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FRAGILE WORDS AND CAYLEY TYPE TRANSDUCERS

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Abstract. We address the problem of finding examples of non-bireversible transducers defining free groups, we show examples of transducers with sink accessible from every state which generate free groups, and, in general, we link this problem to the non-existence of certain words with interesting combinatorial and geometrical properties that we call fragile words. By using this notion, we exhibit a series of transducers constructed from Cayley graphs of finite groups whose defined semigroups are free, and thus having exponential growth.

1. Introduction

Automaton groups became very popular in the last decades because they provide examples of groups with special and exotic properties. In 1980 R. I. Grigorchuk, for example, described the first example of a group of intermediate (i.e. faster than polynomial and slower than exponential) growth giving a positive answer to the so called Milnor's problem. It later appeared that the most natural way to describe this group is by looking at its generating automaton (Mealy machine or transducer). Grigorchuk and his collaborators developped a very exiting research in connection with various mathematical topics. It is worth mentioning here, the deep connections with the theory of profinite groups and with complex dynamics. In particular, many groups of this type satisfy a property of self-similarity, reflected on fractalness of some limit objects associated with them [1,2,12].

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Although the many surprising and interesting results for this class of groups, very little is know from the algebraic, algorithmic and dynamical point of view. Therefore a series of open questions naturally appear in this setting [3, 7, 9, 10, 16]. One of the most intriguing question is what kind of groups may be generated by these Mealy machines. Among these an interesting family is constituted by finitely presented groups such as free groups and free products of finite groups.

In this paper we first tackle the problem of finding free groups generated by automata with a sink state. Indeed, one common feature of all the aforementioned examples of free automata groups, is the fact that they are all defined by transducers which are bireversible [15, 17, 18]. This leads to the question whether it is possible to generate a free group by means of transducers with a sink state. This case represents, in some sense, the opposite of bireversible transducers. This problem has been already adressed in [6] in connection with the dynamics on the boundary. However, in this paper a more combinatorial approach is used. We answer positively to this question by showing how to build an infinite series of Mealy machines with a sink state reachable from every state defining free groups. We show that this reachability condition is equivalent to the existence of g-regular elements on the boundary for every element g in the generated group. In this case the resulting free groups do not act transitively on the corresponding tree, so it leaves open the question of finding a free group generated by a non-bireversible automaton and acting transitively on the sets of words of the same length in the alphabet. In this framework, we propose a combinatorial approach to deal with non-free groups defined by transducers whose sink state is accessible from every state, using certain words which we call fragile. Roughly speaking these are words representing minimal relations such that a small transformation brings them to reduced words. The second part of the paper is devoted to some examples of groups obtained by suitable colorings of the Cayley machines. In particular, we prove that the relations for such groups can be detected by using purely combinatorial properties of the dual automaton. Such observation enables us to study a special class of such machines generating free semigroups, and therefore groups with exponential growth.

2. Preliminaries

In this paper A will denote a finite set, called *alphabet*. A word w over A is a string $w = w_1 \cdots w_n$. The set A^+ (A^*) of all finite non-empty words (words) over A has a structure of free semigroup (monoid) on A with respect to the usual operation of concatenation of words (and with identity the empty word 1). By A^ω we denote the set of words over A infinite to the right. We use the vector notation, and for an element $\underline{u} = u_1 u_2 \cdots u_i \cdots \in A^\omega$ the prefix of length $k > 0$ is denoted by $\underline{u}[k] = u_1 u_2 \cdots u_k$, while the factor $u_i \cdots u_j$ is denoted by $\underline{u}[i, j]$. By $\tilde{A} = A \cup A^{-1}$ we denote the *involutive alphabet* where A^{-1} is the set of *formal* inverses of elements A. For each $u \in \widetilde{A}^*$, we denote by $\overline{u}$ the (unique) reduced word equivalent to u. We say that u is *reduced* whenever $\overline{u} = u$. With a slight abuse in the notation we often identify the elements of the free group on $|A|$ symbols F_A with their reduced representatives. A *finite state Mealy automaton*, shortly a transducer (or a machine), is a 4-tuple $\mathscr{A} = (Q, A, \cdot, \circ)$ where:

- Q is a finite set, called set of *states*;
- A is a finite alphabet;
- $\cdot : Q \times A \to Q$ is the transition (possibly partial) map or *restriction*;
- $\circ : Q \times A \to A$ is the output (possibly partial) map or *action*.

