

Stability, \mathcal{L}_1 performance and state feedback design for linear systems in ice-cream cones

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Abstract

This paper considers linear systems which are positively invariant in a second-order cone (ice-cream cone). Three problems are addressed: (i) stability; (ii) \mathcal{L}_1 performance; (iii) state feedback design for stabilization and optimal \mathcal{L}_1 performance while preserving cone-invariance. We derive necessary and sufficient conditions via Linear Matrix Inequalities (LMI) for the solution of problems (i) and (ii). As for problem (iii), a full parametrization of feasible state feedback gains is provided, along with some LMI relaxations useful to compute a feasible gain. Finally, a numerical example is briefly discussed.

1. Introduction

In various applications, we can find dynamical linear systems characterized by the property that the state trajectories remains in a convex cone for any initial state in the cone. Such systems are dubbed as positively invariant with respect to the cone. A well-studied example is the class of positive linear time-invariant systems, where the cone coincides with the non-negative orthant of the state-space. The related theory is particularly rich, see for instance Briat (2013); Ebihara, Peaucelle, and Arzelier (2014); Farina and Rinaldi (2011); Rantzer (2015); Shen and Lam (2017b); Tanaka and Langbort (2011). Some extensions to positive switched systems have been recently presented in Blanchini, Colaneri, and Valcher (2015); Bolzern and Colaneri (2015).

Stability, and input-output performance analysis of positively invariant systems in general cones were recently studied in Shen and Lam (2017a); Shen and Zheng (2015). An interesting application arises in the rendezvous problem of multiple agents (Bhattacharya, Tiwari, Fung, & Murray, 2009). Specific results for polyhedral cones, defined in terms of linear inequalities, can be found in Chen, Bolzern, Colaneri, Bo, and Du (2018, accepted by CDC 2018). Relatively less attention has been paid to

dynamical systems which are positively invariant with respect to second order cones, defined in terms of quadratic inequalities. In particular, the present paper deals with a special class of second order cones, known in the literature as ice-cream cones. For such systems, stability and some performance indices can be analyzed via cone programming, by applying the general results of Shen and Lam (2017a); Shen and Zheng (2015). However, passing from stability analysis to the design of a stabilizing gain is far from being trivial, due to the constraints induced by the cone invariance requirement.

Remarkably, none of the available contributions provides LMI conditions for stability analysis, neither a full parametrization of the feasible state feedback gains. Moreover, the \mathcal{L}_1 performance index characterization and the corresponding state feedback design problem in ice-cream cones has not been considered before.

The main contributions of this paper include:

- (1) A necessary and sufficient condition for stability and cone invariance in ice-cream cones.
- (2) A necessary and sufficient condition for the existence of a state feedback controller guaranteeing stability and cone invariance of the closed-loop system.
- (3) \mathcal{L}_1 performance evaluation in open loop and \mathcal{L}_1 performance optimization via state feedback.

All the relevant algorithms are expressed in terms of LMI. One of the main difficulties is in the synthesis of the state feedback controller gain ensuring cone invariance in closed-loop. As a matter of fact, while the condition for cone invariance can be expressed in terms of linear programming in open-loop, it becomes nonlinear when the feedback gain has to be designed. Due to this reason, the design procedure cannot rely directly on the open-loop analysis results and is essentially based on quadratic programming tools and appropriate LMI relaxations.”

The paper is organized as follows. Section II gives some useful mathematical preliminaries. Open-loop stability and \mathcal{L}_1 performance analysis are discussed in Section III. Section IV provides a solution to the state feedback design problems. A numerical example is presented in Section V. Some concluding remarks are given in Section VI.

Notation: The symbols \mathbb{R} , \mathbb{R}^n and $\mathbb{R}^{n \times n}$ represent the set of real numbers, the space of vectors of n -tuples of real numbers and the space of $n \times n$ matrices with real entries, respectively. The identity matrix with dimension n is denoted as I_n . Vector e_i is the i th column of I_n . The transpose of matrix A is denoted by A' . A square matrix A is Hurwitz if all its eigenvalues lie in the open left half-plane. The symbol (ξ, ζ) represents the inner product of vectors ξ and ζ . Throughout this paper, we use Q to denote a constant matrix, which is $Q = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}$.

2. Preliminaries

We consider the class of continuous-time linear systems described by

$$\Xi : \dot{x}(t) = Ax(t)$$

where $x(t) \in \mathbb{R}^n$ is the state vector.

2.1. Useful definitions and lemmas

The following definitions and lemmas are required for the subsequent analysis.

Definition 2.1. (Ice-cream cone) (Schneider & Vidyasagar, 1970; Stern & Wolkowicz, 1991) An n -dimensional ice-cream cone \mathcal{K} is defined as

$$\mathcal{K} = \{x \in \mathbb{R}^n : x'Qx \geq 0, x'e_1 \geq 0\}.$$

The symbol $\text{Int}(\mathcal{K})$ will be used to denote the interior of \mathcal{K} , i.e. the set of vectors x satisfying the two inequalities of Definition 2.1 with the strict sign. The dual of \mathcal{K} , i.e. the set of vectors $y \in \mathbb{R}^n$ such that $(y, x) \geq 0, \forall x \in \mathcal{K}$, will be denoted as \mathcal{K}^* . Note that ice-cream cones are self-dual, namely $\mathcal{K}^* = \mathcal{K}$.

Definition 2.2. (Positively invariant system) System Ξ is positively invariant in \mathcal{K} if all state trajectories starting from any initial state $x_0 \in \mathcal{K}$ remain in \mathcal{K} , $\forall t \geq 0$.

Definition 2.3. (Cone invariant matrix) A matrix $A \in \mathbb{R}^{n \times n}$ is said cone invariant in \mathcal{K} if $Ax \in \mathcal{K}, \forall x \in \mathcal{K}$; it is said strictly cone invariant if $Ax \in \text{Int}(\mathcal{K}), \forall x \in \mathcal{K}, x \neq 0$.

