Asian options pricing in Hawkes-type jump-diffusion models

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Abstract

In this paper we propose a method for pricing Asian options in market models with the risky asset dynamics driven by a Hawkes process with exponential kernel. For these processes the couple $(\lambda(t), X(t))$ is affine, this property allows to extend the general methodology introduced by Hubalek *et al.* (2017) for Geometric Asian option pricing to jump-diffusion models with stochastic jump intensity. Although the system of ordinary differential equations providing the characteristic function of the related affine process cannot be solved in closed form, a COS-type algorithm allows to obtain the relevant quantities needed for options valuation. We describe, by means of graphical illustrations, the dependence of Asian options prices by the main parameters of the driving Hawkes process. Finally, by using Geometric Asian options values as control variates, we show that Arithmetic Asian options prices can be computed in a fast and efficient way by a standard Monte Carlo method.

Keywords: Asian options, Option pricing, Jumps clustering, Hawkes processes, Affine Processes, COS Method. JEL Classification: C63, G12, G13.

1. Introduction

Asian options are derivative securities exhibiting an explicit dependence on the average of some underlying asset process on their lifetime. They can be defined according to their payoff structure: if $S(T)$ is the value of the underlying asset at maturity T, K is the strike price, and $A(T)$ is a suitably defined average of the values assumed by the asset during the period considered, the "Average Strike" (also called "Floating Strike") Asian call payoff is provided by the following expression: $(S(T) - A(T))^+$, while the payoff of the "Average Price" (sometimes called "Fixed Strike" or "Average Rate") Asian call is given by: $(A(T) - K)^+$. Different kind of averages can be considered in order to define the Asian options payoff: arithmetic and geometric averages are the most common choices.

Although Asian options are OTC financial products, they are popular derivatives instruments, in particular in energy markets, since they represent a possible hedging strategy against sudden manipulations of the market prices thanks to the appealing property that the averaging procedure can "smooth out" the underlying price process behavior. So they can constitute important tools for portfolio management in commodity markets.

For Arithmetic Asian options the valuation problem, when the underlying price dynamics is described by a Geometric Brownian motion, was investigated in the papers by Geman and Yor

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(1993) and by Dufresne (2001). For Geometric Asian options in a Geometric Brownian motion setting the Average Price call value can be obtained via a straightforward calculation, while the Average Strike call value can be obtained only by a numerical approximation (see Wilmott *et al.* (1995)).

Several different kind of models have been proposed during the last 40 years in order to improve the forecasting performances of the Black-Scholes-Merton model, which is based on a Geometric Brownian motion description of the risky asset dynamics. In particular, models with jumps (see Cont and Tankov (2004)) are able to provide smiles for the implied volatility, although not very realistic for long maturities. Models introducing a stochastic dynamics for the diffusion coefficient of the Brownian motion, named stochastic volatility models, can provide smiles realistic only for long maturities. More sophisticated models introduce both stochastic volatility and jumps in order to provide realistic smiles for all maturities. The model proposed by Bates (1996) combines the features of the jump-diffusion model proposed by Merton (1976) with those of the stochastic volatility model proposed by Heston (1993). Barndorff-Nielsen and Shephard (2001) and Barndorff-Nielsen *et al.* (2002) introduced a model in which the volatility dynamics is described by an Ornstein-Uhlenbeck process driven by a subordinator. Another reasonable attempt to improve the asset price dynamics description has been done by Carr *et al.* (2003) and Carr and Wu (2004), which propose the so-called Time-Changed Lévy models. We mention also the model introduced by Bates (2000), in which an affine (deterministic) dependence is included between the stochastic volatility and the jumps intensity.

All the above mentioned pricing models in which stochastic volatility features have been combined with jumps belong to the large family of the so-called affine models, according to the definition provided by Duffie *et al.* (2003). This class includes almost all the most popular pricing models existing in the literature related to many different type of underlying assets: fixed income securities, credit risk models, equities and commodities. Many relevant features of these models can be described in a unified way by the very general framework provided by the affine process approach. For an extensive treatment of the general properties of affine models and some related technical issues we mention the thesis Keller-Ressel (2008).

In the paper by Hubalek *et al.* (2017) a general methodology for Geometric Asian option pricing is introduced, and for several specific models of affine type a closed-form solutions are obtained for the Riccati equations providing the affine characteristics for the joint dynamics of log-returns, volatility and their average processes, and their joint moment generating functions. This approach allows to compute the Geometric Asian options price by a simple inversion of a Laplace transform, which represents the only numerical step in the pricing procedure and for which several fast and accurate algorithms are available.

Recently, new models have been proposed in order to describe risky assets price dynamics, including jumps with self-exciting features. Evidence has been provided that jumps appear in clusters, this phenomenon is investigated in the paper by Filimonov *et al.* (2014), where a large amount of price sudden movements is shown to be of endogenous type, i.e. they are produced by previous sudden movements. The most popular model way to introduce self-excited jumps is the asset price dynamics is by using Hawkes processes. In the paper by Bacry *et al.* (2013) a limit order book modelling approach is presented, offering a micro-structure foundation for

Hawkes-type models. Ait-Sahalia *et al.* (2015) adopt a Hawkes vector framework in order to describe mutually exciting jumps in the market and Fulop *et al.* (2015) propose a Bayesian learning approach to jumps cluster detection and provide evidence of jumps clustering since the 1987 market crash, which appeared even more pronounced after the 2008 global financial crisis. Rambaldi *et al.* (2015) propose a model based on Hawkes processes in order to describe the foreign exchange markets and Kiesel and Paraschiv (2017) introduce a Hawkes-type model in order to describe the power price dynamics in energy markets. Bernis, Salhi and Scotti Bernis *et al.* (2018) perform a sensitivity analysis with respect to the Hawkes parameters of Collateralized Loan Obligations in a credit risk model driven by a marked Hawkes process.

