every } y \in \partial\Omega. \end{cases}$$

Then, there exists $\delta = \delta(n, \text{diam}(\Omega)) > 0$ such that if $|\Omega| < \delta$, $U \geq 0$ in Ω

The following technical result can be seen as a slight variation of Lemma 2.1 and as an extension of [4, Proposition 2.1] to the case of (special) cooperative systems.

Lemma 2.2. *Let $\xi \in \mathbb{R}^n, r > 0$ be fixed and let*

$$\Omega \subset B_r(\xi) \quad (2.4)$$

be an open set. Moreover, let $x_0 \in \Omega$ be arbitrarily chosen, $A(x) = (a_{ij}(x))$ be as in Lemma 2.1 and let $U = (u_1, \dots, u_m) \in C^2(\Omega \setminus \{x_0\})$ be such that

$$\begin{cases} -(\Delta + A(x))U \geq 0 & \text{in } \Omega \setminus \{x_0\}, \\ \liminf_{x \rightarrow y} U(x) \geq 0 & \text{for every } y \in \partial\Omega, \\ \inf_{\Omega \setminus \{x_0\}} u_i > -\infty & i = 1, \dots, m. \end{cases} \quad (2.5)$$

Then, for sufficiently small $r > 0$, we have

$$U \geq 0, \quad \text{in } \Omega \setminus \{x_0\}.$$

Proof. Without loss of generality, we can assume that $\xi = x_0 = 0$. We then consider the vector-valued function $W := U + \epsilon H$ given by

$$W := \begin{pmatrix} u_1 + \epsilon h \\ u_2 + \epsilon h \\ \vdots \\ u_m + \epsilon h \end{pmatrix},$$

where $h(x) := (-\ln(|x|))^a$, with $a \in (0, 1)$ to be chosen, as in [4]. Obviously, we have that

$$h \geq 0 \text{ on } B_r(0), \text{ and } h \rightarrow +\infty \text{ as } |x| \rightarrow 0. \quad (2.6)$$

Moreover, we claim that

$$\Delta h \leq 0, \quad \text{in } B_r(0) \setminus \{0\},$$

if $r > 0$ is small enough. Indeed, since $a_{ij} \in L^\infty(\Omega)$ for any i, j , we can define

$$C_{ij} := \sup\{a_{ij}(x) : x \in \Omega\} \quad \text{and} \quad K := \max_{i=1, \dots, m} \sum_{j=1}^m C_{ij}.$$

Then, a direct computation shows that

$$\begin{aligned} \Delta h + Kh &= -\frac{a(-\ln(|x|))^{a-1}}{|x|^2}(n-2) + \frac{a(a-1)}{|x|^2}(\ln(|x|))^{a-2} + K(-\ln(|x|))^a \\ &= \begin{cases} -\frac{a(-\ln(|x|))^{a-1}}{|x|^2} \cdot (n-2 + o(|x|)) \\ \frac{(\ln(|x|))^{a-2}}{|x|^2} \cdot (a(a-1) + o(|x|)) \end{cases} \quad (\text{as } |x| \rightarrow 0). \end{aligned}$$

Since $n \geq 2$ and $a \in (0, 1)$, if $r \ll 1$ is sufficiently small we obtain

$$\Delta h + Kh \leq 0 \quad \text{in } B_r(0) \setminus \{0\}. \quad (2.7)$$

Now, by definition of K and since $h \geq 0$ on $\mathbb{R}^n \setminus \{0\}$, from (2.4), (2.5) and (2.7) we get

$$\begin{aligned} [(\Delta + A(x))W]_i &= [(\Delta + A(x))U]_i + \epsilon \left(\Delta h + \left(\sum_{j=1}^m a_{ij}(x) \right) \cdot h \right) \\ &\leq \epsilon (\Delta h + Kh) \leq 0 \quad \text{for } i = 1, \dots, m. \end{aligned} \quad (2.8)$$

Furthermore, recalling (2.6), we have that

$$\bullet \liminf_{x \rightarrow y} W(x) \geq \liminf_{x \rightarrow y} U(x) \geq 0 \text{ for every } y \in \partial\Omega; \quad (2.9)$$

$$\bullet W_i(x) = u_i(x) + \epsilon h(x) \rightarrow +\infty \text{ as } |x| \rightarrow 0 \text{ (since } \inf_{\Omega \setminus \{x_0\}} u_i > -\infty \text{)}. \quad (2.10)$$

