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FIRST ORDER ANALYTICAL SOLUTION FOR DISTANT RETROGRADE ORBITS IN THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

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ABSTRACT

This paper proposes an analytical solution of the first order for distant retrograde orbits in the Circular Restricted Three-Body Problem (CR3BP). To this end, a Maclaurin series expansion on the equation of motions is performed, obtaining a first-order model. Starting from the Hamiltonian formulation of the problem, we apply the theory of canonical perturbations, by addressing the non-perturbed and the perturbed terms separately. The first step is to simplify the problem through a canonical invertible transformation for the non-perturbing part, the second step is to apply the Lie transformation to the perturbed part. This procedure allows to obtain first a mean Hamiltonian that can be analytically solved, then to obtain short-periodic corrections, that take into consideration the short-term fluctuations neglected during the averaging process.

The solutions obtained are compared with the numerical solution simulated with the CR3BP, both in terms of maximum error and in terms of computational speed. Even if the solution shows to be computationally efficient and accurate, the improvement over the solution of Hill on the same problem is not noticeable unless one considers very low mass ratios. The modulation of geometric hypotheses, like the relationship between the size of the orbit and the distance between the primaries, allows to better approximate the contribution of the primary attractor and to increase the range of validity of the analytical solution.

Keywords: Distant retrograde orbits, Circular restricted three-body problem, Canonical perturbation theory, Lie transformation

1 INTRODUCTION

The Distant Retrograde Orbits (DROs) are retrograde orbits that are generated due to the gravitational interaction of the primary with the secondary. They have been identified and classified for the first time by Hènon as the family-f [1].

The peculiarity of DROs is that they always remain in the vicinity of the secondary, consequently they can be exploited for different purposes. In recent years, DROs have been taken into consideration to exploit them in different approaches. For example, to create constellations that allow to increase the reaction time in the case of objects that are directed towards the Earth, [4]. In recent years, DROs have been considered for exploration missions to both the Mars and Jupiter moons and to asteroids, [5], [6]. This study starting from the work by

Lara [2], [3] who applied the Canonical perturbation theory to the Hill model to find an approximate analytic solution for DROs.
In this work, we will continue with Lara's work to try to find a more refined solution to this problem. applying the Canonical perturbation theory, we will find a first order analytical solution for DROs in the Circular Restricted Three-Body Problem (CR3BP). An analytical solution becomes very important in the preliminary phase of the mission design as it allows to reduce the computational time due to the numerical integration. This is advantageous in the optimisation process, used to guarantee the requirements of the mission. An analytical solution allows to better understand the gravitational interactions as well as the dynamics that the primaries generate. This solution will also enable the problem to be better understood at different levels (e.g. in the phase space), making it possible to identify more convenient manoeuvres and station-keeping strategies.

2 DYNAMICAL MODEL

In this context, the Circular Restricted Three-Body Problem describes the motion of a massless body under the gravitational influence of two massive bodies, called the primaries.

2.1 Circular restricted three-body problem

Under the assumptions that the mass of one of the primaries is much larger than the mass of the other and the two primaries orbit around their barycentre in a circular orbit, the normalised equations of motion of the massless body can be derived:

$$\begin{cases} \ddot{x} = 2n\dot{y} + n^2 x - \frac{(1-\mu)(x+\mu)}{\left(\sqrt{(x+\mu)^2+y^2}\right)^3} - \frac{\mu\left(x-(1-\mu)\right)}{\left(\sqrt{\left(x-(1-\mu)\right)^2+y^2}\right)^3} \\ \ddot{y} = -2n\dot{x} + n^2 y - \frac{(1-\mu)y}{\left(\sqrt{(x+\mu)^2+y^2}\right)^3} - \frac{\mu y}{\left(\sqrt{\left(x-(1-\mu)\right)^2+y^2}\right)^3} \end{cases} \tag{2.1}$$

Where x, y represent the cartesian coordinate of the system, n is the non-dimensional mean motion and μ is the mass ration of the system.