Definition 2.4. (Cross positive matrix) (Schneider & Vidyasagar, 1970) A matrix A is called cross positive in \mathcal{K} if for all $y \in \mathcal{K}, z \in \mathcal{K}^*$ satisfying $(z, y) = 0$, it holds that $(z, Ay) \geq 0$.

Definition 2.5. (Strictly cross positive matrix) Schneider and Vidyasagar (1970) A matrix A is called strictly cross positive in \mathcal{K} if for all $y \in \mathcal{K}, z \in \mathcal{K}^*, y \neq 0, z \neq 0$ satisfying $(z, y) = 0$, it holds that $(z, Ay) > 0$.

From now on, we will consider \mathcal{K} as an ice-cream cone and, for brevity, the specification “in \mathcal{K} ” appearing in all definitions will be dropped off. Note that, in view of the above definitions and the property $\mathcal{K} = \mathcal{K}^*$, a matrix A' is (strictly) cross positive if and only if A is (strictly) cross positive.

Lemma 2.6. (Stern & Wolkowicz, 1991) The following statements are equivalent:

- (a) System Ξ is positively invariant.
- (b) Matrix A is cross positive.
- (c) There exists $\xi \in \mathbb{R}$ such that $A'Q + QA + \xi Q \geq 0$.

Lemma 2.7. (Schneider & Vidyasagar, 1970; Stern & Wolkowicz, 1991) The following statements are equivalent:

- (a) Matrix A is strictly cross positive.
- (b) There exists $\xi \in \mathbb{R}$ such that $A'Q + QA + \xi Q > 0$.
- (c) There exists $\alpha \geq 0$ such that $(A + \alpha I_n)$ is strictly cone invariant, i.e. $(A + \alpha I_n)x \in \text{Int}(\mathcal{K}), \forall x \in \mathcal{K}, x \neq 0$.

Remark 1. For system Ξ , A being strictly cross positive is equivalent to the property that all trajectories of system Ξ starting from any nonzero initial state $x_0 \in \mathcal{K}$ lay in $\text{Int}(\mathcal{K}), \forall t > 0$. In particular, trajectories starting from the boundary of the cone evolve in the interior of the cone.

Lemma 2.8. (Shen & Lam, 2017b) If matrix A is cross positive, then A is Hurwitz if and only if there exists a vector $s \in \text{Int}(\mathcal{K})$, such that $-As \in \text{Int}(\mathcal{K})$.

Lemma 2.9. *Matrix A is cone invariant if and only if there exists $\eta \geq 0$ such that*

$$A'e_1 \in \mathcal{K} \quad (1a)$$

$$A'QA \geq \eta Q \quad (1b)$$

Moreover, A is strictly cone invariant if and only if (1a) is replaced by $A'e_1 \in \text{Int}(\mathcal{K})$, and (1b) is satisfied with strict sign.

Proof. Necessity. Assume that A is cone invariant. It means that

$$\begin{aligned} e'_1 Ax &\geq 0, \forall x \in \mathcal{K}, \\ x' A' Q Ax &\geq 0, \forall x \in \mathcal{K}. \end{aligned}$$

Observe that \mathcal{K} is self-dual, so that $A'e_1 \in \mathcal{K}$ and, in view of the S-procedure (Boyd & Vandenberghe, 2004), there exists $\eta \geq 0$ such that $A'QA \geq \eta Q$.

Sufficiency. Assume that $A'e_1 \in \mathcal{K}$, $A'QA \geq \eta Q$. Then, $e'_1 Ax \geq 0, \forall x \in \mathcal{K}$. Let $y = Ax$, then $y'e_1 \geq 0$ and $y'Qy \geq \eta x'Qx \geq 0$. In conclusion, $Ax \in \mathcal{K}, \forall x \in \mathcal{K}$, i.e. A is cone invariant.

Similar arguments can be applied to prove the conditions for strict cone invariance. The detailed proof is therefore omitted. \blacksquare

Lemma 2.10. *Suppose that V_1 is cone invariant and nonsingular, and V_2 is strictly cone invariant. Then, the product V_1V_2 is strictly cone invariant.*

Proof. Define $h = V_2x$ and $y = V_1V_2x = V_1h$. Then $\forall x \in \mathcal{K}, x \neq 0$, we have that $h \in \text{Int}(\mathcal{K})$ by noticing that V_2 is strictly cone invariant. Since V_1 is cone invariant, there exists $\eta \geq 0$ such that $V'_1QV_1 \geq \eta Q$. Moreover, since V_1 is nonsingular, $\eta \neq 0$, i.e. $\eta > 0$. Otherwise, if $\eta = 0$, then we would have that $V'_1QV_1 \geq 0$, or equivalently $Q \geq 0$, which is obviously false. Thus, $\forall x \in \mathcal{K}, x \neq 0$ it follows that

$$y'Qy = h'V'_1QV_1h \geq \eta h'Qh > 0$$

This shows that the product V_1V_2 is strictly cone invariant. \blacksquare

Lemma 2.11. *(Schneider & Vidyasagar, 1970) If matrix A is strictly cross positive and Hurwitz, then there exist $s \in \text{Int}(\mathcal{K})$ and $\alpha_1 > 0$ such that*

$$(A + \alpha_1 I_n)s = 0$$

Moreover, all the eigenvalues λ_i of A different from $-\alpha_1$ are such that $\text{Re}(\lambda_i) < -\alpha_1$.

2.2. Matrix representations of vectors in ice-cream cones

In this subsection, we introduce two useful matrix representations of vectors belonging to an ice-cream cone, named as the arrow-shaped representation $A^{[rw]}[s] : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, and the quadratic representation $A^{[pw]}[z] : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ (Alizadeh & Goldfarb, 2003).