A strict relation holds between Hawkes processes with exponential kernel and continuous branching processes with immigration (CBI); they both belong to the class of affine processes; CBI can be used to successfully model different kind of markets: Jiao, Ma and Scotti Jiao *et al.* (2017) introduce CBI in order to describe the short rates dynamics, while Jiao, Ma, Scotti and Sgarra propose a CBI approach to power markets Jiao *et al.* (2018).

Marked Hawkes processes are compound Poisson processes with stochastic intensity; when the kernel characterizing the self-exciting dynamics is of exponential type, the couple $(\lambda(t), X(t))$ is an affine process. This allows to apply the general framework introduced in Hubalek *et al.* (2017) in order to evaluate Geometric Asian options.

The purpose of the present paper is to propose a fast and efficient pricing method for Asian options in a Hawkes modelling setting. As a reference model we shall adopt the jump-diffusion model proposed by Hainaut and Moraux (2018), but our approach could be easily adapted to other models based on Hawkes processes with exponential kernel. The Riccati equations describing the joint dynamics of log-returns and their arithmetic average, unfortunately, cannot be solved in closed form, but the well-known COS method proposed by Fang and Oosterlee (2008) can be used to get option prices with a reasonable amount of characteristic function evaluations. To our knowledge, this is the first attempt to solve the valuation problem for Asian options in a Hawkes-type modelling framework.

It is well known that, the usage of the Geometric Asian options values as control variates in a Monte Carlo simulation for Arithmetic Asian options pricing allows to reduce dramatically the variance of the simulation step. We shall use this property in order to compute the Arithmetic Asian options price as well and illustrate how the simulation method can be made fast and efficient once the Geometric Asian options values are obtained through the proposed method.

The plan of the work is the following: in Section 2 we introduce our self-exciting jump-diffusion model by following the proposal by Hainaut and Moraux (2018) and we illustrate their affine features. Then we formulate the basic results for Geometric Asian options pricing for jump-diffusion models with stochastic intensity by introducing the required modifications in the approach proposed in Hubalek *et al.* (2017) for affine stochastic volatility models. In Section 3 we derive the closed form solution for the geometric Asian option prices and illustrate the simulation approach adopted for Arithmetic Asian options valuation, in which Geometric Asian options values are in use as a control variate. In Section 4 we discuss our results and provide comparison of performances between the methods, Section 5 concludes.

2. Model setup

We denote by Q^0 and Q^1 respectively the risk-neutral probability measures with the money market account and with the risky asset as Numéraires respectively. We model the log-returns of the underlying under the risk neutral measure Q^0 through the following couple of SDEs:

$$dX(t) = \left(r - \frac{\sigma^2}{2} - \mathrm{E}[e^J - 1]\lambda(t)\right) dt + \sigma dW(t) + d\left(\sum_{i=1}^{N(t)} J_i\right) \tag{1}$$

$$d\lambda(t) = \alpha(\theta - \lambda(t))dt + \eta dL(t) \tag{2}$$

where $X(t) := \ln S(t)$, $S(t)$ denotes the price of the underlying at time t, r is the risk-less rate, σ is the diffusion coefficient, J_i is the size of the i-th jump, $N(t) \sim \mathrm{Poisson}(\lambda(t))$ and $L(t)$ is a jump process defined as:

$$L(t) = \sum_{i=1}^{N(t)} |J_i|$$

with $J_i \sim \mathcal{DE}(p, \rho^+, \rho^-)$, where $\mathcal{DE}(p, \rho^+, \rho^-)$ is the double exponential distribution, with probability of positive jumps denoted with p, average sizes of positive and negative shocks given, respectively, by $\frac{1}{\rho^+}$ and $\frac{1}{\rho^-}$. In agreement with the present notation $X(0) = \ln S(0)$.

Remark 1. *We model directly the asset dynamics under the risk-neutral measure Q^0. A measure change preserving the model structure and the relations between parameters under the historical measure P and the risk-neutral measure Q^0 is proposed in Hainaut and Moraux (2018), together with an estimation method and a hedging strategy based on European options trading.*

Proposition 1. *(Hainaut and Moraux (2018))*

Given the model specified by equations 1 and 2, consider $t \in [0, T]$ the joint characteristic function of $(X(T)$,$\lambda(T))$ is given by:

$$\mathrm{E}^0[e^{u_1 X(T) + u_2 \lambda(T)} | X(0), \lambda(0)] = \exp\left(A(0,T) + \lambda(0)B(0,T) + u_1 X(0)\right), \tag{3}$$

for all $(u_1, u_2) \in i\mathbb{R}^2$ where $A(t,T)$ and $B(t,T)$ are given by the solution of the following ODEs system:

$$\begin{cases} \frac{\partial A}{\partial t} = F(u_1, B), & B(T,T) = u_2 \\ \frac{\partial B}{\partial t} = R(u_1, B), & A(T,T) = 0 \end{cases} \tag{4}$$

where $\mathrm{E}^0[\cdot]$ indicates that the expectation is taken with respect to the risk neutral measureQ^0, $F(u_1, B) = -\alpha\theta B - \left(r - \frac{\sigma^2}{2}\right) u_1 - \frac{\sigma^2}{2} u_1^2$, $R(u_1, B) = -\alpha B + k u_1 - (\psi(B\eta, u_1) - 1)$, $k = \mathrm{E}^0[e^J - 1]$ and $\psi(z_1, z_2) = p\frac{\rho^+}{\rho^+ - (z_1 + z_2)} + (1-p)\frac{\rho^-}{\rho^- - (z_1 - z_2)}$ is the joint moment generating function of the distribution of the jumps size and their absolute value.