$$\mu = \frac{m_2}{m_1+m_2} \tag{2.2}$$

2.1.1 Hamiltonian function

To obtain the Hamiltonian function, the variational principle can be used [7], [8]. The equations are obtained by performing the Legendre transformation of the Lagrangian function, using the generalised velocities as variables. In this way the phase space variables are introduced, and the dynamics generated by a Hamiltonian function is described. Consequently, the Hamiltonian function turns out to be:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + n\left(yp_x - xp_y\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \tag{2.3}$$

where p_x, p_y represent the conjugate moments and r_1, r_2 represent the distance of the third body from the primary and the secondary attractor.

2.2 First order term of the circular restricted three-body problem

The CR3BP is not an integrable system. To simplify the CR3BP and enhance the dynamics in the vicinity of the secondary, we assume the mass of the primary is much larger than that of the secondary ($\mu \to 0$). The previous hypothesis is known as Hill and leads to the homonymous model. To obtain the new model, used in this study, it is necessary to move the reference system from the centre of gravity of the system to the secondary attractor normalising the variables by the coefficient proportional to the $\mu^{1/3}$, as described in [9], [10], to better observe the dynamics in the vicinity of the secondary. Finally, a series development of Mclaurin is performed on the variable μ of the equations of motion, arrested at the first order. Then we obtain the following Hamiltonian function:

$$H = \frac{1}{2}\left((p_x + ny)^2 + \left(p_y - nx\right)^2\right) - \frac{3}{2}n^2x^2 - \frac{\mu}{r_2} + \frac{n^2}{p}x\left(x^2 - \frac{3}{2}y^2\right) \tag{2.4}$$

where $p = \frac{1}{\mu^{1/3}}$ represent the distance between the primaries. This new model will be called CR3BP-1$_{13}$.

3 PERTURBATION APPROACH

To apply the Canonical perturbation theory, as done by Lara for the Hill's problem, we need to separate the Hamiltonian into a non-perturbed, H_0, and a perturbed part, H_1:

$$H = H_0 + \varepsilon H_1 \to \begin{cases} H_0 = \frac{1}{2}\left(p_x^2 + p_y^2\right) - nxp_y + nyp_x - n^2x^2 + \frac{1}{2}n^2y^2 \\ H_1 = -\frac{\mu}{r_2} + \frac{n^2}{p}x\left(x^2 - \frac{3}{2}y^2\right) \end{cases} \tag{3.1}$$

where ε is a formal small parameter which is used to manifest that the effect of H_1, the perturbation, is much smaller than H_0.

3.1 Canonical transformation

The study of the solutions of a system of differential equations can take place through the search for a transformation of coordinates under whose action the system assumes a particularly simple form. First of all, the problem is to find a class of transformations $(x, y, p_x, p_y) = \mathcal{T}(\phi, q, \Phi, Q)$ invertible such that the system of the Hamilton equations, relative to the Hamiltonian function, is transformed into the system written in Canonical variables.

3.1.1 Hamilton-Jacobi equation

The method of the generating function allows to find a suitable Canonical transformation starting from the Hamiltonian function. To determine an appropriate generating function, a partial differential equation, called (complete) Hamilton-Jacobi equation, [11]–[13] need to be solved. In our case the Hamilton-Jacobi equation for non-perturbed part only is:

$$\frac{1}{2}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + ny\frac{\partial W}{\partial x} - nx\frac{\partial W}{\partial y} - n^2x^2 + \frac{1}{2}n^2y^2 = \mathcal{K}(\Phi, Q) \tag{3.2}$$

Where Φ, Q are the Canonical conjugate momenta, so the generating function can be written as:

$$W = \frac{nx - 2Q}{2n}\sqrt{2n\Phi - (nx - 2Q)^2} + y(Q - nx) + \Phi \tan^{-1}\left(\frac{nx - 2Q}{\sqrt{2n\Phi - (nx - 2Q)^2}}\right) \tag{3.3}$$

which allows to generate the following Canonical transformation:

$$\begin{cases} x=\frac{Q+k\sqrt{2n\Phi}\sin\phi}{kn} \\ y=2kq+2\sqrt{\frac{2\Phi}{n}}\cos\phi \\ p_x=-2knq-\sqrt{2n\Phi}\cos\phi \\ p_y=\frac{-Q-2k\sqrt{2n\Phi}\sin\phi}{2k} \end{cases} \tag{3.4}$$

where $k = 3/4$ is the normalisation term.