2.2.1. Arrow-shaped representation

Let \mathcal{S} be the set of symmetric matrices of the form

$$\begin{aligned} A^{[rw]}[s] &= e_1 s' + s e_1' - e_1' s Q \\ &= \begin{bmatrix} e_1' s & e_2' s & \cdots & e_n' s \\ e_2' s & e_1' s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_n' s & 0 & \cdots & e_1' s \end{bmatrix} \end{aligned}$$

where $s \in R^n$. Notice that, thanks to the Schur Lemma, $A^{[rw]}[s] \geq 0$ (with $A^{[rw]}[s] \neq 0$) if and only if

$$e_1' s > 0, \quad (e_1' s)^2 - \sum_{i>1} (e_i' s)^2 \geq 0$$

Hence, there is a bijective correspondence between vectors $s \in \mathcal{K}, s \neq 0$ and (non-null) positive semidefinite matrices $A^{[rw]}[s] \in \mathcal{S}$, i.e. a vector s belongs to the cone if and only if $A^{[rw]}[s] \geq 0$. Similarly, there is a bijective correspondence between vectors $s \in \text{Int}(\mathcal{K})$ and positive definite matrices $A^{[rw]}[s] \in \mathcal{S}$, i.e. a vector s belongs to the interior of the cone if and only if $A^{[rw]}[s] > 0$. Since $A^{[rw]}[s]$ is a linear function of s , these properties are important because the check on s being in the cone, or in its interior, can be expressed in terms of LMI, instead of using the quadratic condition implied by Definition 2.1.

As a further property, notice that

$$s = A^{[rw]}[s] e_1$$

Moreover, it can be shown that $A^{[rw]}[s] > 0$ implies that $(A^{[rw]}[s])^{-1} e_1$ is a vector in $\text{Int}(\mathcal{K})$. More in general, matrix $(A^{[rw]}[s])^{-1}$ is cone invariant.

2.2.2. Quadratic representation

Given $z \in \mathbb{R}^n$, define $A^{[pw]}[z]$ as

$$A^{[pw]}[z] = 2zz' - (z'Qz)Q$$

Notice that, by inspection, it results that

$$A^{[pw]}[z] Q A^{[pw]}[z] = (z'Qz)^2 Q \quad (2)$$

Vectors $s \in \mathcal{K}$ are generated by vectors $z \in R^n$ as follows:

$$s = \begin{bmatrix} z'z \\ 2(z'e_1)(z'e_2) \\ \vdots \\ 2(z'e_1)(z'e_n) \end{bmatrix} = A^{[pw]}[z] e_1 = A^{[rw]}[s] e_1 \quad (3)$$

Vice versa, given $s \in \mathcal{K}$, we can obtain $z \in R^n$ from the following equations:

$$\begin{aligned} (z'e_1)^2 &= \frac{1}{2}(s'e_1 + \sqrt{s'Qs}) \\ z'e_i &= \frac{s'e_i}{2z'e_1} \quad (i > 1) \end{aligned} \tag{4}$$

Notice that, if $s \in \mathcal{K}$, it results that $z'Qz = \sqrt{s'Qs}$. Therefore, $s \in \text{Int}(\mathcal{K})$ implies that $z \in \text{Int}(\mathcal{K})$, provided that the positive root is taken for the first entry of z .

Some important properties of matrix $A^{[pw]}[z]$ will be useful in the sequel. First, it can be shown that matrix $A^{[pw]}[z]$ is cone invariant, for any $z \in R^n$. Indeed, let $w = A^{[pw]}[z]p$ with $p \in \mathcal{K}$. Then, using (2),

$$w'Qw = p'A^{[pw]}[z]QA^{[pw]}[z]p = (z'Qz)^2 p'Qp \geq 0$$

Moreover, matrix $A^{[pw]}[z]$ is nonsingular provided that $z'Qz \neq 0$. In such a case, $(A^{[pw]}[z])^{-1}$ is cone invariant. Finally, $A^{[pw]}[z]$ is positive definite if and only if $z \in \text{Int}(\mathcal{K})$.

3. Stability and performance

In this section, stability and \mathcal{L}_1 performance analysis for the dynamical linear system Ξ will be discussed. Recall that Q represents the characteristic matrix of the cone, as introduced in Definition 2.1.

3.1. Stability

First, a necessary and sufficient condition is provided to check stability and positive invariance of system Ξ .

Theorem 3.1. *The matrix A is cross positive and Hurwitz (or equivalently, system Ξ is positively invariant and asymptotically stable) if and only if there exist $s \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$ such that*

$$A'Q + QA + \xi Q \geq 0 \tag{5a}$$

$$A^{[rw]}[s] > 0 \tag{5b}$$

$$A^{[rw]}[As] < 0 \tag{5c}$$

Proof. In view of Lemma 2.6, condition (5a) is equivalent to cross-positivity of A .

Moreover, (5b) and (5c) can be rewritten as

$$\begin{bmatrix} e'_1 s & e'_2 s & \cdots & e'_n s \\ e'_2 s & e'_1 s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e'_n s & 0 & \cdots & e'_1 s \end{bmatrix} > 0$$

$$\begin{bmatrix} -e'_1 A s & e'_2 A s & \cdots & e'_n A s \\ e'_2 A s & -e'_1 A s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e'_n A s & 0 & \cdots & -e'_1 A s \end{bmatrix} > 0$$

Via the Schur Lemma, these two inequalities are equivalent to $s \in \text{Int}(\mathcal{K})$ and $-As \in \text{Int}(\mathcal{K})$, respectively. According to Lemma 2.8, A is Hurwitz. \blacksquare

Remark 2. Conditions (5a)-(5c) are given in terms of LMI, with vector $s \in \mathbb{R}^n$ and the scalar ξ as unknowns. Hence, standard LMI tools can be used to check their feasibility.