Gathering together (2.8), (2.9) and (2.10) (and by possibly shrinking r), we are entitled to apply the weak minimum principle in Lemma 2.1, obtaining $W = U + \epsilon H \geq 0$ in $\Omega \setminus \{0\}$. From this, taking the limit as $\epsilon \rightarrow 0^+$, we conclude that $U \geq 0$ in $\Omega \setminus \{0\}$. \square

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected subset of \mathbb{R}^n and let $A(x) = (a_{ij}(x))$ be a $m \times m$ -valued map such that $a_{ij} \in L^\infty(\Omega)$ and*

$$a_{ij} \geq 0 \text{ on } \Omega \quad \text{for every } i, j \in \{1, \dots, m\} \text{ with } i \neq j. \quad (2.11)$$

Moreover, let $U = (u_1, \dots, u_m) \in C^2(\Omega; \mathbb{R}^m)$ be such that

$$\begin{cases} \Delta U + A(x) \cdot U \geq 0, & \text{on } \Omega, \\ U \leq 0 & \text{on } \Omega. \end{cases} \quad (2.12)$$

If there exists $x_0 \in \Omega$ such that $u_k(x_0) = 0$ (for some $k \in \{1, \dots, m\}$), then

$$u_k \equiv 0 \quad \text{on } \Omega.$$

Proof. We consider the \mathbb{R}^m -valued function $V \in C^2(\Omega; \mathbb{R}^m)$ defined by

$$V(x) := e^{-\alpha x_1} U(x) \quad (x \in \Omega),$$

where $\alpha > 0$ is a suitable real constant which will be chosen later on. For every fixed $k \in \{1, \dots, m\}$ and every $x \in \Omega$ we have

$$\Delta v_k(x) = \alpha^2 v_k(x) + e^{-\alpha x_1} \Delta u_k(x) - 2\alpha e^{-\alpha x_1} \partial_{x_1} u_k(x);$$

thus, owing to the first inequality in (2.12) we get

$$\begin{aligned} \Delta v_k + \sum_{j=1}^m a_{kj}(x) v_j &= \alpha^2 v_k + e^{-\alpha x_1} \left(\Delta u_k + \sum_{j=1}^m a_{kj}(x) u_j \right) - 2\alpha e^{-\alpha x_1} \partial_{x_1} u_k \\ &\geq \alpha^2 v_k - 2\alpha e^{-\alpha x_1} \partial_{x_1} u_k = -\alpha^2 v_k - 2\alpha \partial_{x_1} v_k. \end{aligned}$$

From this, by combining the second inequality in (2.12) with (2.11) we obtain

$$L_\alpha v_k + (\alpha^2 + a_{kk}(x)) v_k \geq \sum_{\substack{j=1 \\ j \neq k}}^m a_{kj}(x) (-v_j) \geq 0,$$

where $L_\alpha := \Delta + 2\alpha \partial_{x_1}$. Now, since the function a_{kk} is bounded on Ω (by assumption), it is possible to choose $\alpha > 0$ in such a way that

$$\alpha^2 + a_{kk} \geq 0 \quad \text{on } \Omega;$$

with this choice of α we then get (see also (2.12))

$$L_\alpha v_k \geq -(\alpha^2 + a_{kk}(x)) v_k \geq 0.$$

We are ready to conclude: the operator L_α being uniformly elliptic in Ω , the classical Strong Maximum Principle holds for L_α ; as a consequence, since

$$v_k = e^{-\alpha x_1} u_k \leq 0 \text{ on } \Omega,$$

if $u_k(x_0) = 0$ (for some $x_0 \in \Omega$) we have $v_k(x_0) = 0$, and hence $v_k \equiv 0$ on Ω . This obviously implies that $u_k \equiv 0$ on Ω , and the proof is complete. \square

Finally, in order to make the paper self-contained, we state and prove a Hopf Lemma for weakly cooperative elliptic systems.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open subset of \mathbb{R}^n and let $A(x) = (a_{ij}(x))$ be a $m \times m$ -valued map such that $a_{ij} \in L^\infty(\Omega)$ and*

$$a_{ij} \geq 0 \text{ on } \Omega \quad \text{for every } i, j \in \{1, \dots, m\} \text{ with } i \neq j. \quad (2.13)$$

Moreover, let $U = (u_1, \dots, u_m) \in C^2(\Omega; \mathbb{R}^m)$ be such that

$$\begin{cases} \Delta U + A(x) \cdot U \geq 0, & \text{on } \Omega, \\ U < 0 & \text{on } \Omega. \end{cases} \quad (2.14)$$

If there exists $x_0 \in \partial\Omega$ such that $u_k(x_0) = 0$ (for some $k \in \{1, \dots, m\}$) and if Ω satisfy an interior ball condition at x_0 , then

$$\frac{\partial u_k}{\partial \nu}(x_0) > 0, \quad (2.15)$$

provided that the outer normal derivative does exist.