3.2 Perturbed solution

The DROs have an elliptical shape and the centre at the following coordinates with respect to the secondary mass:

$$\begin{cases} x_c = \frac{Q}{kn} \\ y_c = 2kq \end{cases} \tag{3.5}$$

where q represent the Canonical coordinate, introducing the following auxiliary variables:

$$\chi = \frac{y_c}{a} \quad \gamma = \frac{1}{an\Phi} \quad B = bn$$
$$\sigma = \frac{x_c}{2b} \quad a = 2b \quad b = \sqrt{\frac{2\Phi}{n}} \tag{3.6}$$

where b is the semi-minor axis of the DRO, a is the semi-major axis, in Figure 3.1 the reader can see what the geometric interpretation of the variables that have just been described is.

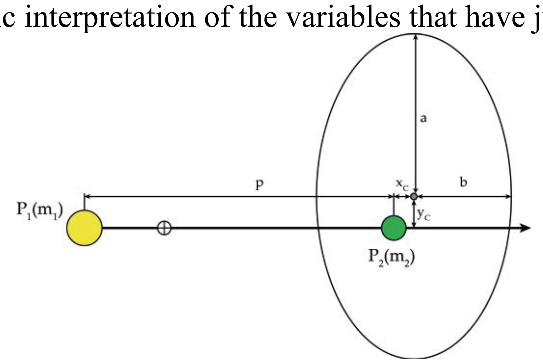

Figure 3.1: Geometric interpretation of the auxiliary variables.

hence the Hamiltonian takes the following form:

$$\mathcal{K} = \mathcal{K}_{0\,Hill} + \varepsilon\mathcal{K}_1 \tag{3.7}$$

with

$$\begin{cases} \mathcal{K}_0 = n\Phi(1-3\sigma^2) \\ \mathcal{K}_1 = \begin{cases} -\frac{(n\Phi)\gamma}{\sqrt{\sigma^2+\chi^2+\sigma\sin\phi+2\chi\cos\phi+\Delta^2}} \\ \frac{b(2\sigma+\sin\phi)}{p}(8\sigma(\sigma+\sin\phi)-5-12\chi(\chi+2\cos\phi)-7\cos 2\phi) \end{cases} \end{cases} \tag{3.8}$$

3.2.1 Lie transformation

Equation 3.7 can be expanded into Maclaurin series for $\chi \to 0$, assuming also that $\sigma = O[\chi^2]$. These assumptions are defined in DRO because the abscissa of the centre of the ellipse x_c suffers a small variation, secondly χ remains limited and less than 1. In addition, to ensure that the perturbing term is of a lower order than the non-perturbation one, we introduce the following assumption $\gamma = O[\chi^4]$. With these assumptions, we can apply the following replacement,

$$\begin{cases} \sigma = \varepsilon^2 \chi \\ \gamma = \varepsilon^4 \chi \end{cases} \tag{3.9}$$

like Lara in [2], [3]. In addition, we add a hypothesis concerning the distance between the primaries, in fact, this must be sufficiently large with respect to the oscillation of the centre of the orbit. This assumption in normalised terms translates into the form $b/p = O[\chi^2]$. Now applying the Lie transformation, the mean Hamiltonian ($\langle \mathcal{K} \rangle$) takes the following form:

$$\begin{aligned} \langle \mathcal{K} \rangle = {} & n\Phi - \tfrac{1}{2}Q^2 - \sqrt{\tfrac{n}{2\Phi}}\widetilde{K} + \tfrac{1}{4}\sqrt{\tfrac{n^3}{2\Phi^3}}\left(\tilde{E} - \widetilde{K}\right)q^2 + \tfrac{1-2\widetilde{K}}{8\Phi^2} + \tfrac{(14\tilde{E}-11\widetilde{K})}{128}\sqrt{\tfrac{n^5}{2\Phi^5}}q^4 + \tfrac{\widetilde{K}-4\tilde{E}}{9\pi\sqrt{2n\Phi^3}}Q^2 + \\ & \tfrac{1}{4320np^4\sqrt{\Phi}}\Big[40\sqrt{3}p^3Q\left(\sqrt{2n}\left(1367\widetilde{K} - 2216\tilde{E}\right) + 12\sqrt{\Phi}(8Q^2 - 27n^2q^2) - 216n\sqrt{\Phi^3}\right) - \\ & p^2\Big(5\sqrt{2n}\widetilde{K}(2187nq^2 - 55379\Phi) + 4\sqrt{\Phi}\Big(55552\sqrt{2n\Phi}\tilde{E} + 405(27n^3q^4 + 180n^2q^2\Phi - \\ & 14n\Phi^2 544Q^2\Phi)\Big)\Big) + 9720\sqrt{3}pQ\sqrt{\Phi^3}(36nq^2 - 509\Phi) + 810\sqrt{\Phi^5}(91854nq^2 - \\ & 39191\Phi)\Big] \end{aligned} \tag{3.10}$$

where $\widetilde{K} = \frac{K(k^2)}{\pi}$, $\tilde{E} = \frac{E(k^2)}{\pi}$ and where $K(k^2)$ and $E(k^2)$ represent complete elliptic integrals of the first and second types, q represent the Canonical coordinate and Φ, Q are the Canonical conjugate momenta. Consequently, the equations of motion obtained from this Hamiltonian turn out to be:

$$\begin{cases} \dot{\phi} = \tilde{C}_{\phi-a} + \tilde{C}_{\phi-b}q^2 + \tilde{C}_{\phi-c}q^4 + \tilde{C}_{\phi-d}Q^2 \\ \dot{q} = \tilde{C}_{q-a}Q + \varepsilon\,\tilde{C}_{q-a\varepsilon}Q \\ \dot{\Phi} = 0 \\ \dot{Q} = \tilde{C}_{Q-a}q + \varepsilon\,\tilde{C}_{Q-a\varepsilon}q^3 \end{cases} \tag{3.11}$$

where the 2nd in the 4th equations, which are coupled, can be solved by Lindstedt-Poincaré technique, [2], [3], obtaining:

$$\begin{cases} q(t) = q_{9,0}(1 + \omega_1 t) + q_{9,1}(1 + \omega_1 t) \\ Q(t) = Q_{9,0}(1 + \omega_1 t) + Q_{9,1}(1 + \omega_1 t) \end{cases} \tag{3.11}$$

At the end the 1st equation is resolved by quadrature after replacing in the previous ones.

4 MODEL VALIDATION

We will now compare both numerical and analytical models with respect to the numerical CR3BP model that is taken as a reference in this work. In this analysis we will consider the family-f of orbits by Hénon [1], also called DRO, in different synodic systems. To better understand the comparison, the problem is reduced by representing it in a 2-D graph. Each orbit is represented as a point corresponding to the maximum error, along an orbital period, of the analytical model with respect to the numerical one. To do this, we introduce the following error variables:

- Absolute error: It represents the absolute error obtained in position with respect to the correct numerical model (CR3BP) over one orbital period:

$$\epsilon_{abs,r}[0,T] = \left\|\underline{r}[0,T] - \underline{r}_{true}[0,T]\right\| \tag{4.1}$$