Errais *et al.* (2010) show that (X, λ) is an affine process, we aim to exploit this result extending the validity of Proposition 1 to the following quantities:

$$Y(T) := \int_0^T X(s)ds, \quad \Lambda(T) := \int_0^T \lambda(s)ds. \tag{5}$$

We get the following theoretical result:

Proposition 2. *If (X, λ) is an affine model with functional characteristics (F, R), then the joint characteristic function of $(X(T), \lambda(T), Y(T), \Lambda(T))$ is given by:*

$$\mathrm{E}^0[e^{u_1 X(T) + u_2 \lambda(T) + u_3 Y(T) + u_4 \Lambda(T)} | X(0), \lambda(0)] = \exp\Big(A(0,T) + (u_1 + u_3 T) X(0) + B(0,T)\lambda(0)\Big), \tag{6}$$

for all $(u_1, u_2, u_3, u_4) \in i\mathbb{R}^4$ where A and B are given by the solution of the following ODEs system:

$$\begin{cases} \frac{\partial A}{\partial t} = F(u_1 + u_3 t, B), \quad A(T,T) = 0 \\ \frac{\partial B}{\partial t} = R(u_1 + u_3 t, B) - u_4, \quad B(T,T) = u_2 \end{cases} \tag{7}$$

and $F(\cdot,\cdot)$ and $R(\cdot,\cdot)$ are as in Proposition 1.

Proof. Since (X, λ) is an affine process, the proof follows from Hubalek *et al.* (2017, Proposition 3) ■

This result enables the pricing of European and fixed strike geometric Asian options through simple changes of arguments, we are going to detail this in next section.

As far as the Average Strike options are concerned, we need to extend to the present case the results on the change of Numéraire obtained in Hubalek *et al.* (2017, Section 4). By recalling that Q^0 and Q^1 denote respectively the risk-neutral probabilities with the money market account and the risky asset as Numéraires, the results can be reformulated as follows:

Lemma 1. *If (X, λ) is affine under Q^0 with functional characteristics F^0 and R^0, then it is affine under Q^1 with functional characteristics F^1 and R^1 given by*

$$F^1(u_1, u_2) = F^0(u_1 + 1, u_2), \quad R^1(u_1, u_2) = R^0(u_1 + 1, u_2). \tag{8}$$

Lemma 2. *If (X, λ) is an affine model, then the joint law of $(X_t, Y_t = \int_0^t X_s ds)$ under Q^1 is described by*

$$E^1[e^{u_1 X(t) + u_2 Y(t)} | X(0), \lambda(0)] = \exp\left(C(t, u_1, u_2) + X(0)(u_1 + u_2 t) + \lambda(0) D(t, u_1, u_2)\right) \tag{9}$$

for all $(u_1, u_2) \in i\mathbb{R}^2$, where

$$\partial_t C(t, u_1, u_2) = F^0(u_1 + 1 + u_2 t, D(t, u_1, u_2)) \qquad C(0, u_1, u_2) = 0 \tag{10}$$

$$\partial_t D(t, u_1, u_2) = R^0(u_1 + 1 + u_2 t, D(t, u_1, u_2)) \qquad D(0, u_1, u_2) = u_2, \tag{11}$$

and E^1 denotes expectection with respect to the probability measure Q^1. Moreover, the Riccati equations can be extended to all parameters values in the effective domain (for the definition of *the effective domain we refer to Hubalek* et al. *(2017)).*

Proof. The proof of the two previous statements follows step by step the proof provided in Hubalek *et al.* (2017), by assuming the stochastic intensity λ as a state variable instead of the variance. ■

In next section we are going to illustrate how the general results presented in this section can be applied in order to get geometric Asian options prices.

Remark 2. *In the following, in agreement with Hainaut and Moraux (2018), we always assume the relevant characteristic/moment generating functions and cumulants to exist finite for the parameters values considered. Conditions on the affine characteristics granting moments and cumulants existence can be found in Keller-Ressel (2011).*

3. Option pricing

In this section we show how to price efficiently European and Geometric (fixed and floating strike) Asian options for the model specified in 1 and 2. We start by identifying the characteristic functions which will be used to get the price of the various derivative instruments, then we illustrate the COS method for computing the value of an option given the characteristic function and show that the truncation range can be determined analytically (even if the characteristic function is not known analytically). Finally, we discuss arithmetic Asian options pricing.

3.1. Option pricing given the characteristic function: the COS method

Let's start recalling that $S(T) := S(0)e^{X(T)}$ and defining the geometric average of the price process at maturity as $G(T) := S(0)e^{\frac{Y(T)}{T}}$. The price of European and fixed and average strike Geometric Asian options is given, respectively, by:

$$P_E = e^{-rT}\mathrm{E}^0[(S(T)-K)^+] = e^{-rT}\int_{-\infty}^{\infty}(e^x - K)^+ f_X(x)dx \tag{12}$$

$$P_{GFS} = e^{-rT}\mathrm{E}^0[(G(T)-K)^+] = e^{-rT}\int_{-\infty}^{\infty}(e^{\frac{y}{T}} - K)^+ f_Y(y)dx \tag{13}$$

$$P_{GAS} = e^{-rT}\mathrm{E}^1[(1-e^{Z(T)})^+] = e^{-rT}\int_{-\infty}^{\infty}(1-e^z)^+ f_Z(z)dz, \tag{14}$$

where r is the risk-less rate, T is the option maturity, K is the strike price, $S(0)$ is the starting price, $Z(T) := \frac{Y(T)}{T} - X(T)$ and $f_{\cdot}(\cdot)$ is the probability density function (henceforth pdf). Proposition 2 and Lemmas 1 and 2 can be used to derive the characteristic function of $X(T)$, $Y(T)$, $Z(T)$ through simple changes of arguments. Let's start considering the case of European and fixed strike geometric Asian options: in the case where $u_2 = u_3 = u_4 = 0$ (respectively, $u_1 = u_2 = u_4 = 0$) Proposition 2 allows to identify the characteristic function of the log-returns (integrated log-returns) which can be inverted numerically to obtain the price of the European (geometric Asian) option. Similarly, consider replacing $u_1 = -1$ and $u_2 = \frac{u}{T}$ into 9, Lemma 2 identifies the characteristic function under Q^1 of $Z(T)$.