Proof. We consider once again the \mathbb{R}^m -valued map $V \in C^2(\Omega; \mathbb{R}^m)$ defined by

$$V := e^{-\alpha x_1} U \quad \text{on } \Omega.$$

If $k \in \{1, \dots, m\}$ is arbitrarily fixed and if $\alpha > 0$ is chosen in such a way that $\alpha^2 + a_{kk} \geq 0$ on Ω (remind that, by assumption $a_{kk} \in L^\infty(\Omega)$), by arguing exactly as in the proof of Theorem 2.3 we infer that

$$L_\alpha v_k \geq 0 \quad \text{on } \Omega,$$

where $L_\alpha := \Delta + 2\alpha \partial_{x_1}$. Since the operator L_α is uniformly elliptic in Ω , the classical Hopf Lemma holds for L_α (see, e.g., [14, Lemma 3.4]); as a consequence, since $v_k = e^{-\alpha x_1} u_k < 0$ on Ω (see (2.14)), if there exists a point $x_0 \in \partial\Omega$ such that $v_k(x_0) = 0$ (and if the outer normal derivative of u_k at x_0 exists) we have

$$\frac{\partial v_k}{\partial \nu}(x_0) = -\alpha \nu_1(x_0) v_k(x_0) + e^{-\alpha x_1} \frac{\partial u_k}{\partial \nu}(x_0) = e^{-\alpha x_1} \frac{\partial u_k}{\partial \nu}(x_0) > 0.$$

This obviously implies the desired (2.15), and the proof is complete. \square

3. Proof of Theorem 1.1. In the present setting, we can now perform the proof of Theorem 1.1. For simplicity we separate the proof of the first claim in Theorem 1.1, which is the core of the argument, from the proof of the second claim, which is mostly straightforward.

Proof of Theorem 1.1 - (1). First of all we observe that, for every fixed $\lambda \in (0, 1)$, the functions $v_i^{(\lambda)}$ satisfy in the open set $\Sigma_\lambda \setminus \{0_\lambda\}$ the following (system of) PDEs

$$-\Delta v_i^{(\lambda)} = -\Delta u_i^{(\lambda)} + \Delta u_i = \sum_{j=1}^m c_{ij}(x; \lambda) v_k^{(\lambda)}(x), \quad i = 1, \dots, m,$$

where, the $c_{ij}(x; \lambda)$ are suitable non-negative functions bounded above by a certain positive constant $M > 0$, due to (f1). Moreover, we remind that

$$U = (u_1, \dots, u_m) \equiv 0 \text{ on } \partial\Omega \quad (3.1)$$

and

$$U = (u_1, \dots, u_m) > 0 \quad \text{in } \Omega \setminus \{0\}. \quad (3.2)$$

We then observe that, since λ is *strictly positive*, the reflection of $\partial\Sigma_\lambda \cap \partial\Omega$ with respect to the hyperplane T_λ is entirely contained in Ω (remind the structural assumptions satisfied by Ω). As a consequence, by (3.1) and (3.2), we derive that (for $i = 1, \dots, m$)

$$v_i^{(\lambda)} \geq 0 \text{ on } \partial\Sigma_\lambda \setminus \{0_\lambda\} \text{ but } v_i^{(\lambda)} \not\equiv 0 \text{ on the same set.} \quad (3.3)$$

We explicitly point out that, if $\lambda \neq 1/2$, we have that $0_\lambda \notin \partial\Sigma_\lambda$.