- Relative error: It represents the relative error obtained in position with respect to the correct numerical model over one orbital period:

$$\epsilon_{rel,r}[0,T] = \frac{\epsilon_{abs,r}[0,T]}{\left\|\underline{r}[0,T]-\underline{r}_{true}[0,T]\right\|} \tag{4.2}$$

- Maximum error: It represents the maximum of the relative error over one orbit:

$$\epsilon_{rel-max,r} = max\left(\epsilon_{rel,r}[0,T]\right) \tag{4.3}$$

4.1 Results for the model with the assumption $O[\chi^2]$

This case represents the solution proposed in the previous section associated with the assumption $b/p = O[\chi^2]$. This assumption guarantees a good trade-off between the accuracy of the solution and the fact that it allows to develop a lot of terms of the Lie series without encountering problems in the analytical integration, e.g. to determine the mean Hamiltonian or the generating function. Figure 4.1 represents the trend of the maximum error along one orbital period. In particular, the blue lines represent the error that the Hill model has with respect to CR3BP, instead the red ones the error that the CR3BP-1_{13} model has, always, to the CR3BP model. In particular, the dash-dot represent the analytical solution. Consequently, the model is particularly reliable and usable for low mass ratio systems, so that for further decreasing μ the maximum error related to the CR3BP-1_{13} model (dash-dot red line) is lower than that of Hill (dash-dot blue line).
For Figure 4.1, the error reaches a minimum of about 0.003 for $A_x = 7$, the error has a zone between $A_x = 7$ and $A_x = 11$ where it remains constant and then increases again, in the first part (on the left) the red line, for the analytical solution, remain equal to the blue one.

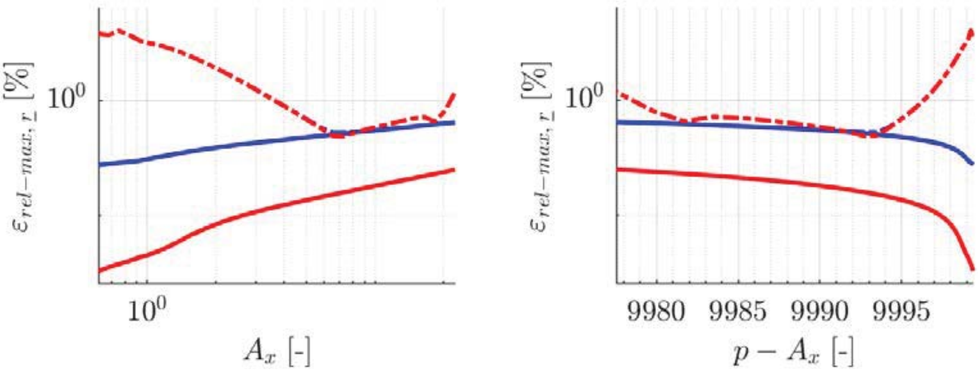

Figure 4.1: Trend of the maximum error as the size of the orbit for Sun-Alauda system increases with the assumption $b/p = O[\chi^2]$.

4.2 Comparison between different assumptions

Several models were produced in this study. In addition to the one proposed in the previous paragraph, models with the hypothesis $b/p = O[\chi^5]$ the $b/p = O[\chi^3]$ and $b/p = O[\chi^1]$ were considered. These hypotheses were tested by observing the validity limits. The verification phase consisted in simulating the different models obtained for various systems, i.e. when the mass parameter μ varies, in particular the values corresponding to the Sun system (Earth-Moon) (about 10^{-6}), Mars-Deimos (about 10^{-9}) and finally to Sun-Alauda ($\mu = 10^{-12}$). To better understand the validity of these hypotheses, observe the comparison between the parameters in Figure 4.2. These images represent the trend of the assumption $b/p = O[\chi^j]$, previously reported, when the size of the orbit (A_x) increases and the mass parameter m varies. The plots show the trend and above all the intensity of the Big-O hypothesis ($b/p = O[\chi^j]$) when the orbit increases in size i.e. approaches the primary attractor. The colors are associated with the order, j, based on the hypothesis used. From these figures we can deduce that the more j is small, the more the $O[\chi^j]$ assumption is verified. This can be explained by the fact that the more the orbit increases in size, the closer it gets to the primary attractor and consequently the dynamics brings the deviation of the centre of the orbit to be more and more comparable with the distance between the primaries and no longer a fraction of the latter.