These results open the doors to options pricing through standard inversion algorithms, such as FFT and COS methods. Hainaut and Moraux (2018) propose pricing formulas for European calls and puts based on FFT, but the COS method is usually preferable because of its exponential convergence to the true solution (while its computational complexity is linear), see Fang and Oosterlee (2008). The most obvious consequence is that option price can be estimated through a smaller number of evaluations of the characteristic functions, this is particularly important when it is given by time consuming numerical techniques such as, for example, solution of an ODEs system as in the present case. We briefly recall here the main features of the COS method developed by Fang and Oosterlee (2008).

Given a characteristic function (denoted with $\phi(\cdot)$), the probability density function (pdf in the following) can be computed as follows:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\phi(u)du. \tag{15}$$

Several algorithms can be used to solve the integral in 15, we refer to Fang and Oosterlee (2008) for a review. Among them, the best performances are obtained through the COS method, where the inverse Fourier integral in 15 is computed via Cosine Expansion and the pdf is approximated as follows:

$$f(x) = \sum_{k=1}^{\infty} F_k \cos\left(k\pi\frac{x-a}{b-a}\right) + \frac{1}{b-a} \approx \sum_{k=1}^{N-1} F_k \cos\left(k\pi\frac{x-a}{b-a}\right) + \frac{1}{b-a}, \tag{16}$$

where $F_k = \frac{2}{b-a}\text{Real}\left(\phi\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right)$ and $[a,b]\in\mathbb{R}$ is chosen such that:

$$\int_a^b e^{iux} f(x)ds \approx \int_{\mathbb{R}} e^{iux} f(x)dx. \tag{17}$$

In other words, in order to implement the COS method is necessary to truncate the domain of the pdf through a suitable choice of a and b. In order to do that, Fang and Oosterlee (2008) propose the following formulas:

$$a = c_1 - L\sqrt{c_2+\sqrt{c_4}}, \quad b = c_1 + L\sqrt{c_2+\sqrt{c_4}} \tag{18}$$

where L can be chosen arbitrary large and c_i denotes the i-th cumulant of $\ln\left(\frac{S(T)}{K}\right)$.

Since cumulants can be expressed as functions of moments, the choice of the truncation range is related to the moments of log-returns. In many Lévy models they can be computed easily due to the availability of a handy formula for the characteristic function[1]. In the next subsection we show how to compute them in the present model. This is important since, as pointed out by Fang and Oosterlee (2008, pag. 13), if $L = 10$ then formula 18 gives a truncation

[1]Simple formulas for the Merton jump diffusion, double exponential jump diffusion, Variance Gamma, NIG models are provided in Cont and Tankov (2004).

error around 10^{-12}, enlarging the interval $[a, b]$ would require larger N to reach the same level of accuracy.

Remark 3. *The choice of N impacts on the efficiency of the method as the characteristic function of log-returns must be evaluated $N-1$ times. This aspect is crucial when the characteristic function must be computed through time consuming numerical techniques such as, for example, by solving an ODEs system.*

Hence, given the characteristic function of log-returns, the pdf can be recovered from formula 16. Given the pdf, the price of the option can be computed from equations 12, 13 and 14 by computing the integral numerically. Alternatively, starting from 16, Fang and Oosterlee (2008, Formula 19) also provide direct explicit formulas for the price of European call and put options which don't require any numerical integration (we consider this approach throughout the paper but, also in this case, Remark 3 is valid).

3.2. Truncation range computation

In this section we show how to compute the moments of log-returns in the model specified in 1 and 2 given the characteristic function in 3. These will be then used for computing a proper truncation range for implementation of COS method for European options pricing. For sake of brevity and clarity, we restrict our attention to the cumulants of log-returns, but the same argument applies also to $Y(T)$ and $Z(T)$. Taking the logarithm of 3 one gets the cumulant generating function:

$$\psi(u_1, u_2, u_3, u_4) = A(0,T) + (u_1 + u_3 T)X(0) + B(0,T)\lambda(0). \tag{19}$$

By setting $u_1 = u$, $u_2 = u_3 = u_4 = 0$, we get the cumulants of log-returns[2] according to:

$$k_n = \left.\frac{\partial^n \psi(u)}{\partial u^n}\right|_{u=0} = \left.\frac{\partial^n A(0,T)}{\partial u^n}\right|_{u=0} + \left.\frac{\partial^n u}{\partial u^n}\right|_{u=0} \cdot X(0) + \left.\frac{\partial^n B(0,T)}{\partial u^n}\right|_{u=0} \cdot \lambda(0), \tag{20}$$

with $\left.\frac{\partial^n u}{\partial u^n}\right|_{u=0} \cdot X(0) = \begin{cases} X(0) & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2 \end{cases}$ and

$$\begin{cases} \frac{\partial}{\partial t}\left(\left.\frac{\partial^n A(t,T)}{\partial u^n}\right|_{u=0}\right) & = \left.\frac{\partial^n F(u,B)}{\partial u^n}\right|_{u=0}, \quad A(T,T) = 0 \\ \frac{\partial}{\partial t}\left(\left.\frac{\partial^n B(t,T)}{\partial u^n}\right|_{u=0}\right) & = \left.\frac{\partial^n R(u,B)}{\partial u^n}\right|_{u=0}, \quad B(T,T) = 0 \end{cases}. \tag{21}$$

Tedious maths show that this system can be solved analytically for each $n \in \mathbb{N}^+$ (we don't report here analytical solution, Mathematica® snippets are available upon request). Finally, we stress that, taking the logarithm of 9 one gets the cumulant generating function of $Z(T)$, which can be used for the case of Average Strike. Given cumulants, moments can be computed