3.2. Performance analysis

Now we characterize the \mathcal{L}_1 performance of system Ξ through the index

$$J(x_0) = \int_0^\infty v'x(t)dt, \quad v \in \mathcal{K} \quad (6)$$

with $x(0) = x_0 \in \mathcal{K}$ as initial state. Note that $v'x(t)$ remains nonnegative $\forall t \geq 0$ provided that system Ξ is positively invariant. Based on the stability criterion presented in Theorem 3.1, a necessary and sufficient condition for checking stability, positive invariance and boundedness of $J(x_0)$ can be derived as follows.

Theorem 3.2. *Given an initial state $x_0 \in \mathcal{K}$, a vector $v \in \mathcal{K}$ and a positive scalar γ , system Ξ is asymptotically stable, positively invariant and $J(x_0) < \gamma$ if and only if there exist $s \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$ such that*

$$A'Q + QA + \xi Q \geq 0 \quad (7a)$$

$$A^{[rw]}[s] > 0 \quad (7b)$$

$$A^{[rw]}[As + x_0] < 0 \quad (7c)$$

$$v's < \gamma \quad (7d)$$

Proof. Inequality (7a) is the necessary and sufficient condition for system Ξ being positively invariant. If system Ξ is asymptotically stable, the performance index $J(x_0)$ can be computed as

$$J(x_0) = -v'A^{-1}x_0$$

Sufficiency. Note that $x_0 \in \mathcal{K}$ is equivalent to $A^{[rw]}[x_0] \geq 0$. Thus, (7c) implies $A^{[rw]}[As] < 0$. By using also (7b) and (7a), we can conclude that system Ξ is positively

invariant and asymptotically stable. Next, notice that $-A^{-1}$ is cone invariant (Shen & Lam, 2017a) and $-(As + x_0) \in \mathcal{K}$. Then

$$J(x_0) = v'(-A^{-1}x_0) = v'[-A^{-1}(As + x_0) + s] < v's < \gamma$$

Necessity. Since $J(x_0) < \gamma$, there always exists a sufficiently small $\epsilon > 0$ such that $(1 + \epsilon)J(x_0) < \gamma$. Now, define $s = -(1 + \epsilon)A^{-1}x_0$. Then we have that $As + x_0 = -\epsilon x_0$ and $v's = (1 + \epsilon)J(x_0) < \gamma$, and (7d) is satisfied. Finally, observe that the assumptions that A is cross positive and Hurwitz imply $-A^{-1}$ being cone invariant. Since $x_0 \in \mathcal{K}$, both (7b) and (7c) hold. ■

Remark 3. Note that conditions (7a)-(7d) are in LMI form. Hence, the value of the performance index can be computed by solving the convex problem of minimization of $\gamma > 0$ satisfying all inequalities.

4. Feedback design

In this section, we consider the input-driven system

$$\Xi_u : \dot{x}(t) = Ax(t) + Bu(t)$$

where $u(t) \in \mathbb{R}^m$ is the input vector. We aim at designing a state-feedback gain K for system Ξ_u , with $u(t) = Kx(t)$, in order to guarantee both positive invariance and asymptotic stability of the closed-loop system. This amounts to imposing that the matrix $A + BK$ is cross positive and Hurwitz. In addition, we will consider the design of a state-feedback gain guaranteeing also \mathcal{L}_1 performance, namely $J(x_0) < \gamma$, where $J(x_0)$ is defined in (6).

4.1. Stabilization

For the stabilization problem in the cone, a direct application of the conditions of Theorem 3.1 does not lead to a problem formulation viable to an easy solution even when cross positivity is replaced by strict cross positivity. For this reason, we will resort to Lemma 2.7 to enforce strict cross positivity of $A + BK$ by looking for a nonnegative scalar α such that $A + BK + \alpha I_n$ is strictly cone invariant.

A necessary and sufficient condition to guarantee strict cross positivity and Hurwitz stability of the closed-loop system matrix $A + BK$ is given in the following result.

Theorem 4.1. *A necessary and sufficient condition for the existence of K such that $A + BK$ is strictly cross positive and Hurwitz is that there exist nonsingular $P = P' \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{m \times n}$, $\eta \geq 0$, $\alpha > 0$ such that*

$$A^{[rw]}[Pe_1] > 0 \tag{8a}$$

$$A^{[rw]}[(AP + BH)e_1] < 0 \tag{8b}$$

$$(AP + BH + \alpha P)Q(AP + BH + \alpha P)' > \eta Q \tag{8c}$$

$$PQP = Q \tag{8d}$$

$$A^{[rw]}[(AP + BH + \alpha P)e_1] \geq 0 \tag{8e}$$

Any admissible state feedback gain K is then given by $K = HP^{-1}$.

Proof. Sufficiency. Assume that (8a)-(8e) hold and let $K = HP^{-1}$, $s = Pe_1$.

According to Lemma 2.9, (8a) and (8d) indicate that P is cone invariant. The same occurs for P^{-1} by noticing that $P^{-1} = QPQ$ and $P^{-1}e_1 = QPQe_1 = QPe_1 \in \mathcal{K}$.

Conditions (8c) and (8e) imply that $P(A + BK + \alpha I_n)'$ is strictly cone invariant. According to Lemma 2.10, $(A + BK + \alpha I_n)' = P^{-1}P(A + BK + \alpha I_n)'$ is strictly cone invariant, as well. From Lemma 2.7, $(A + BK)'$ is strictly cross positive, and, equivalently, $A + BK$ is strictly cross positive.

Finally, (8a) and (8b) indicate that

$$\begin{aligned} s &\in \text{Int}(\mathcal{K}) \\ -(A + BK)s &\in \text{Int}(\mathcal{K}) \end{aligned} \tag{9}$$

Hence $A + BK$ is Hurwitz.