When a state $q \in Q$ is fixed, we denote by $q\cdot : A \to Q$, $q\circ : A \to A$ the associated (partial) maps. In this paper we assume both $q\cdot : A \to Q$ and $q\circ : Q \to A$ maps. The transducer $\mathscr{A}$ is said to be invertible if, for all $q \in Q$, the transformation $q\circ$ is a permutation of A. Sometimes, it is easier to represent a Mealy automaton using a graph theoretic approach. Indeed, we may visualize the transducer $\mathscr{A}$ as an $A \times A$-labelled digraph with vertex set Q and edges of the form $q\xrightarrow{a|b}q'$ whenever $q \cdot a = q'$ and $q \circ a = b$. The maps $q\cdot$ and $q\circ$ may be extended to A^* inductively by

$$q \cdot (a_1 \dots a_n) = (q \cdot a_1) \cdot (a_2 \cdots a_n)$$

and

$$q \circ (a_1 \cdots a_n) = q \circ a_1 \left((q \cdot a_1) \circ (a_2 \cdots a_n)\right)$$

Analogously one can naturally extend these maps to Q^*. From the algebraic point of view the action Q^* over A^* gives rise to a finitely generated semigroup $\mathcal{S}(\mathscr{A})$ generated by the graph endomorphisms $\mathscr{A}_q$, $q \in Q$, of the rooted tree identified with A^* defined by $\mathscr{A}_q(u) = q \circ u$, $u \in A^*$. For $q_1, \dots, q_m \in Q$ we may use the shorter notation $\mathscr{A}_{q_m \cdots q_1} = \mathscr{A}_{q_m} \cdots \mathscr{A}_{q_1}$. An important role in group theory is played by groups defined by invertible transducers, for more details we refer the reader to [12]. In the case of invertible transducers all the maps $\mathscr{A}_q$, $q \in Q$, are automorphisms of the rooted regular tree identified with A^*, and the group generated by these automorphisms is denoted by $\mathcal{G}(\mathscr{A})$ (with identity $\mathbb{1}$). Henceforth a generator $\mathscr{A}_q$ of $\mathcal{G}(\mathscr{A})$ (or $\mathcal{S}(\mathscr{A})$) is identified with the element $q \in Q$, and its inverse with the formal inverse $q^{-1} \in Q^{-1} = \{q^{-1} : q \in Q\}$. Note that the actions of the inverses Q^{-1} are given by the inverse (transducer) $\mathscr{A}^{-1}$ having Q^{-1} as the set of vertices, and by swapping input with output: $q^{-1}\xrightarrow{a|b}p^{-1}$ in $\mathscr{A}^{-1}$ if and only if $q\xrightarrow{b|a}p$ is an edge in $\mathscr{A}$. The action of $\mathcal{G}(\mathscr{A})$ on A^*, in the case when $\mathscr{A}$ is invertible (or $\mathcal{S}(\mathscr{A})$ in the more general case), can be naturally extended to the action on the *boundary* A^ω of the tree. Two important classes of transducers that we consider throughout the paper are the *reversible* and *bireversible* machines. A transducer $\mathscr{A}$ is called reversible whenever

$$q \cdot a = p \cdot a \quad \text{implies} \quad q = p,$$

and it is bireversible if it is reversible in the input and the output, that is

$$q \cdot a = p \cdot b \quad \text{and } q \circ a = p \circ b, \quad \text{implies} \quad q = p.$$

The dual of a transducer $\mathscr{A} = (Q, A, \cdot, \circ)$ is the automaton $\partial\mathscr{A} = (A, Q, \circ, \cdot)$. It is simply obtained by exchanging the role of the set of states and the alphabet. The dual of an invertible transducer is in general non-invertible but it is reversible [5].

3. Transducers with sink-state

We consider the rather broad class $\mathcal{S}_a$ of invertible transducers $\mathscr{A} = (Q, A, \cdot, \circ)$ having a sink-state (shortly sink), i.e., a state e with transitions $e \xrightarrow{x|x} e$, $x \in A$. We also make the extra assumption that the sink $e \in Q$ is accessible from every state ("a" of $\mathcal{S}_a$ stands for accessible).
Automata with sink were considered by Sidki [14], who defined a notion of complexity measured by the number of directed paths avoiding the sink-state in the automaton. This led him to the notion of polynomial automata, for which he managed to prove that they do not contain any nonabelian free group. We address the more general question whether or not a nonabelian free group may be generated by a transducer containing a sink-state. We show that this problem has a simple solution, but for the transitive case it appears more difficult. Moreover, there is an interesting connection between non-freeness of such automata groups and the existence of certain words, that we name *fragile*. These words possess some interesting combinatorial features that deserve to be studied.
We now characterize the class $\mathcal{S}_a$ in terms of regular elements (see, for instance [13]). Let $g \in \mathcal{G}(\mathscr{A})$. We recall that an element $\underline{w} \in A^\omega$ is g-regular, if there exists a prefix $\underline{w}[n]$ of $\underline{w}$ such that $g \cdot \underline{w}[n] = e$. If for every $n \geq 1$, $g\underline{w}[n] \neq e$ then $\underline{w}$ is said to be g-singular. Given an automaton $\mathscr{A} = (Q, A, \cdot, \circ)$, we may define its *reduction automaton* $\mathcal{R}(\mathscr{A}) = (\mathcal{R}(Q), A, \widehat{\cdot}, \widehat{\circ})$ as the smallest automaton where we identify all the maximal strongly connected components that induce the trivial transformation and glue them into one single sink-state e. It is straightforward to check that $\mathcal{G}(\mathscr{A}) \simeq \mathcal{G}(\mathcal{R}(\mathscr{A}))$. With the notion of g-regular (g-singular) elements the class $\mathcal{S}_a$ has the following property.