[2]Similarly, posing $u_3 = u$, $u_1 = u_2 = u_4 = 0$ one gets the cumulants of $Y(T)$.

analytically using Faá di Bruno's formula for high derivatives of composite functions:

$$\mathrm{E}[X^n] = \sum_{i=1}^{n} \mathrm{B}_{n,i}(k_1, ..., k_{n-i+1}),$$

where $\mathrm{B}_{n,i}$ are the incomplete Bell polynomials. A first application of this result concerns the calculation of the proper truncation range for implementation of the COS method. We note that moments of $\ln\left(\frac{S(T)}{K}\right)$ can be expressed as function of the moments of X:

$$z_n := \mathrm{E}\left[\left(\ln\left(\frac{S(T)}{K}\right)\right)^n\right] = \mathrm{E}\left[\left(\ln\left(\frac{S(0)}{K}\right) + X\right)^n\right]. \tag{22}$$

By solving equation 22 for $n = \{1, 2, 3, 4\}$ one can compute the cumulants in 18 according to:

$$c_1 = z_1, \quad c_2 = z_2 - z_1^2, \quad c_3 = z_3 - 3z_2z_1 + 2z_1^3, \quad c_4 = z_4 - 4z_3z_1 - 3z_2^2 + 12z_2z_1^2 - 6z_1^4. \tag{23}$$

By substituting these into 18 one gets the truncation range.

3.3. Pricing Arithmetic Asian options

Since the distribution of the arithmetic average is unknown even under very simple model assumptions for the underlying's price (e.g. geometric Brownian motion), simple exact closed form solutions does not exist for the price of arithmetic average Asian options. Several techniques based on moments, lower and upper bounds and Monte Carlo have been proposed in literature to approximate their value under different dynamics for the price process (see, for example, Fusai and Kyriakou (2016)). In this paper, since we are dealing with a very involved dynamics for the log-returns, we consider Monte Carlo methods. The simulation of a Hawkes process is not a trivial task, Ogata (1981) and Dassios and Zhao (2013) present exact schemes, with the latter outperforming the former in terms of runtime speed. Despite that, such method is still very time consuming, as a result, we choose to simulate the SDE in 2 using the Euler scheme. This choice reduces drastically the computing time, but introduces a large discretization error. Consequently, crude Monte Carlo simulation is not a convenient choice and can be only used as benchmark (a very fine discretization grid and a large number of simulations is needed to achieve a good accuracy). Runtime-accuracy performances can be highly improved by using control variates methods, which are easily implementable in our context thanks to the availability of a semi-closed form solution for the price of geometric (fixed and floating) strike Asian option (see Kemna and Vorst (1990) and Fu *et al.* (1999)).

More specifically, let's denote with $A(T) := \frac{1}{T}\int_0^T S(t)dt$ the arithmetic average of the price process, the value of fixed and average strike arithmetic Asian option can be computed, respectively, as:

$$P_{AFS} = e^{-rT}\mathrm{E}^0\left[(A(T) - K)^+ + \zeta_1\left((G(T) - K)^+ - P_{GFS}\right)\right], \tag{24}$$

$$P_{AAS} = e^{-rT}\mathrm{E}^0\left[(S(T) - A(T))^+ + \zeta_2\left((S(T) - G(T))^+ - P_{GAS}\right)\right] \tag{25}$$

where P_{GFS} and P_{GAS} are computed as in 13 and 14, the control variates coefficients $\zeta_{1,2}$ are estimated, following Glasserman (2004) and Cont and Tankov (2004), running pilot simulations[3].

4. Numerical results

Numerical results are provided in this section. Computations are done using Matlab®(Version R2017b) in Microsoft Windows 10®running on a machine equipped with Intel(R) Core(TM) i7-6700HQ CPU @2.60GHz and 16 GB of RAM.

We start by identifying some parameter settings taken from literature for the model specified by equations 1 and 2. By using the *peaks over treshold* method, Hainaut and Moraux (2018) calibrate the model specified by equations 1 and 2 on the S&P 500 index, we consider three different parameter settings taken from their results (these are summarized in Table 1). We note immediately that jumps are very frequent (parameter η is very high) and that since positive jumps are rarer than negative, the resulting distribution of the log-returns (and, as a consequence, of the average returns) presents elevated kurtosis and heavy left tail. The resulting probability density function for the parameter settings in Table 1 is shown in Figure 1.

Secondly, we implement the following numerical exercise: compute the price of European and geometric Asian (fixed and floating strike) call options through Monte Carlo (which is used as benchmark) and COS method. Following Hubalek and Sgarra (2011) we consider five different strikes $K = \{80, 90, 100, 110, 120\}$ and three different maturities $T = \{1, 2, 3\}$ years. Prices are computed using the COS method with characteristic functions as in Proposition 2 and Lemma 2, and ODEs systems characterizing the characteristic functions are solved numerically using an explicit Runge-Kutta (4,5) formula[4]. Absolute and relative tolerances are set equal to, respectively, 10^{-9} and 10^{-10}.

The truncation range is computed as in formula 18 and infinite summations are truncated at $N = 2^7$. In Table 2 we show the price of the European call option computed through Monte Carlo simulation and the COS method. We note that both procedures are very accurate with the COS price always falling into the confidence interval provided by Monte Carlo. In Table 3 we show the same results regarding the geometric (fixed and floating strike) Asian call option. We note that Monte Carlo is less accurate in this situation (this is particularly evident looking at the price of the floating strike Asian options) and the bias is certainly due to time discretization.

We also investigate the sensitivity of the price of the fixed strike Geometric Asian call options on the Hawkes process parameters (α, θ and η), see Figure 2. We note that the parameter α (which indicates the speed of mean reversion of the Hawkes process) impacts negatively the call option price, indeed, the higher α, the lower the expected number of jumps (and, consequently, the variance of the distribution at maturity of the average of log-returns). A similar argument holds for the long run mean jump intensity θ and the parameter η (which controls the magnitude of jumps in the marked Hawkes process).