Gathering all these facts we see that, for every fixed $\lambda \in (0, 1)$, the \mathbb{R}^m -valued map V_λ satisfies on $\Sigma_\lambda \setminus \{0_\lambda\}$ the following elliptic system of PDEs:

$$\begin{cases} \Delta V_\lambda + A(x; \lambda) V_\lambda = 0, & \text{in } \Sigma_\lambda \setminus \{0_\lambda\}, \\ v_i^{(\lambda)} \geq 0, \ v_i^{(\lambda)} \not\equiv 0, & \text{on } \partial\Sigma_\lambda \setminus \{0_\lambda\} \end{cases} \quad (\text{for } i = 1, \dots, m), \quad (3.4)$$

where $A(x; \lambda)$ is given by

$$A(x; \lambda) = (c_{ij}(x; \lambda))_{i,j=1,\dots,m}, \quad (3.5)$$

and, by virtue of assumption (f1), it holds that

$$c_{ij}(x; \lambda) \geq 0 \text{ for } i \neq j \text{ on } \Sigma_\lambda \setminus \{0_\lambda\}. \quad (3.6)$$

Furthermore, since u_1, \dots, u_m are positive and continuous on $\overline{\Omega} \setminus \{0\}$ and, for every fixed $\lambda \in (0, 1)$, the set $\overline{\Sigma}_\lambda$ is compactly contained in $\overline{\Omega} \setminus \{0\}$, we also have

$$\inf_{\Sigma_\lambda \setminus \{0_\lambda\}} v_i^{(\lambda)} \geq \inf_{\Sigma_\lambda \setminus \{0_\lambda\}} (-u_i) > -\infty, \quad \text{for every } i = 1, \dots, m. \quad (3.7)$$

It is immediate to check that the system in (3.4) is (weakly) cooperative and satisfies the assumptions of Lemma 2.1. This implies that, for λ very close to 1, $V_\lambda \geq 0$ in Σ_λ (note that, if $\lambda \sim 1$, then $0_\lambda \notin \overline{\Omega}$). Moreover, by Theorem 2.3 we have

$$V_\lambda > 0 \quad \text{in } \Sigma_\lambda. \quad (3.8)$$

We can then define

$$\mathcal{I} := \{\lambda \in (0, 1) : V_t > 0 \text{ in } \Sigma_t \setminus \{0_t\} \ \forall t \in (\lambda, 1)\} \quad \text{and} \quad \mu := \inf \mathcal{I}.$$

We list below some useful properties of μ .

- (i) $\mu \in [0, 1)$. This is a straightforward consequence of (3.8).
- (ii) $V_t > 0$ on $\Sigma_t \setminus \{0_t\}$ for every $t \in (\mu, 1)$. Indeed, let $t \in (\mu, 1)$ be fixed. Since $\mu = \inf \mathcal{I}$, it is possible to find $\lambda \in \mathcal{I}$ such that $\mu < \lambda < t$. As a consequence, since $\lambda \in \mathcal{I}$, we conclude that $V_t > 0$ on $\Sigma_t \setminus \{0_t\}$, as claimed.
- (iii) $V_\mu \geq 0$ on $\Sigma_\mu \setminus \{0_\mu\}$. Indeed, if $\xi \in \Sigma_\mu \setminus \{0_\mu\}$ is fixed, there exists a small $\epsilon_0 > 0$ such that $\xi \in \Sigma_{\mu+\epsilon} \setminus \{0_{\mu+\epsilon}\}$ for every $\epsilon \in [0, \epsilon_0)$. Thus, by (ii) we have

$$V_{\mu+\epsilon}(\xi) > 0 \quad \text{for every } \epsilon \in [0, \epsilon_0).$$

Taking the limit as $\epsilon \rightarrow 0^+$, we conclude that $V_\mu(\xi) \geq 0$, as claimed.

- (iv) If $\mu > 0$, then $V_\mu > 0$ on $\Sigma_\mu \setminus \{0_\mu\}$. Indeed, by the previous point (iii) we know that $V_\mu \geq 0$ on $\Sigma_\mu \setminus \{0_\mu\}$; moreover, assuming that $\mu > 0$, we know that V_μ satisfies (3.4) on the same set (which is open and connected); we are then entitled to apply the strong maximum principle in Theorem 2.3, which ensures that

$$V_\mu > 0 \text{ on } \Sigma_\mu \setminus \{0_\mu\}.$$

The goal is to show that

$$\mu = 0. \quad (3.9)$$

Indeed, if this is the case, by the above (iii) we have (for $x \in \Sigma_0 = \Omega \cap \{x_1 > 0\}$)

$$U(-x_1, x_2, \dots, x_n) = U_0(x) \geq U(x) = U(x_1, \dots, x_n).$$

Thus, by applying this result to the function $\Omega \ni x \mapsto W(x) := U(-x_1, \dots, x_n)$ (which solves the same system of PDEs in (1.1)), we conclude that, for every $x \in \Omega \cap \{x_1 > 0\}$,

$$U(x_1, \dots, x_n) = W(-x_1, \dots, x_n) \geq W(x_1, \dots, x_n) = U(-x_1, \dots, x_n), \quad (3.10)$$

and this proves that U is symmetric in x_1 , as desired.