Proposition 1. *If $\mathscr{A} \in \mathcal{S}_a$ with $\mathscr{A} = (Q, A, \cdot, \circ)$, then every element $g \in \mathcal{G}(\mathscr{A})$ has a g-regular element in A^ω. Conversely if the action of every element $g \in \mathcal{G}(\mathscr{A})$ has a g-regular element in A^ω then $\mathcal{R}(\mathscr{A}) \in \mathcal{S}_a$.*

Proof. The proof is by induction on the length m of an element $q_1 \cdots q_m \in \widetilde{Q}^*$, representing some element $g \in \mathcal{G}(\mathscr{A})$ via the canonical map $\widetilde{Q}^* \to \mathcal{G}(\mathscr{A})$. We consider the two following cases.

- If $q_m \in Q$, then since $\mathscr{A} \in \mathcal{S}_a$ there is a $u \in A^*$ such that $q_m \cdot u = e$. Hence we get $q_1 \cdots q_m \cdot u = q'_1 \cdots q'_{m-1} e$. By the induction hypothesis there is a $w \in A^*$ such that $q'_1 \cdots q'_{m-1} \cdot w = e^{m-1}$, whence $q_1 \cdots q_m \cdot uw = e^m$, and so $uwA^\omega = \{uw\eta : \eta \in A^\omega\}$ is a set of g-regular elements.
- If $q_m \in Q^{-1}$, then there is a $u \in A^*$ such that $q_m^{-1} \cdot u = e$. Since $\mathscr{A}$ is invertible consider $u' = q_m^{-1} \circ u$. Thus, $q_m \cdot u' = e$ and the rest of the proof proceeds like in the previous case.

Conversely, let us show that if $\mathcal{R}(\mathscr{A}) \notin \mathcal{S}_a$, then there exists $g \in \mathcal{G}(\mathscr{A})$ that does not admit any g-regular sequence. The first condition implies that either $\mathcal{R}(\mathscr{A})$ has a unique sink, but the sink is not accesible from every state, or $\mathcal{R}(\mathscr{A}) = \mathscr{A}$ has no sink-state. The latter case gives that for every state $q \in Q$, q has no q-regular elements. The former case implies that there exists a state $q \in \mathcal{R}(Q)$ with the property that $q \,\widehat{\cdot}\, w \neq e$ for every $w \in A^*$. In particular q does not admit any q-regular sequence. □

For every $q \in Q$, define the erasing homomorphism $\epsilon_q : \widetilde{Q}^* \to \widetilde{Q}^*$ as the homomorphism that sends q, q^{-1} to the empty word 1, and does not affect other letters. In this context, a word $w \in \widetilde{Q}^*$ is called trivial if $\overline{\epsilon_e(w)} = 1$. The *support* of $w \in \widetilde{Q}^*$ is the smallest subset $Q' \subseteq Q$ such that $w \in \widetilde{Q'}^*$; we say that w has m *occurrences* if the cardinality of the support of w is m.

Definition 1 (Fragile and fully fragile words). *Let $\mathscr{A} = (Q, A, \cdot, \circ)$ be an invertible transducer in $\mathcal{S}_a$ with sink-state e. A non-trivial word $w \in \widetilde{Q}^*$ is called fragile if there is $a \in A$ such that $\overline{w \cdot a}$ is trivial. If $\overline{w \cdot a} = 1$ for all $a \in A$, we call w fully fragile.*

The following proposition shows the crucial role that fragile words play in non-free automata groups defined by transducers with a sink-state.