Finally, we exploit availability of Geometric Asian call options values in order to price Arithmetic Asian call options through the control variates method (see formulas 24 and 25).

[3]In particular we are using here 10^2 pilot simulations.
[4]We use the built-in Matlab® function `ode45`.

The numerical results are reported in Table 4. The usage of control variates method allows to reduce significantly the number of simulations required to get a good estimate of the price (indeed we use 10^5 simulations instead of 10^6 as in Tables 2 and 3) and to reduce the time discretization bias.

5. Conclusions

With the aim of taking into account the jump clustering phenomena widely observed in financial markets, we model the log-returns dynamics of the underlying asset through a self-exciting jump diffusion model of Hawkes-type and, by exploiting the affine features of the model considered, we derive the characteristic function of the arithmetic average of log-returns in the form of the solution of an ODEs system. By starting with this general result as a building block, we derive semi-closed form solutions for Geometric (fixed and floating strike) Asian options under this model by applying the COS method. We evaluate accuracy and efficiency of such approach through an extensive numerical study based on the usage of Monte Carlo simulation as a benchmark. Numerical results show that the proposed pricing method is fast and accurate. Finally, we show that this closed form solutions can be easily incorporated into a Monte Carlo simulation as control variables, allowing for an efficient pricing of Arithmetic Asian options as well.

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	r	σ	α	θ	η	p	ρ^+	ρ^-
A	0.05	0.12	14.71	4.68	244.82	0.36	42.74	-46.17
B	0.05	0.12	14.71	5.57	291.36	0.37	35.58	-39.01
C	0.05	0.12	14.71	6.44	337.08	0.37	30.47	-33.90

Table 1: Parameter settings in literature.

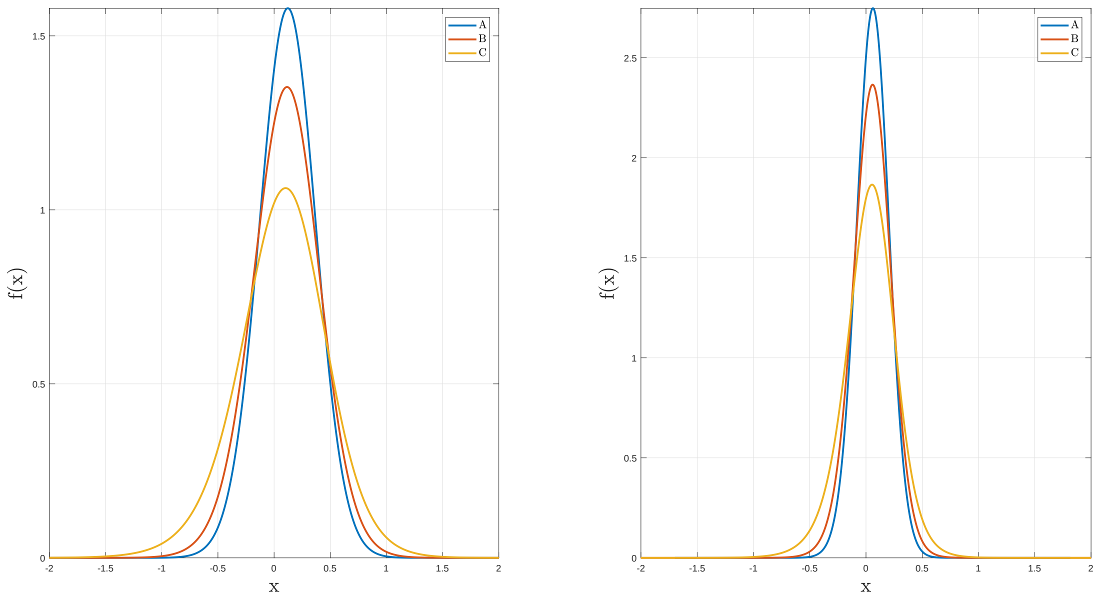

Figure 1: Probability density functions of $X(T)$ (left subplot) and $\frac{Y(T)}{T}$ (right subplot) for parameter settings in Table 1 and final date $T = 3$ years.

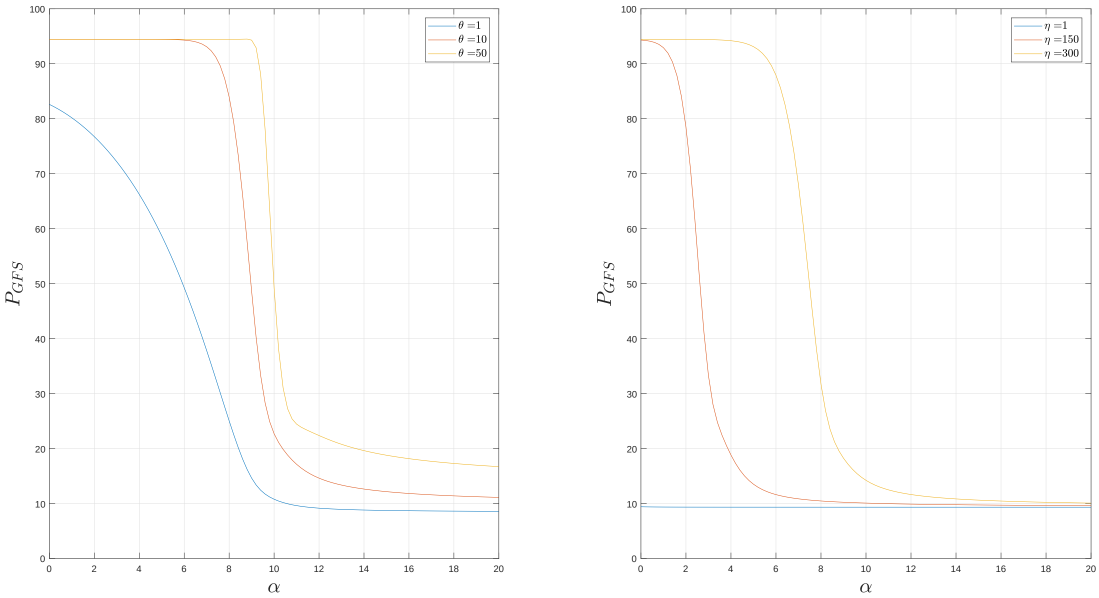

Figure 2: Price of fixed strike Geometric Asian options for different α and θ (left subplot) and η (right subplot). Other parameters are as in parameter setting "C" (see Table 1) with maturity $T = 3$ years.