As in [4], we prove (3.9) by contradiction considering three possible cases:

- **Case I:** $\mu \in (1/2, 1)$;
- **Case II:** $\mu = 1/2$;
- **Case III:** $\mu \in (0, 1/2)$.

Case I: We note that, in this case, $0_\mu \notin \Sigma_\mu$, so that there is no effect of the singularity. If $\delta > 0$ is as in the statement of Lemma 2.1, we choose a compact set $K \subset \Sigma_\mu$ such that

$$|\Sigma_\mu \setminus K| < \frac{\delta}{2}.$$

By definition of μ , and since $K \subset \Sigma_\mu$ is compact, it holds that

$$v_i^{(\mu)} \geq \min_K v_i^{(\mu)} =: m_i > 0, \quad \text{in } K \quad (\text{for any } i = 1, \dots, m).$$

We claim that there exists a small $\epsilon_0 > 0$, which we may assume to be smaller than $\mu - 1/2$, with the following property: for every $\epsilon \in [0, \epsilon_0]$, one has

$$|\Sigma_{\mu-\epsilon} \setminus K| < \delta, \quad (3.11)$$

and

$$V_{\mu-\epsilon} > 0, \quad \text{in } K. \quad (3.12)$$

While (3.11) follows by continuity, the claim in (3.12) can be proved by arguing as follows: we set $\rho := \text{dist}(K, 0_\mu) > 0$ and we consider the compact set

$$K' := \{x \in \bar{\Omega} : \|x\| \geq \rho/2\} \subseteq \bar{\Omega} \setminus \{0\}.$$

It is very easy to see that, if R_λ is as in (2.1) and $\epsilon_0 < \rho/4$, then

$$R_{\mu-\epsilon}(x) \in K' \quad \text{for every } x \in K \text{ and every } \epsilon \in [0, \epsilon_0];$$

as a consequence, since U is uniformly continuous on K' , by shrinking ϵ_0 we get

$$v_i^{(\mu-\epsilon)} = (u_i(2\mu - 2\epsilon - x_1, x') - u_i(2\mu - x_1, x')) + v_i^{(\mu)} \geq \frac{m_i}{2} > 0,$$

for every $i = 1, \dots$, and every $\epsilon \in [0, \epsilon_0]$. In particular, this implies that

$$V_{\mu-\epsilon} > 0 \quad \text{on } \partial(\Sigma_{\mu-\epsilon} \setminus K). \quad (3.13)$$

Now, since $V_{\mu-\epsilon}$ satisfies the system of PDEs

$$\Delta V_{\mu-\epsilon} + A(x; \mu - \epsilon) V_{\mu-\epsilon} = 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus K \subseteq \Sigma_{\mu-\epsilon} \setminus \{0_{\mu-\epsilon}\},$$

on account of (3.11) and (3.13) we are entitled to apply Lemma 2.1, which gives

$$V_{\mu-\epsilon} \geq 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus K.$$

By combining this last fact with (3.12) we then get

$$V_{\mu-\epsilon} \geq 0 \quad \text{on } \Sigma_{\mu-\epsilon} \quad (\text{for } \epsilon \in [0, \epsilon_0]).$$

As a consequence, taking into account (3.4) and (3.6), we infer from Theorem 2.3 that

$$V_{\mu-\epsilon} > 0 \quad \text{on } \Sigma_{\mu-\epsilon} \quad (\text{for } \epsilon \in [0, \epsilon_0]),$$

which contradicts the definition of μ . This excludes the case $\mu \in (1/2, 1)$.

Case II: In this case, we need to distinguish two possible sub-cases. If the $0_{1/2}$ does not lie in $\bar{\Omega}$, then there is no singularity involved and we can argue exactly as in Case I. We then have to rule out just the case $0_{1/2} \in \partial\Omega$. Let us consider a positive constant $\rho > 0$ sufficiently small and let us define the set $D_\rho \subset \Sigma_{1/2}$

$$D_\rho := \{y \in \Sigma_{1/2} : \text{dist}(y, \partial\Sigma_{1/2}) \geq \rho\}.$$