Necessity. Assume that K is such that $A + BK$ is strictly cross positive and Hurwitz. Then there exists $s \in \mathbb{R}^n$ such that (9) is satisfied. Let

$$P = \frac{A^{[pw]}[z]}{z'Qz}$$

where $z \in \text{Int}(\mathcal{K})$ is such that $A^{[pw]}[z]e_1 = A^{[rw]}[s]e_1 = s$, see eq. (4). It is obvious that $PQP = Q$, by recalling (2), so that (8d) holds. By defining $H = KP$, then

$$\begin{aligned} A^{[rw]}[Pe_1] &= \frac{1}{z'Qz} A^{[rw]}[s] > 0 \\ A^{[rw]}[(AP + BH)e_1] &= \frac{1}{z'Qz} A^{[rw]}[(A + BK)s] < 0 \end{aligned}$$

which lead to (8a) and (8b), respectively.

Since $A + BK$ is strictly cross positive, so is $(A + BK)'$. Then, there exists $\alpha \geq 0$ such that $(A + BK + \alpha I_n)'$ is strictly cone invariant. Since P is cone invariant, the strict cone invariance of $P(A + BK + \alpha I_n)' = (AP + BH + \alpha P)'$ follows (see Lemma 2.10). Finally, from Lemma 2.9, (8c) and (8e) hold. ■

Note that since (8c) and (8d) are quadratic in matrix P , they are not easy to be solved. In order to simplify the computation, we present an equivalent relaxed condition for (8c) by introducing a new parameter α_1 , which can be interpreted as the absolute value of the dominant closed-loop eigenvalue.

Corollary 4.2. *A necessary and sufficient condition for the existence of K such that $A + BK$ is strictly cross positive and Hurwitz is that there exist nonsingular $P = P' \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{m \times n}$, $\eta \geq 0$, $\alpha > \alpha_1 > 0$ such that*

$$A^{[rw]}[Pe_1] > 0 \tag{10a}$$

$$\begin{bmatrix} 2(\alpha - \alpha_1)^2 p_1 p_1' - \eta Q & AP + BH + \alpha P \\ (AP + BH + \alpha P)' & I_n \end{bmatrix} > 0 \tag{10b}$$

$$PQP = Q \tag{10c}$$

$$(AP + BH + \alpha_1 P)e_1 = 0 \tag{10d}$$

where $p_1 = Pe_1$. Then, any admissible state feedback gain K is given by the formula $K = HP^{-1}$.

Proof. Sufficiency. Assume that (10a)-(10d) hold. Define $K = HP^{-1}$. Condition (10a) coincides with (8a), and condition (10c) coincides with (8d). Equation (10d) can be rewritten as

$$(AP + BH)e_1 = -\alpha_1 Pe_1$$

Then, condition (8b) is satisfied by noticing that $\alpha_1 > 0$ and $A^{[rw]}[Pe_1] > 0$.

From the Schur Lemma, condition (10b) is equivalent to

$$2(\alpha - \alpha_1)^2 Pe_1 e_1' P - FF' > \eta Q \quad (11)$$

where $F = (AP + BH + \alpha P)$. Note that $Q = 2e_1 e_1' - I_n$, so that

$$-FF' = FQF' - 2Fe_1 e_1' F'$$

Moreover, by noticing that $(AP + BH + \alpha_1 P)e_1 = 0$, we have $(\alpha - \alpha_1)Pe_1 = (AP + BH + \alpha_1 P)e_1 + (\alpha - \alpha_1)Pe_1 = Fe_1$. Then

$$-2Fe_1 e_1' F' = -2(\alpha - \alpha_1)^2 Pe_1 e_1' P$$

Therefore, (11) is equivalent to $FQF' > \eta Q$, which is condition (8c).

Finally, (10d) implies that

$$\begin{aligned} & (AP + BH + \alpha P)e_1 \\ &= (AP + BH + (\alpha + \alpha_1)P)e_1 - \alpha_1 Pe_1 \\ &= (\alpha - \alpha_1)Pe_1 \end{aligned}$$

Hence condition (8e) holds. According to Theorem 4.1, $A + BK$ is strictly cross positive and Hurwitz.

Necessity. Assume that there exists a state feedback gain such that $A + BK$ is Hurwitz and strictly cross positive. Then, in view of Lemma 2.11, there exist $s \in \text{Int}(\mathcal{K})$ and $\alpha_1 > 0$ such that $(A + BK + \alpha_1 I)s = 0$. Define $P = \frac{A^{[pw]}[z]}{z'Qz}$, where $z \in \text{Int}(\mathcal{K})$ is such that $A^{[pw]}[z]e_1 = A^{[rw]}[s]e_1 = s$, see (4). Obviously, $PQP = Q$ and $Pe_1 = \frac{s}{z'Qz} \in \text{Int}(\mathcal{K})$, which imply (10a) and (10c). Moreover, (10d) follows by noticing that

$$(AP + BH + \alpha_1 P)e_1 = \frac{(A + BK + \alpha_1 I)s}{z'Qz} = 0$$

Finally, by reversing the arguments applied in the sufficiency part, inequality (10b) holds. \blacksquare

In Corollary 4.2, equality condition (10c) is still quadratic in P . In order to simplify the computation, we can choose $P = \frac{A^{[pw]}[s]}{s'Qs}$, so that, in view of (2), condition (10c) is trivially satisfied. Then, an equivalent necessary and sufficient condition for state feedback design can be formulated.

Corollary 4.3. *A necessary and sufficient condition for the existence of K such that $A + BK$ is strictly cross positive and Hurwitz is that there exist $s \in \mathbb{R}^n$, $H \in \mathbb{R}^{m \times n}$, $\eta \geq 0$ and $\alpha > \alpha_1 > 0$ such that*

$$A^{[rw]}[s] > 0 \quad (12a)$$

$$\begin{bmatrix} \Upsilon - \eta Q & \Phi \\ \Phi' & I \end{bmatrix} > 0 \quad (12b)$$

$$(AR + BH + \alpha_1 R - AQ - \alpha_1 Q)e_1 = 0 \quad (12c)$$

where

$$\begin{aligned} R &= \frac{2ss'}{s'Qs}, \quad r_1 = Re_1 \\ \Upsilon &= 2(\alpha - \alpha_1)^2(r_1 - e_1)(r_1 - e_1)' \\ \Phi &= AR + BH + \alpha R - AQ - \alpha Q \end{aligned} \quad (13)$$

Then any admissible state feedback gain K is given in the form $K = H(R - Q)^{-1}$.