	$T=1$		$T=2$		$T=3$	
	95% C.I.	COS	95% C.I.	COS	95% C.I.	COS
			A			
$K=80$	(24.0513-24.1074)	24.0884	(28.5492-28.6411)	28.1007	(31.7868-31.8841)	31.8421
$K=90$	(15.3780-15.4297)	15.4152	(21.0221-21.1077)	20.1798	(24.3599-24.4520)	24.4117
$K=100$	(8.3931-8.4356)	8.4297	(14.7122-14.7887)	13.5210	(17.9296-18.0138)	17.9752
$K=110$	(3.8473-3.8778)	3.8785	(9.8167-9.8824)	8.4509	(12.6910-12.7653)	12.7309
$K=120$	(1.5052-1.5246)	1.5241	(6.2912-6.3457)	4.9547	(8.6674-8.7310)	8.7028
Time	115.1754	1.0911	212.1326	1.0728	368.1458	1.0472
			B			
$K=80$	(24.2826-24.3473)	24.3422	(28.5492-28.6411)	28.6035	(32.4618-32.5757)	32.5067
$K=90$	(15.9370-15.9961)	15.9928	(21.0221-21.1077)	21.0759	(25.4458-25.5534)	25.4915
$K=100$	(9.2439-9.2935)	9.2918	(14.7122-14.7887)	14.7649	(19.3928-19.4919)	19.4373
$K=110$	(4.7346-4.7724)	4.7742	(9.8167-9.8824)	9.8657	(14.4075-14.4968)	14.4480
$K=120$	(2.1988-2.2257)	2.2275	(6.2912-6.3457)	6.3318	(10.4701-10.5489)	10.5071
Time	115.0458	1.1361	213.0240	1.0222	384.1418	1.0120
			C			
$K=80$	(24.9868-25.0674)	25.0330	(29.8733-29.9914)	29.9447	(34.2306-34.3800)	34.2774
$K=90$	(17.1450-17.2189)	17.1854	(22.9766-23.0872)	23.0442	(27.8994-28.0414)	27.9413
$K=100$	(10.8308-10.8950)	10.8673	(17.1704-17.2717)	17.2340	(22.4243-22.5576)	22.4591
$K=110$	(6.3776-6.4307)	6.4083	(12.5303-12.6212)	12.5877	(17.8197-17.9434)	17.8490
$K=120$	(3.6082-3.6509)	3.6318	(8.9902-9.0706)	9.0398	(14.0420-14.1558)	14.0666
Time	114.5119	1.0286	228.2773	1.0843	354.4452	1.0414

Table 2: Price and confidence interval of European call options for different strikes and maturities calculated through Monte Carlo simulation and the COS method. Parameter setting as in Table 1 and initial price is $S = 100$. Monte Carlo simulation is implemented using 10^6 simulations and discretizing the time grid with $1000 \cdot T$ equally spaced points. COS is implemented truncating infinite summations at 2^7, time is expressed in seconds.

	$T=1$		$T=2$		$T=3$	
	95% C.I.	COS	95% C.I.	COS	95% C.I.	COS
			A			
$K=80$	(21.2443-21.2761)	21.2705	(22.4207-22.4646)	22.4542	(23.5066-23.5588)	23.5395
$K=90$	(11.9546-11.9848)	11.9971	(13.8471-13.8881)	13.8929	(15.5058-15.5547)	15.5501
$K=100$	(4.4014-4.4236)	4.4702	(6.8445-6.8775)	6.9026	(8.8503-8.8915)	8.9056
$K=110$	(0.8707-0.8812)	0.9187	(2.5748-2.5963)	2.6254	(4.2865-4.3170)	4.3392
$K=120$	(0.1031-0.1068)	0.1169	(0.7450-0.7567)	0.7743	(1.7697-1.7896)	1.8084
AS	(4.7818-4.8068)	4.7545	(7.6911-7.7301)	7.6840	(10.2945-10.3458)	10.2930
Time - FS	115.0179	1.5225	212.1204	1.4365	368.0770	1.3121
Time - AS	114.9270	1.0027	212.0176	1.4562	367.9541	1.6107
			B			
$K=80$	(21.2016-21.2380)	21.2537	(22.3959-22.4464)	22.4557	(23.5062-23.5666)	23.5551
$K=90$	(12.0750-12.1089)	12.1853	(14.0917-14.1382)	14.1983	(15.8227-15.8784)	15.9092
$K=100$	(4.7785-4.8039)	4.9527	(7.3992-7.4370)	7.5511	(9.5109-9.5584)	9.6314
$K=110$	(1.1896-1.2035)	1.3214	(3.1763-3.2028)	3.3244	(5.0800-5.1168)	5.2063
$K=120$	(0.2266-0.2332)	0.2828	(1.1533-1.1699)	1.2556	(2.4438-2.4702)	2.5489
AS	(5.3466-5.3766)	5.2231	(8.5373-8.5840)	8.4416	(11.3337-11.3952)	11.2552
Time - FS	115.0533	2.0634	213.0202	1.7715	384.0936	1.6361
Time - AS	114.9365	1.0327	212.9111	1.3264	383.9907	1.6529
			C			
$K=80$	(21.1833-21.2270)	21.3327	(22.4272-22.4893)	22.6529	(23.5980-23.6731)	23.8174
$K=90$	(12.3513-12.3911)	12.6791	(14.6083-14.6647)	14.9734	(16.4880-16.5568)	16.8109
$K=100$	(5.3968-5.4277)	5.8742	(8.3759-8.4230)	8.8429	(10.7006-10.7606)	11.0980
$K=110$	(1.7385-1.7583)	2.1507	(4.2506-4.2869)	4.7130	(6.4915-6.5411)	6.9008
$K=120$	(0.5110-0.5229)	0.7561	(1.9950-2.0215)	2.3645	(3.7471-3.7867)	4.1116
AS	(6.4647-6.5055)	6.0223	(10.2901-10.3548)	9.8779	(13.5475-13.6340)	13.1777
Time - FS	114.5002	2.2244	228.2661	2.2353	354.3970	1.7965
Time - AS	114.3982	1.0126	228.1672	1.3011	354.2999	1.6413