Proposition 2. *Let $\mathscr{A} \in \mathcal{S}_a$. If $\mathcal{G}(\mathscr{A})$ is not free, then there is a shortest non-trivial relation $w \in \widetilde{Q}^*$ that is a fragile word. On the other hand, if $w \in \widetilde{Q}^*$ is a fully fragile word, then w is a relation of $\mathcal{G}(\mathscr{A})$.*

Proof. Take any shortest non-trivial relation $w \in \widetilde{Q}^*$, and assume $w = q_k^{f_k} \cdots q_1^{f_1}$, for some $f_i \in \{1, -1\}$. Since q_1 (q_1^{-1} in $\mathscr{A}^{-1}$) is connected to the sink-state e (e^{-1}) by some suitable word $u_1 \cdots u_\ell = u \in A^*$, we get $w \cdot u = z_k \cdots z_2 e^{f_1}$. Therefore, since $z_k \cdots z_2 e^{f_1}$ is still a relation by Theorem 2 of [5] and $|\epsilon_e(w \cdot u)| < k$, by minimality, we get that $z_k \cdots z_2 e^{f_1}$ is necessarily trivial. Hence, there is a $j \in \{1, \ldots, \ell - 1\}$ such that $w \cdot (u_1 \cdots u_j)$ is fragile. The last statement is a consequence of the definition of strongly fragile word. □

3.1. **A particular series of automata with sink-state and their fully fragile words.** In this subsection we present a series of automata with sink such that, combined with already known transducers generating free groups give rise to transducers with a sink-state defining free groups. Although, these automata are the simplest cases of transducers with sink, the characterization of their fragile words (called *strongly fragile*) is purely combinatorial, and their description appears as a non-trivial problem.

Definition 2. *Given two transducers $\mathscr{A} = (Q, A, \cdot, \circ), \mathscr{B}$ on the same set of states Q, we say that $\mathscr{B}$ dually embeds into $\mathscr{A}$, in symbols $\mathscr{B} \hookrightarrow_d \mathscr{A}$, if $\partial\mathscr{B}$ is a proper sub-automaton of $\partial\mathscr{A}$.*

Note that with the above condition if B is the alphabet of $\mathscr{B}$, then the actions of $\mathscr{B}$ are simply the actions of $\mathscr{A}$ restricted to the alphabet B. For this reason, without loss of generality, we may write $\mathscr{B} = (Q, B, \cdot, \circ)$.

Lemma 1. *Given two invertible transducers* $\mathscr{A} = (Q, A, \cdot, \circ)$*,* $\mathscr{B} = (Q, B, \cdot, \circ)$ *such that* $\mathscr{B} \hookrightarrow_d \mathscr{A}$ *there is an epimorphism* $\psi\colon \mathcal{G}(\mathscr{A}) \twoheadrightarrow \mathcal{G}(\mathscr{B})$*.*

Proof. Let $\mathcal{G}(\mathscr{A}) = F_Q/N$, $\mathcal{G}(\mathscr{B}) = F_Q/M$, and

$$\mathcal{N} \subseteq \bigcap_{a\in A} L\left((\partial\mathscr{A})^-, a\right), \quad \mathcal{M} \subseteq \bigcap_{b\in B} L\left((\partial\mathscr{B})^-, b\right)$$

be the sets as in Theorem 2 of [5]. More precisely $\mathcal{N}$ and $\mathcal{M}$ represent the set of words in $\widetilde{Q}^*$ corresponding to relations. Such words label cycles (at each state) of the enriched dual automata $(\partial\mathscr{A})^-$ and $(\partial\mathscr{B})^-$, respectively. And so they are contained int the languages recognised by the corresponding automaton.

The fact that $\partial\mathscr{B}$ is a proper sub-automaton of $\partial\mathscr{A}$ implies that $\mathcal{N} \subseteq \mathcal{M}$. Hence, $N \leq M$ from which the statement follows. □

Note that an analogous lemma for semigroup automata holds. Furthermore, the previous result is a particular case of a more general statement. Indeed, given an invertible automaton $\mathscr{B}$ and the invertible automaton $\mathscr{A}$ obtained from $\mathscr{B}$ by extending the alphabet and defining the transitions and output functions for the extra letters arbitrarily (keeping the invertibility), there is an natural epimorphism from $\mathcal{G}(\mathscr{A})$ to $\mathcal{G}(\mathscr{B})$.

The *sink transducer on an alphabet* A is $\mathscr{E} = (\{e\}, A, \cdot, \circ)$ with $e \cdot a = e$, $e \circ a = a$, for all $a \in A$. Given a transducer $\mathscr{A} = (Q, A, \cdot, \circ)$ such that $e \notin Q$, we denote by $\mathscr{A}^e = \mathscr{A} \sqcup \mathscr{E}$. Note that adding a sink-state does not change the group, i.e., $\mathcal{G}(\mathscr{A}^e) \simeq \mathcal{G}(\mathscr{A})$. We now introduce a class of auxiliary automata with some interesting features. We present them via their duals. Let Q be a finite set, then $\partial\mathscr{S}_Q = (Q, Q \cup \{e\}, \circ, \cdot)$, where the actions (see Fig. 1) are defined by:

$$q \circ x = q, \quad q \cdot x = \begin{cases} e, & \text{if } x = q \\ x, & \text{otherwise} \end{cases}$$

for all $q \in Q$. Note that with these actions $\partial\mathscr{S}_Q$ is reversible and so the action of $\mathscr{S}_Q$ is invertible