Proof. The proof is immediate since, letting $P = R - Q$, conditions (12a)-(12c) are equivalent to conditions (10a), (10b) and (10d) of Corollary 4.2. Moreover condition (10c) of Corollary 4.2 is trivially verified since $P = \frac{A^{[pw]}[s]}{s'Qs}$. ■

Note that conditions of Corollary 4.3 cannot be easily handled due to the following reasons:

- (1) Matrix Υ is quadratic with respect to vector r_1 .
- (2) Matrix R is a rank one nonlinear function of s .
- (3) The term $(\alpha - \alpha_1)^2$ in (12b) is nonlinear with respect to α and α_1 .

In order to address these difficulties, we assume, without loss of generality, that $s'e_1 = 1$, and introduce the following relaxations:

- (1) The quadratic equation defining Υ is replaced by the inequality

$$\begin{bmatrix} \Upsilon & \sqrt{2}(\alpha - \alpha_1)(r_1 - e_1) \\ \sqrt{2}(\alpha - \alpha_1)(r_1 - e_1)' & 1 \end{bmatrix} \geq 0$$

- (2) The term ss' is replaced by $\frac{s'Qs}{2}R$ with

$$\begin{bmatrix} R & s & 0_{n,n} \\ s' & 1 & s' \\ 0_{n,n} & s & 2I_n \end{bmatrix} \geq 0$$

A practical way to find a feasible gain through LMI is summarized in the following algorithm:

Algorithm 1:

Step 1. Select the absolute value $\alpha_1 > 0$ of the dominant closed loop eigenvalue and $\alpha > \alpha_1$ sufficiently large.

Step 2. For $\beta \in (0, 1)$ solve the following optimization problem with respect to the unknowns $s \in \mathbb{R}^n, \eta \geq 0, H \in \mathbb{R}^{m \times n}, R > 0, \Upsilon \geq 0$:

$$\min \beta \text{Trace}(R) + (1 - \beta) \text{Trace}(\Upsilon)$$

$$\begin{aligned} A^{[rw]}[s] &> 0 \\ \begin{bmatrix} \Upsilon - \eta Q & \Phi \\ \Phi' & I_n \end{bmatrix} &> 0 \\ (AR + BH + \alpha_1 R - AQ - \alpha_1 Q)e_1 &= 0 \\ \begin{bmatrix} R & s & 0_{n,n} \\ s' & 1 & s' \\ 0_{n,n} & s & 2I_n \end{bmatrix} &\geq 0 \\ \begin{bmatrix} \Upsilon & \sqrt{2}(\alpha - \alpha_1)(r_1 - e_1) \\ \sqrt{2}(\alpha - \alpha_1)(r_1 - e_1)' & 1 \end{bmatrix} &\geq 0 \end{aligned}$$

with $s'e_1 = 1, r_1 = Re_1$ and $\Phi = AR + BH + \alpha R - AQ - \alpha Q$.

Step 3. Check the conditions

$$\text{Trace}(\Upsilon - 2(\alpha - \alpha_1)^2(r_1 - e_1)'(r_1 - e_1)) = 0, \quad \text{Trace}(R - \frac{2}{s'Qs}s's) = 0$$

with a prescribed numerical tolerance. If they all hold for some $\beta \in (0, 1)$, go to Step 4. Otherwise, no gain exists such that $A + BK$ is Hurwitz, strictly cross positive and the dominant eigenvalue is $-\alpha_1$.

Step 4. Compute the state feedback gain as $K = H(R - Q)^{-1}$.

Remark 4. The algorithm above is intended to find a feedback gain that, besides guaranteeing positive invariance and stability, assigns the dominant eigenvalue of $A + BK$ to be $-\alpha_1$. If one is interested only in positive invariance and stability, the algorithm could be run with decreasing values of α_1 .

4.2. \mathcal{L}_1 performance

The results presented in the previous subsection can be extended to solve the problem of state feedback design preserving positive invariance and stability while guaranteeing an upper bound of the \mathcal{L}_1 performance index $J(x_0)$ defined in (6). Suppose that the prescribed upper bound is given by $\gamma > 0$. By using Theorems 3.2 and 4.1, the following result straightforwardly follows.

Theorem 4.4. *A necessary and sufficient condition for the existence of K such that $A + BK$ is strictly cross positive and Hurwitz and $J(x_0) < \gamma$ is that there exist*

nonsingular $P = P' \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{m \times n}$, $\eta \geq 0$, $\alpha > 0$ such that

$$A^{[rw]}[Pe_1] > 0 \quad (15a)$$

$$A^{[rw]}[(AP + BH)e_1 + x_0] < 0 \quad (15b)$$

$$(AP + BH + \alpha P)Q(AP + BH + \alpha P)' > \eta Q \quad (15c)$$

$$PQP = Q \quad (15d)$$

$$A^{[rw]}[(AP + BH + \alpha P)e_1] \geq 0 \quad (15e)$$

$$v'Pe_1 < \gamma \quad (15f)$$

Any admissible state feedback gain K is then given by $K = HP^{-1}$.

The following corollaries are the counterparts of Corollaries 4.2 and 4.3. Notice that it is not required to assign the dominant eigenvalue $-\alpha_1$.