Table 3: Price and confidence interval of Geometric (fixed and floating strike) Asian call options for different strikes and maturities calculated through Monte Carlo simulation and the COS method. Legend: "FS" denotes "Fixed Strike", "AS" denotes "Average Strike". Parameter setting as in Table 1 and initial price is $S = 100$. Monte Carlo simulation is implemented using 10^6 simulations and discretizing the time grid with $1000 \cdot T$ equally spaced points. COS is implemented truncating infinite summations at 2^7, time is expressed in seconds.

	$T=1$			$T=2$			$T=3$		
	Price	SE $\cdot 10^5$	95% C.I.	Price	SE $\cdot 10^5$	95% C.I.	Price	SE $\cdot 10^5$	95% C.I.
				A					
$K=80$	21.3591	0.2717	(21.3575-21.3608)	22.7841	0.6134	(22.7803-22.7879)	24.0815	0.9161	(24.0758-24.0872)
$K=90$	12.0743	0.3353	(12.0722-12.0763)	14.1945	0.6795	(14.1903-14.1987)	16.0526	1.0277	(16.0463-16.0590)
$K=100$	4.5373	0.4301	(4.5346-4.5399)	7.1622	0.8264	(7.1571-7.1673)	9.3531	1.1462	(9.3460-9.3602)
$K=110$	0.9676	0.3950	(0.9651-0.9700)	2.8228	0.9250	(2.8171-2.8286)	4.7074	1.2150	(4.6999-4.7150)
$K=120$	0.1315	0.3394	(0.1294-0.1336)	0.8875	0.8663	(0.8821-0.8928)	2.0706	1.1831	(2.0633-2.0779)
AS	4.6855	0.3852	(4.6831-4.6879)	7.4333	0.7735	(7.4285-7.4381)	9.8557	1.2261	(9.8481-9.8633)
Time - FS		13.2173			26.5489			59.0626	
Time - AS		12.4671			25.0717			37.6292	
				B					
$K=80$	21.4056	0.3696	(21.4033-21.4078)	22.9016	0.7821	(22.8967-22.9064)	24.2635	1.3712	(24.2550-24.2720)
$K=90$	12.3136	0.4271	(12.3110-12.3163)	14.5958	0.8681	(14.5905-14.6012)	16.5550	1.4746	(16.5458-16.5641)
$K=100$	5.0572	0.5528	(5.0537-5.0606)	7.8899	0.9723	(7.8838-7.8959)	10.2040	1.5753	(10.1942-10.2138)
$K=110$	1.3925	0.6387	(1.3885-1.3965)	3.5906	1.0223	(3.5842-3.5969)	5.6863	1.6242	(5.6762-5.6964)
$K=120$	0.3030	0.7985	(0.2981-0.3080)	1.4321	0.8834	(1.4266-1.4375)	2.9134	1.5680	(2.9037-2.9231)
AS	5.1130	0.5356	(5.1096-5.1163)	8.1028	0.9736	(8.0967-8.1088)	10.6837	1.6420	(10.6736-10.6939)
Time-FS		13.1373			24.7512			38.7536	
Time-AS		12.5227			23.0554			37.1071	
				C					
$K=80$	21.6181	0.6944	(21.6138-21.6224)	23.3743	1.4701	(23.3652-23.3835)	24.9273	3.2365	(24.9073-24.9474)
$K=90$	12.9109	0.8226	(12.9058-12.9160)	15.6038	1.6275	(15.5937-15.6139)	17.8027	3.3642	(17.7819-17.8236)
$K=100$	6.0463	0.9975	(6.0401-6.0525)	9.3784	1.7325	(9.3677-9.3892)	11.9704	3.4279	(11.9491-11.9916)
$K=110$	2.2626	1.0850	(2.2558-2.2693)	5.1484	1.7614	(5.1375-5.1594)	7.6505	3.4157	(7.6293-7.6716)
$K=120$	0.8031	1.1175	(0.7962-0.8100)	2.6882	1.7041	(2.6776-2.6987)	4.7280	3.2948	(4.7075-4.7484)
AS	5.8049	1.0628	(5.7983-5.8115)	9.3295	2.0474	(9.3168-9.3422)	12.2271	4.8515	(12.1970-12.2571)
Time - FS		15.7604			23.0520			41.6911	
Time - AS		13.5132			22.7089			37.5917	

Table 4: Price and confidence interval of Arithmetic (fixed and floating strike) Asian call options for different strikes and maturities calculated through Monte Carlo simulation with Geometric average counterpart used as control variable (see formulas 24 and 25). Legend: "FS" denotes "Fixed Strike", "AS" denotes "Average Strike". Parameter setting as in Table 1 and initial price is $S = 100$. Monte Carlo simulation is implemented using 10^5 simulations and discretizing the time grid with $1000 \cdot T$ equally spaced points. COS is implemented truncating infinite summations at 2^7, time is expressed in seconds.