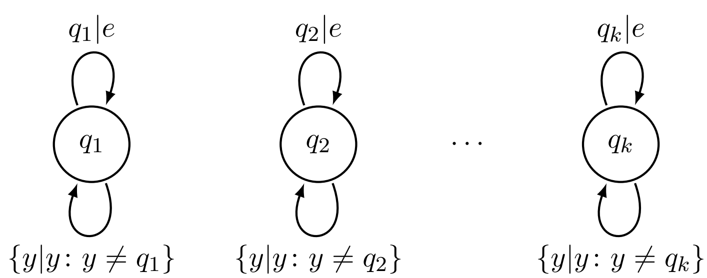

FIGURE 1. The transducer $\partial\mathscr{S}_Q$ with $Q = \{q_1, \ldots, q_k\}$.

with a sink e. We have also the following easy fact.

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Proposition 3. *Let $\mathscr{B} = (Q, B, \cdot, \circ)$ be an invertible transducer. Put $\partial\mathscr{A}_Q = \partial\mathscr{B}^e \sqcup \partial\mathscr{S}_Q$. Then, $\mathscr{A}_Q \in \mathcal{S}_a$ is an automaton with sink and $\mathcal{G}(\mathscr{B})$ is a quotient of $\mathcal{G}(\mathscr{A}_Q)$. In particular, for any invertible transducer $\mathscr{B}$ defining a free group F_m, the transducer $\mathscr{A}_Q$ defines a free group F_m.*

Proof. It follows from Lemma 1 and the definitions. □

In particular taking any transducer in [15] one is able to explicitly exhibit such machines. In general, if $\mathscr{S}_Q$ dually embeds into any other transducer, then its minimal defining relations are all fragile words. Instead, minimal defining relations of $\mathscr{S}_Q$ are fully fragile words that may be described in a purely combinatorial way. Indeed, we say that a word $w \in \widetilde{Q'}^*$, with support $Q' \subseteq Q$, is called *strongly fragile* on the set Q' if

$$\overline{\epsilon_q(w)} = 1, \ \forall q \in Q'$$

Although these words have this simple description, it seems that they are difficult to characterize. We present here a method to construct some of them using commutators. If we order arbitrarily the set $Q' = \{q_1, \ldots, q_m\}$, where $m \geq 2$, we may define the set $\mathcal{C}(Q')$ of *commutator words over Q'* given by the following properties. Let $\mathcal{C}_1$ be the set of words of type

$$[q_i^e, q_j^{e'}] := q_i^e q_j^{e'} q_i^{-e} q_j^{-e'} \ \forall i \neq j, \ e, e' \in \mathbb{Z} \setminus \{0\}$$

and, inductively, let $\mathcal{C}_i$ be the set of words $[q_i^e, v^{e'}] := q_i^e v^{e'} q_i^{-e} v^{-e'}$, $e, e' \in \mathbb{Z} \setminus \{0\}$, where q_i is not an occurrence of v and $v \in \mathcal{C}_{i-1}$. We put $\mathcal{C}(Q') := \mathcal{C}_{m-1}$. This construction generates some strongly fragile words as the following proposition shows.

Proposition 4. *The following facts hold:*

i) *The set $\mathcal{C}(Q')$ is a subset of the set of strongly fragile words over $Q' \subseteq Q$.*
ii) *If $|Q'| = m$, the shortest strongly fragile words over Q' which are in the set $\mathcal{C}(Q')$ have length $3(2^{m-1}) - 2$.*

Proof. i). Let us prove the statement by induction on $|Q'| = m \geq 2$. For $m = 2$, we have that $\mathcal{C}(Q')$ is the set of commutator words in q_1 and q_2. It is straightforward to check that, for every $w \in \mathcal{C}(Q')$ one has $\overline{\epsilon_{q_1}(w)} = \overline{\epsilon_{q_2}(w)} = 1$. Suppose now that *i)* is true for $|Q'| = m - 1$. Let w be a commutator word on m occurrences, $w \in \mathcal{C}(Q') := \mathcal{C}_{m-1}$. By definition $w = [q_i^e, v^{e'}]$, for some $v \in \mathcal{C}_{m-2}$ that does not contain q_i as occurrence. Notice that v is a commutator word on $Q' \setminus \{q_i\}$. Then $\overline{\epsilon_{q_i}(w)} = \overline{v^{e'} v^{-e'}} = 1$ and for $j \neq i$, one gets $\overline{\epsilon_{q_j}(w)} = \overline{q_i^e q_i^{-e}} = 1$, since $\overline{\epsilon_{q_j}(v^{e'})} = \overline{\epsilon_{q_j}(v^{-e'})} = 1$.
ii). Clearly the shortest words are obtained when the exponents e's belong to $\{-1, +1\}$. For $m = 2$ we get $4 = 3(2^{2-1}) - 2$. By induction we have

$$|w| = 4 + 2|v| = 2 + 2(3(2^{m-2}) - 2) = 3(2^{m-1}) - 2$$

□

In what follows, we show that the commutator words are not the only strongly fragile words over Q'. It would be interesting to find upper and lower bounds for the shortest strongly fragile words on a set Q. This may be interesting when one considers the problem of determining upper bounds that ensure the existence of relations, and the problem of finding the length of the shortest non-trivial relation.