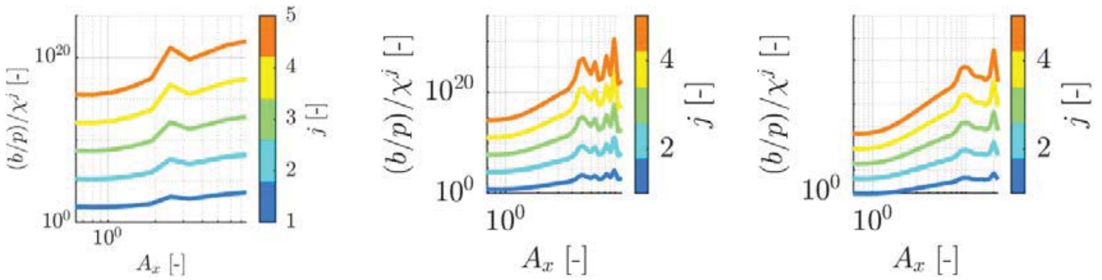

Figure 4.2: Trend of the assumption $b/p = O[\chi^j]$ for Sun-Alauda system as the size of the orbit (A_x) increases (from left to right Earth-Moon, Mars-Deimos and Sun-Alauda).

5 CONCLUSION

The mathematical model used here is the planar CR3BP in a simplified version. This new model, little discussed in the literature, is very interesting as it allows to improve the Hill problem and can be exploited even in regions closer to the primary attractor.

The work produced a series of analytical solutions for the first term of the CR3BP using various assumptions. In particular, the hypothesis that produces an improvement is that involving $b/p = O[\chi^2]$. This occurs because the contribution of the primary attractor, even if approximated, must be considered predominant with respect to the influence of the secondary. Through the assumption, the Canonical perturbation theory allows, first of all, to consider the contribution of the primary in a more incisive way with respect to the secondary. Secondly, the assumption guarantees to avoid the coupling between the elliptical integrals allowing to solve analytically the problem by considering different terms of the canonical perturbation theory.

However, the analytical solution, proposed by Lara, for the Hill model compared to the new solution proposed in this study turns out to be very similar, in particular, in the vicinity of the secondary attractor. This happens because, getting closer to the secondary, the contribution of the primary can be considered as a disturbance. On the other hand, in the Hill model, the contribution of the primary attractor is correctly solved, according to the well-known Clohessy-Wiltshire equations [15]. Unfortunately, this does not appear to be true in the new model CR3BP-1_{13}, where the additional terms relating to the primary attractor are considered as deductions of the perturbations. Therefore, the closer the third body to the primary attractor, the less these terms can be considered as perturbations. From the previous considerations, the most restrictive hypothesis is certainly that on b/p. This assumption involves several couplings that make the use of the Lie transformation complicated, even at a low level of the theory. As already mentioned above, the terms of the Lie transform (which are a function of incomplete elliptic integrals of the first and second type) are already coupled at low orders of the perturbative theory. This coupling makes the analytical integration of these elliptical integrals not feasible, reducing the strength of the theory itself. However, the model obtained in this study is more advantageous from the point of view of computational time. However, it is slowed down by the calculation of elliptic integrals, in particular, the incomplete ones, which need iterative algorithms to be solved. In any case, the errors that the analytical model produces with respect to the numerical one, are acceptable with respect to the numerical one. Moreover, the more the mass ratio decreases, the more the range of validity of the new model CR3BP-1_{13} increases. This improvement depends on the fact that the distance between the primaries is no longer infinite (as considered in the Hill model) but finite and therefore applicable to more realistic cases.

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