Since $D_\rho \subset \Sigma_{1/2}$, we have $V_{1/2} > 0$ in D_ρ ; We then define the set $A_\rho \subset \bar{\Omega}$ as

$$A_\rho := \{z \in \bar{\Omega} : |z - 0_{1/2}| = \rho/2\}.$$

By definition, $D_\rho \cap A_\rho = \emptyset$; moreover, it follows from the above (iv) that

$$V_{1/2} > 0, \quad \text{in } D_\rho \cup A_\rho.$$

Now, by arguing as in the proof of (3.12), we can show that there exists $\epsilon_0 = \epsilon_0(\rho) > 0$ such that, for any $\epsilon \in [0, \epsilon_0]$, we have $0_{1/2-\epsilon} \in \Omega \cap B_{\rho/2}(0_{1/2})$ and

$$V_{1/2-\epsilon} > 0, \quad \text{in } D_\rho \cup A_\rho; \tag{3.14}$$

as a consequence, since (3.14) obviously implies that

$$V_{1/2-\epsilon} \geq 0, \quad \text{on } \partial(\Sigma_{1/2-\epsilon} \setminus (D_\rho \cup B_{\rho/2}(0_{1/2}))),$$

and, by (3.4), the map $V_{1/2-\epsilon}$ satisfies the system of PDEs

$$\Delta V_{1/2-\epsilon} + A(x; 1/2 - \epsilon)V_{1/2-\epsilon} = 0 \quad \text{on } \Sigma_{1/2-\epsilon} \setminus (D_\rho \cup B_{\rho/2}(0_{1/2})),$$

by eventually shrinking ρ we are entitled to apply Lemma 2.1, which gives

$$V_{1/2-\epsilon} \geq 0, \quad \text{on } \Sigma_{1/2-\epsilon} \setminus (D_\rho \cup B_{\rho/2}(0_{1/2})). \tag{3.15}$$

Gathering together (3.14) and (3.15), for every $\epsilon \in [0, \epsilon_0]$ we then get

$$V_{1/2-\epsilon} \geq 0, \quad \text{in } \Sigma_{1/2-\epsilon} \setminus B_{\rho/2}(0_{1/2}). \tag{3.16}$$

We now proceed in this second step by showing that, setting

$$G := \Omega \cap B_{\rho/2}(0_{1/2}) \subseteq \Sigma_{1/2-\epsilon},$$

we have $V_{1/2-\epsilon} > 0$ on $G \setminus \{0_{1/2-\epsilon}\}$. In fact, G is clearly contained in $B_\delta(0_{1/2})$; moreover, $V_{1/2-\epsilon} \geq 0$ on $\partial G \subseteq \Sigma_{1/2-\epsilon} \setminus B_{\rho/2}(0_{1/2})$ and, by (3.7), we have

$$\inf_G v_i^{(\mu-\epsilon)} > -\infty \quad \text{for every } i = 1, \dots, m;$$

we are then entitled to apply Lemma 2.2, which gives

$$V_{1/2-\epsilon} \geq 0 \quad \text{on } G \setminus \{0_{1/2-\epsilon}\},$$

as claimed. By combining this last fact with (3.16) we obtain

$$V_{1/2-\epsilon} \geq 0 \quad \text{on } \Sigma_{1/2-\epsilon} \quad (\text{for } \epsilon \in [0, \epsilon_0]),$$

thus, again by taking into account (3.4) and (3.6), we deduce from Theorem 2.3 that

$$V_{1/2-\epsilon} > 0 \quad \text{on } \Sigma_{1/2-\epsilon} \quad (\text{for } \epsilon \in [0, \epsilon_0]).$$

This proves that even the case $\mu = \frac{1}{2}$ is not possible.

Case III: To prove that also this case is not possible, we argue essentially as in the previous Case II. First of all, given any $\rho > 0$ such that $\text{dist}(0_\mu, \partial\Sigma_\mu) > \rho$, we define

$$D_\rho := \{y \in \Sigma_\mu : \text{dist}(y, \partial\Sigma_\mu) \geq \rho\}, \quad K_\rho := D_\rho \setminus B_{\frac{\rho}{2}}(0_\mu) \subseteq \Sigma_\mu.$$

Since K_ρ is compact and, by (iv), the function V_μ is continuous and strictly positive on $K_\rho \subseteq \Sigma_\mu \setminus \{0_\mu\}$, it is possible to find $m_0 > 0$ such that

$$v_i^{(\mu)} \geq m_0 \quad \text{on } K_\rho \quad (\text{for every } i = 1, \dots, m).$$