Corollary 4.5. *A necessary and sufficient condition for the existence of K such that $A + BK$ is strictly cross positive and Hurwitz and $J(x_0) < \gamma$ is that there exist nonsingular $P = P' \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{m \times n}$, $\eta \geq 0$, $\alpha > 0$ such that*

$$A^{[rw]}[Pe_1] > 0 \quad (16a)$$

$$\begin{bmatrix} 2(\alpha p_1 - x_0)(\alpha p_1 - x_0)' - \eta Q & AP + BH + \alpha P \\ (AP + BH + \alpha P)' & I_n \end{bmatrix} > 0 \quad (16b)$$

$$PQP = Q \quad (16c)$$

$$(AP + BH)e_1 + x_0 + p_1 = 0 \quad (16d)$$

$$v'p_1 < \gamma \quad (16e)$$

where $p_1 = Pe_1$. Then, any admissible state feedback gain K is given by the formula $K = HP^{-1}$.

Corollary 4.6. *A necessary and sufficient condition for the existence of K such that $A + BK$ is strictly cross positive and Hurwitz and $J(x_0) < \gamma$ is that there exist $s \in \mathbb{R}^n$, $H \in \mathbb{R}^{m \times n}$, $\eta \geq 0$ and $\alpha > 0$ such that*

$$A^{[rw]}[s] > 0 \quad (17a)$$

$$\begin{bmatrix} \Upsilon_x - \eta Q & \Phi \\ \Phi' & I \end{bmatrix} > 0 \quad (17b)$$

$$(AR + BH + R - AQ - Q)e_1 + x_0 = 0 \quad (17c)$$

$$v'(r_1 - e_1) < \gamma \quad (17d)$$

where

$$\begin{aligned} R &= \frac{2ss'}{s'Qs}, \quad r_1 = Re_1 \\ \Upsilon_x &= 2(\alpha(r_1 - e_1) - x_0)(\alpha(r_1 - e_1) - x_0)' \\ \Phi &= AR + BH + \alpha R - AQ - \alpha Q \end{aligned} \quad (18)$$

Then any admissible state feedback gain K is given in the form $K = H(R - Q)^{-1}$.

For α sufficiently large, an LMI algorithm that implements a relaxation of the conditions of Corollary 4.6 can be worked out following a similar rationale as presented in the stabilization algorithm above.

Algorithm 2:

Step 1. Select $\alpha > 0$ sufficiently large.

Step 2. For $\beta \in (0, 1)$ solve the following optimization problem with respect to the unknowns $s \in \mathbb{R}^n, \eta \geq 0, H \in \mathbb{R}^{m \times n}, R > 0, \Upsilon_x \geq 0$:

$$\min \beta \text{Trace}(R) + (1 - \beta) \text{Trace}(\Upsilon_x)$$

$$\begin{aligned} A^{[rw]}[s] &> 0 \\ \begin{bmatrix} \Upsilon_x - \eta Q & \Phi \\ \Phi' & I_n \end{bmatrix} &> 0 \\ (AR + BH + \alpha_1 R - AQ - \alpha_1 Q)e_1 + x_0 &= 0 \\ v'(r_1 - e_1) &< \gamma \\ \begin{bmatrix} R & s & 0_{n,n} \\ s' & 1 & s' \\ 0_{n,n} & s & 2I_n \end{bmatrix} &\geq 0 \\ \begin{bmatrix} \Upsilon_x & \sqrt{2}(\alpha(r_1 - e_1) - x_0) \\ \sqrt{2}(\alpha(r_1 - e_1)' - x_0') & 1 \end{bmatrix} &\geq 0 \end{aligned}$$

with $s'e_1 = 1, r_1 = Re_1$ and $\Phi = AR + BH + \alpha R - AQ - \alpha Q$.

Step 3. Check the conditions $\Upsilon_x = 2(\alpha(r_1 - e_1) - x_0)(\alpha(r_1 - e_1) - x_0)', R = \frac{2}{s'Qs}ss'$ with a prescribed numerical tolerance. If they all hold for some $\beta \in (0, 1)$, go to Step 4. Otherwise, no gain exists such that $A + BK$ is Hurwitz and strictly cross positive.

Step 4. Compute the state feedback gain as $K = H(R - Q)^{-1}$.

Remark 5. For any given $x_0 \in \mathcal{K}$, by iterating Algorithm 2 with decreasing values of γ , one can find $K = K(x_0)$ that solves the minimization problem

$$J^o(x_0) = \inf_{K \in \mathcal{S}} -v'(A + BK)^{-1}x_0$$

where \mathcal{S} is the set of all state feedback gains such that the closed-loop system matrix $A + BK$ is Hurwitz and strictly cross positive. In order to evaluate the robust performance yielded by any $K \in \mathcal{S}$, for different values of the initial state x_0 in the cone with the same norm $\|x_0\| = \zeta$, it can be observed that the guaranteed performance is

$$J_g(K) := \max_{x_0 \in \mathcal{K}, \|x_0\| = \zeta} -v'(A + BK)^{-1}x_0 = \zeta \|v'(A + BK)^{-1}\|$$

Indeed, for any $K \in \mathcal{S}$, it turns out that $-(A' + K'B')^{-1}v \in \mathcal{K}$, so that the worst performance is attained when x_0 is aligned with such a vector.

5. Numerical example

Example 5.1. Consider the 3-dimensional system Ξ_u with

$$A = \begin{bmatrix} 1.3000 & -0.1586 & -0.5266 \\ 0.5936 & -0.7791 & -0.6855 \\ 0.7902 & -0.1948 & -1.9184 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.8147 & 0.9134 \\ 0.9058 & 0.6324 \\ 0.1270 & 0.0975 \end{bmatrix}$$

By applying Algorithm 1 with $\alpha_1 = 0.1$ and $\alpha = 5$, the optimal solution gives

$$s = [1 \quad 0.0133 \quad 0.2059]'$$

Both matrices R and Υ pass the check in Step 3 with the given tolerance 10^{-7} . The state feedback gain K , denoted here as K_s , is

$$K_s = H(R - Q)^{-1} = \begin{bmatrix} 1.6005 & -11.6150 & 0.5294 \\ -2.7859 & 10.0405 & -0.3071 \end{bmatrix}$$

and $A + BK_s$ is strictly cross positive and Hurwitz.