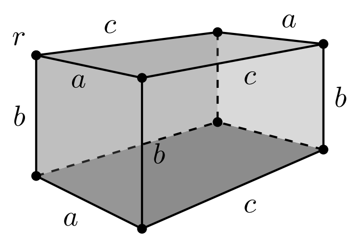

FIGURE 2. The rooted 2-simplicial complex $(\mathcal{Q}, r)$.

We now describe a geometric way to build some examples, however we do not know a characterization of all the strongly fragile words, especially the shortest ones, and this seems a non-trivial problem. For simplicity of exposition we restrict ourself to the alphabet $A = \{a, b, c\}$, but the following construction may be generalized to larger alphabets. Consider the rooted 2-simplicial complex $(\mathcal{Q}, r)$ such that the geometrical realization of the 2-simplicial complex $\mathcal{Q}$ is the unit cube in $\mathbb{R}^3$ with orthonomal bases $\{\mathbf{i}_x, x \in A\}$, and let us fix a root (base-point) r among the 0-faces of $\mathcal{Q}^0$. Let us label the 1-faces $\mathcal{Q}^1$ in such a way that the 1-faces parallel $\mathbf{i}_x$ are labelled by x, $x \in A$ (see Fig. 2). Consider now the set Γ_r of closed paths in the 1-simplicial complex $\mathcal{Q}^1$ starting from r; an element $p \in \Gamma_r$ may be represented as

$$r = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_i} v_i \xrightarrow{e_{i+1}} v_{i+1} \cdots \xrightarrow{e_{m-1}} v_{m-1} \xrightarrow{e_m} v_m = r$$

where $v_i \in \mathcal{Q}^0$, and $e_i \in \mathcal{Q}^1$ such that v_i, v_{i+1} are the 0-faces of e_{i+1}; in case the only repeated vertex is v_0 we speak of a cycle. We may define a labelling map $\mu : \Gamma_r \to \widetilde{A}^*$ defined by $\mu(p) = \mu(e_1) \cdots \mu(e_m)$ where $\mu(e_i) = x$ if $v_{i+1} = v_i + \mathbf{i}_x$, and $\mu(e_i) = x^{-1}$ if $v_i = v_{i+1} + \mathbf{i}_x$ for $x \in A$. For each face $\sigma \in \mathcal{Q}^2$ contained in an affine plane perpendicular to $\mathbf{i}_c$, and for a cycle $v_0 \xrightarrow{u} v_0$ in σ^1 we have that either u is a cyclic shift of $aba^{-1}b^{-1}$ or $bab^{-1}a^{-1}$; with respect to the orientation $\mathbf{i}_c$, this corresponds to a clockwise, or a counter-clockwise travel in σ^1 around $\mathbf{i}_c$. Hence, we may define a map ϕ from the set Λ_r of closed paths $p \in \Gamma_r$ of the form

$$p = r \xrightarrow{s} v_0 \xrightarrow{u} v_0 \xrightarrow{s^{-1}} r$$

for some some $s \in \widetilde{A}^*$ and any cycle $v_0 \xrightarrow{u} v_0$ in σ^1, for some $\sigma \in \mathcal{Q}^2$, into $\mathbb{R}^3$ sending p to either $\mathbf{i}_x$ or $-\mathbf{i}_x$ whether or not $v_0 \xrightarrow{u} v_0$ is a cycle that is traveling clockwise around the normal $\mathbf{i}_x$ to the face σ. The set Λ_r may be extended to a submonoid Λ_r^* of Γ_r by concatenation of closed paths. Consequently, the map ϕ may be naturally extended via $\phi(p_1 p_2) = \phi(p_1) + \phi(p_2)$. The following proposition shows another way to generate strongly fragile words.