From this, by arguing as in Case II, we infer the existence of a small $\epsilon_0 = \epsilon_0(\rho) > 0$ such that, for every $\epsilon \in [0, \epsilon_0]$, the point $0_{\mu-\epsilon}$ lies in D_ρ and

$$V_{\mu-\epsilon} > 0 \quad \text{on } K_\rho. \quad (3.17)$$

In particular, since $B_{\rho/2}(0_\mu)$ lies in the interior of D_ρ , we get

$$V_{\mu-\epsilon} \geq 0 \quad \text{on } \partial(\Sigma_{\mu-\epsilon} \setminus D_\rho),$$

for every $\epsilon \in [0, \epsilon_0]$. As a consequence, since $V_{\mu-\epsilon}$ solves the system of PDEs

$$(\Delta + A(x; \mu - \epsilon)) V_{\mu-\epsilon} = 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus D_\rho,$$

if ρ has been chosen in such a way that $|\Sigma_{\mu-\epsilon_0} \setminus D_\rho|$ is sufficiently small, we can apply the weak maximum principle in Lemma 2.1 and obtain

$$V_{\mu-\epsilon} \geq 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus D_\rho \quad \text{for every } \epsilon \in [0, \epsilon_0]. \quad (3.18)$$

Gathering together (3.18) and (3.17) we conclude that, for every $0 \leq \epsilon \leq \epsilon_0$,

$$V_{\mu-\epsilon} \geq 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus B_{\frac{\rho}{2}}(0_\mu).$$

We now turn to prove that, for every fixed $\epsilon \in [0, \epsilon_0]$, we have $V_{\mu-\epsilon} \geq 0$ throughout the punctured ball $B_{\rho/2}(0_\mu) \setminus \{0_{\mu-\epsilon}\}$, so that (see the above (3.18))

$$V_{\mu-\epsilon} \geq 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus \{0_{\mu-\epsilon}\}. \quad (3.19)$$

To this end, we argue as in the previous case: indeed, it suffices to apply Lemma 2.2, with $A(x; \mu - \epsilon)$ as in (3.5), on $B_{\rho/2}(0_\mu)$. We are now ready to conclude: by virtue of the above (3.19) and since, by (3.4), we have

$$v_i^{\mu-\epsilon} \not\equiv 0 \quad \text{on } \partial\Sigma_{\mu-\epsilon} \setminus \{0_{\mu-\epsilon}\} \quad (\text{for any } i = 1, \dots, m),$$

we derive from the strong maximum principle in Theorem 2.3 that

$$V_{\mu-\epsilon} > 0 \quad \text{on } \Sigma_{\mu-\epsilon} \setminus \{0_{\mu-\epsilon}\}.$$

The proof of (3.9) is therefore complete. \square

Proof of Theorem 1.1 - (2). We now complete the proof of Theorem 1.1 by showing the monotonicity of u with respect to x_1 (if $x_1 > 0$). First of all, by (3.9), for every fixed $\lambda \in (0, 1)$ we have that $V_\lambda > 0$ on $\Sigma_\lambda \setminus \{0_\lambda\}$. On the other hand, since (by definition)

$$v_i^{(\lambda)} \equiv 0 \quad \text{on } T_\lambda \quad \text{for every } i = 1, \dots, m,$$

we are entitled to apply the Hopf-type Lemma 2.4, which gives (note that $\Sigma_\lambda \setminus \{0_\lambda\}$ certainly satisfies the interior ball condition at any point of $T_\lambda \cap \Omega$)

$$0 < \frac{\partial v_i^{(\lambda)}}{\partial x_1}(x) = -2 \frac{\partial u_i}{\partial x_1}(x) \quad \text{for every } x \in T_\lambda \cap \Omega.$$

This ends the proof of Theorem 1.1. \square

Appendix A. A combinatorics observation. This appendix considers the setting in (1.2) and (1.3), and checks the following combinatorics remark:

Lemma A.1. *We have that $\alpha_1, \dots, \alpha_m \geq 0$ if and only if $s_0(\alpha), \dots, s_m(\alpha) \geq 0$*

Proof. Obviously, if $\alpha_1, \dots, \alpha_m \geq 0$, we have that $s_0(\alpha), \dots, s_m(\alpha) \geq 0$, due to (1.3).