Note that no feasible solution can be found when α_1 is greater than 1.01.

Consider now the \mathcal{L}_1 performance design problem with $x_0 = [1 \quad 0.5 \quad -0.1]'$, $v = [1 \quad 0.5 \quad 0.5]'$. The previous design yields $J^o(x_0) = -v'(A + BK_s)^{-1}x_0 = 12.7$. It is interesting to compute the minimum attenuation level γ_o that can be achieved with the given x_0 . By iteratively applying Algorithm 2 with decreasing values of $\gamma < 12.7$, it turns out that $\gamma_o = J_o(x_0) = 1.12$, with s given by

$$s = [1 \quad -0.1012 \quad 0.1511]'$$

and the state feedback gain matrix K , denoted here as K_p , given by

$$K_p = \begin{bmatrix} -0.8741 & -13.5718 & -1.6056 \\ -1.7652 & 12.6846 & 2.6091 \end{bmatrix}$$

Matrices R, Υ_x pass the check with tolerance 10^{-7} and $A + BK_p$ is strictly cross positive, Hurwitz and the attenuation level γ^o is attained. The time-evolutions of $v'x(t)$ and $x(t)'Qx(t)$ for both gain matrices K_s and K_p are shown in Figures 1, 2.

In view of Remark 5, one can easily compute the guaranteed performance (with $\zeta = \|x_0\| = 1.1225$), associated with both K_s and K_p , resulting in $J_g(K_s) = 14.6741$ and $J_g(K_p) = 1.4201$.

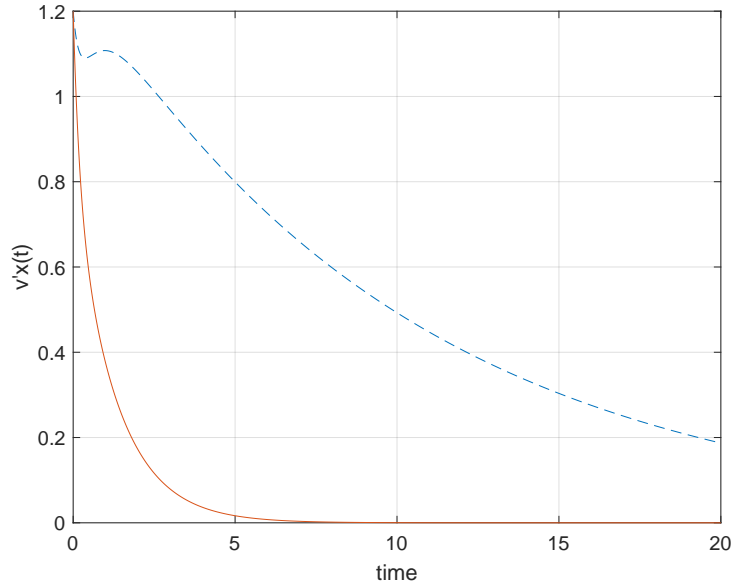


Figure 1. Time evolution of $v'x(t)$ for different gain matrices K : stabilization design (dashed) and performance design (solid).

6. Conclusions

In this paper, we have investigated on stability and \mathcal{L}_1 performance of time-invariant linear systems in ice-cream cones. State feedback design guaranteeing both stability and cone invariance has been also worked out. Necessary and sufficient conditions based on LMI have been established, which apparently are not available in the literature to date. It is remarkable that the theory developed in this paper with respect to ice-cream cones can be easily adapted to cope with general ellipsoidal cones, where matrix Q is replaced by $\bar{Q} = \text{diag}\{1, -Z\}$, where $Z \in \mathbb{R}^{(n-1) \times (n-1)}$ is positive definite. Indeed, it is sufficient to apply the state transformation $\xi = Tx$, with $T = \text{diag}\{1, Z^{-1/2}\}$ and use the results of the paper on the transformed system. In perspective, further investigation on different indices of input-output performance and the extension to Markov jump linear systems in ice-cream cones is foreseen.

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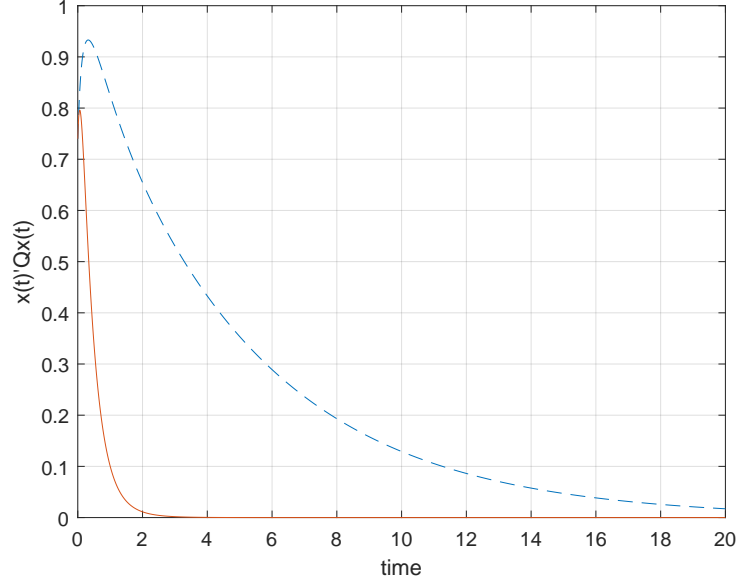


Figure 2. Time evolution of $x(t)'Qx(t)$ for different gain matrices K : stabilization design (dashed) and performance design (solid).

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