Proposition 5. *Every non-reduced word $\mu(p) \in \widetilde{A}^*$ such that $p \in \Lambda_r^*$ satisfies $\phi(p) = 0$ is strongly fragile.*

Proof. We just give a sketch of the proof and we leave the details to the reader. Take any $\mu(p) \in \widetilde{A}^*$ with $p \in \Lambda_r^*$ as in the statement. Note that, for any $p_1 \in \Lambda_r$ such that $\phi(p_1) \in \{\mathbf{i}_x, -\mathbf{i}_x\}$, $x \in A$, we get $\overline{\epsilon_y(p_1)} = 1$ for all $y \in A \setminus \{x\}$. Hence, for any $x \in A$, it is not difficult to check that

$$\overline{\epsilon_x(\mu(p))} = \overline{\mu\left(\epsilon_x\left(q_1 \cdots q_k\right)\right)}$$

for some $q_1, \ldots, q_k \in \Lambda_r$ with $\phi(q_i) \in \{\mathbf{i}_x, -\mathbf{i}_x\}$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \phi(q_i) = 0$. Now, it is not difficult to check that if $q_i, q_j \in \Lambda_r$ have the property that $\phi(q_i) = -\phi(q_j)$, then $\overline{\epsilon_x\left(\mu(q_1)\right)} = \overline{\epsilon_x\left(\mu(q_2)\right)}^{-1}$. From this fact and $\sum_{i=1}^{k} \phi(q_i) = 0$ we obtain $\overline{\epsilon_x(\mu(p))} = \overline{\mu\left(\epsilon_x\left(q_1 \cdots q_k\right)\right)} = 1$, and this concludes the proof. □

For instance, we can immediately compute the following strongly fragile word:

$$\begin{aligned} w = \overline{(ab^{-1}cbc^{-1}a^{-1})(ab^{-1}a^{-1}b)(cb^{-1}aba^{-1}c^{-1})(cb^{-1}c^{-1}b)} = \\ = ab^{-1}cbc^{-1}b^{-1}a^{-1}bcb^{-1}aba^{-1}b^{-1}c^{-1}b \end{aligned}$$

We leave open the problem of characterizing strongly fragile words. Maybe, generalizing the previous geometric construction it would be possible to find a way to characterize them, and/or to give bounds on the length of the shortest one.

3.2. **Cayley type transducers.** In this subsection we introduce a series of automata in the class $\mathcal{S}_a$. We introduce them as duals of suitable "colorings" of Cayley type of transducers. The idea is to "color" a Cayley automaton in such a way that one obtains a reversible transducer which is interpreted as the dual transducer of an invertible one generating a group. This coloring approach may be found also in [4]. The coloring that we present here is in some way the "easiest", since besides the transitions entering and exiting from the state corresponding to the identity element, the others act like the identity. Using the notion of fragile words, we show that the semigroups generated by these automaton are free.

Since both the states and the alphabet are elements of a group we need to carefully fix the notation. Let G be a finite group with neutral element e. For any $g \in G$ we denote by g^{-} the unique inverse of g in G. As usual, the product of n elements $g_1, \ldots, g_n$ in G is $g_1 g_{2n}$. When we are interested in strings of elements in G, without invoking the composition law of the group G, we "parenthesize" the elements of G and we put $(G) = \{(g) : g \in G\}$, hence an element of $(G)^*$ is of the form $(g_1)(g_2)\cdots(g_n)$, where $g_i \in G$.

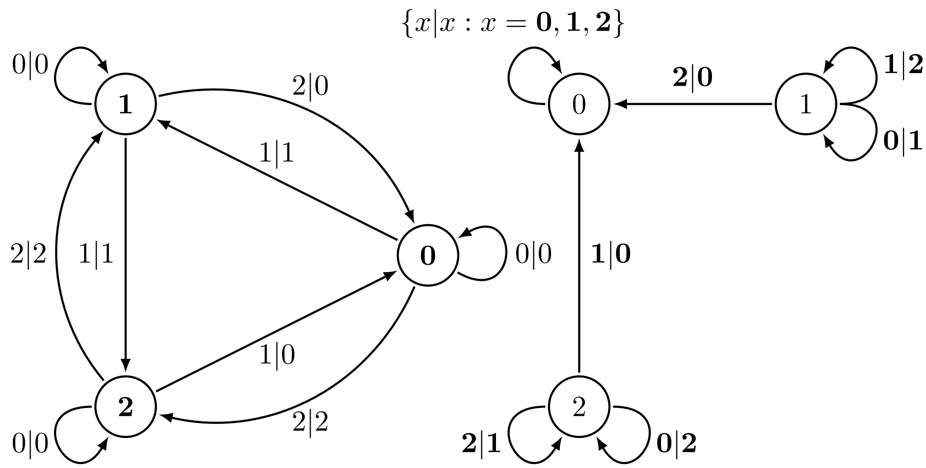

FIGURE 3. On the left the transducer $\mathscr{C}(\mathbb{Z}_3)$, on the right its dual $\partial\mathscr{C}(\mathbb{Z}_3)$.

The *0-transition Cayley machine* $\mathscr{C}(G) = (\mathbf{G}, (G), \circ, \cdot)$ is the transducer defined on the alphabet $\mathbf{G} = \{\mathbf{g} : \ g \in G\}$ whose transitions are of the form

- $\mathbf{g}\xrightarrow{(x)|(x)}\mathbf{gx}$ for all $g, x \in G$ such that $g \neq x^{-}$;
- $\mathbf{g}\xrightarrow{(x)|(e)}\mathbf{e}$ for all $g, x \in G$ such that $g = x^{-}$.