Now, we prove that

$$\text{if } s_0(\alpha), \dots, s_{m-1}(\alpha) \geq 0 \text{ then necessarily } \alpha_1, \dots, \alpha_m \geq 0. \quad (\text{A.1})$$

To this end, we first prove (A.1) under the additional assumption that

$$\alpha_i \neq 0 \text{ for all } i \in \{1, \dots, m\}. \quad (\text{A.2})$$

To prove (A.1) under the additional assumption in (A.2), we argue by induction over m . If $m = 1$, we have that $s_0(\alpha) = \alpha_1$, thus the claim is obvious.

Hence, to perform the inductive step, we suppose now that

$$(\text{A.1}) \text{ holds true for } m \text{ replaced by } m-1, \quad (\text{A.3})$$

and we establish (A.1) for $m \geq 2$.

Up to reordering coordinates, we may suppose that

$$\alpha_1 \geq \dots \geq \alpha_m. \quad (\text{A.4})$$

Therefore, if $\alpha_m \geq 0$ we are done, and we can accordingly assume also that

$$\alpha_m < 0. \quad (\text{A.5})$$

We also set $\alpha' := (\alpha_1, \dots, \alpha_{m-1}) \in \mathbb{R}^{m-1}$. We recall (1.3) and observe that, for every $k \in \{1, \dots, m-1\}$,

$$\begin{aligned} s_k(\alpha) &= \sum_{1 \leq i_1 < \dots < i_{m-k} \leq m-1} \alpha_{i_1} \dots \alpha_{i_{m-k}} + \sum_{1 \leq i_1 < \dots < i_{m-1-k} \leq m-1} \alpha_{i_1} \dots \alpha_{i_{m-1-k}} \alpha_m \\ &= s_{k-1}(\alpha') + \alpha_m s_k(\alpha'), \end{aligned}$$

with the notation that the sum over the void set of indexes equals to 1. Consequently, for every $k \in \{1, \dots, m-1\}$,

$$s_{k-1}(\alpha') = s_k(\alpha) - \alpha_m s_k(\alpha'). \quad (\text{A.6})$$

We claim that

$$s_k(\alpha') \geq 0, \quad \text{for all } k \in \{0, \dots, m-1\}. \quad (\text{A.7})$$

To prove this, we argue by backward induction over k . First of all, we know that $s_{m-1}(\alpha') = 1$. Then, suppose that $s_j(\alpha') \geq 0$ for all $j \in \{k, \dots, m-1\}$, for some $k \in \{1, \dots, m-1\}$. Hence, by (A.5),

$$\alpha_m s_k(\alpha') \leq 0$$

and therefore, by (A.6),

$$s_{k-1}(\alpha') = s_k(\alpha) - \alpha_m s_k(\alpha') \geq s_k(\alpha) \geq 0.$$

This completes the inductive step and it proves (A.7).

Then, by (A.7), we are in the position of using (A.3) and deduce that $\alpha_1, \dots, \alpha_{m-1} \geq 0$. Hence, by (A.2), we have that $\alpha_1, \dots, \alpha_{m-1} > 0$. But then

$$0 \leq s_0(\alpha) = \alpha_1 \dots \alpha_m < 0,$$

which is a contradiction. This shows that (A.5) cannot hold, and so, by (A.4),

$$\alpha_1 \geq \dots \geq \alpha_m \geq 0,$$

and thus (A.1) is established, under assumption (A.2).

To remove the additional assumption in (A.2), let now assume that $\alpha_{m-\ell+1} = \dots = \alpha_m = 0$, with $\alpha_i \neq 0$ for all $i \in \{1, \dots, m-\ell\}$, for some $\ell \in \{0, \dots, m\}$. Then, letting $\alpha^* := (\alpha_1, \dots, \alpha_{m-\ell})$, we have that

$$\begin{aligned} \sum_{k=0}^m s_k(\alpha) t^k &= \det \begin{pmatrix} \alpha_1 + t & 0 & \dots & 0 \\ 0 & \alpha_2 + t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_m + t \end{pmatrix} \\ &= t^\ell \det \begin{pmatrix} \alpha_1 + t & 0 & \dots & 0 \\ 0 & \alpha_2 + t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{m-\ell} + t \end{pmatrix} \\ &= t^\ell \sum_{k=0}^{m-\ell} s_k(\alpha^*) t^k. \end{aligned}$$

This gives that

$$0 \leq s_{k+\ell}(\alpha) = s_k(\alpha^*),$$

for all $k \in \{0, \dots, m-\ell\}$.

Since we have already proved (A.1) is established under assumption (A.2), we deduce that $\alpha_1, \dots, \alpha_{m-\ell} \geq 0$, which completes the proof of (A.1). \square

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