Recall that $\mathscr{C}(G)^{-}$ denotes the enriched transducer of $\mathscr{C}(G)$, that acts on the rooted tree $\widetilde{(G)}^{*}$ where the set of formal inverses of (G) is given by $(G)^{-1} = \{(g)^{-1} : \ (g) \in (G)\}$. With this notation, the inverse transitions of $\mathbf{g}\xrightarrow{(x)|(x)}\mathbf{gx}$, $\mathbf{g}\xrightarrow{(x)|(e)}\mathbf{e}$ are $\mathbf{gx}\xrightarrow{(x)^{-1}|(x)^{-1}}\mathbf{g}$, $\mathbf{e}\xrightarrow{(x)^{-1}|(e)^{-1}}\mathbf{g}$, respectively.

Similarly, we define the *bi-0-transition Cayley machine* $\widetilde{\mathscr{C}}(G) = (\mathbf{G}, (G), \circ, \cdot)$ with transitions given by:

- $\mathbf{g}\xrightarrow{(x)|(x)}\mathbf{gx}$ for all $g, x \in G$ such that $g \neq x^{-}$ and $g \neq e$;
- $\mathbf{g}\xrightarrow{(x)|(e)}\mathbf{e}$ for all $g, x \in G$ such that $g = x^{-}$ and $g \neq e$;
- $\mathbf{e}\xrightarrow{(x)|(e)}\mathbf{x}$ for all $x \in G$.

Notice that all transitions except those passing through the state corresponding to the identity do not change the input letter.
We stress once again the fact that g represents an element of the finite group G and (g) an element of the alphabet (G) of the Cayley machine.

Lemma 2. *Let* $\mathbf{h}$ *be a state of* $\mathscr{C}(G) = (\mathbf{G}, (G), \circ, \cdot)$ *or* $\widetilde{\mathscr{C}}(G) = (\mathbf{G}, (G), \circ, \cdot)$. *Then* $\mathbf{h}\circ(g_1)\cdots(g_k) = \mathbf{h}$ *if and only if* $g_1 \cdots g_k = e$.

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Proof. It is enough to observe that a closed path in $\mathscr{C}(G) = (\mathbf{G}, (G), \circ, \cdot)$ (or $\widetilde{\mathscr{C}}(G) = (\mathbf{G}, (G), \circ, \cdot)$) corresponds to a closed path in the Cayley graph of G with respect to the generating set G. This implies that any closed path is a relation of G. □

Lemma 3. *Let $u = (u_1)^{e_1}(u_2)^{e_2} \cdots (u_n)^{e_n}$, $e_i \in \{1, -1\}$, be a relation of minimal length in $\mathcal{G}(\partial\mathscr{C}(G))$ (or $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$). Then, u is fragile, i.e., there exists $\mathbf{h} \in \mathbf{G}$ such that $v := \overline{\epsilon_{(e)}(\mathbf{h} \cdot u)} = 1$.*

Proof. It follows from Proposition 2. □

Now we give the self-similar presentation of the elements of $\mathcal{G}(\partial\mathscr{C}(G))$ and $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$. Since the group G is supposed to be finite we may order its elements as $G = \{g_0 = e, g_1, \ldots, g_{|G|}\}$. The group $\mathcal{G}(\partial\mathscr{C}(G))$ acts on the set $\mathbf{G}^* \sqcup \mathbf{G}^\omega$ and is generated by (G). In a natural way, any (g_i) induces a permutation $\sigma_i \in Sym(|\mathbf{G}|)$ on the set $\mathbf{G}$ defined by $\sigma_i(\mathbf{h}) = \mathbf{h} \circ (g_i)$. Let i^- be the index such that $g_i g_{i^-} = e$. With this notation, we get the *self-similar* representation

$$(g_i) = ((g_i), \ldots, (g_i), \underbrace{(e)}_{i^-}, (g_i), \ldots, (g_i))\sigma_i.$$

For the group $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$, we have

$$(g_i) = ((e), \ldots, (g_i), \underbrace{(e)}_{i^-}, (g_i), \ldots, (g_i))\sigma_i.$$

Proposition 6. *For any non trivial finite group G, the semigroup $\mathcal{S}(\partial\mathscr{C}(G))$ is free.*

Proof. Suppose contrary to our claim that $\mathcal{S}(\partial\mathscr{C}(G))$ is not free. Hence there are $u = (u_1) \cdots (u_n)$ and $v = (v_1) \cdots (v_k)$, such that $u = v$ and $(u_i), (v_i) \in (G)$ and $n \geq 1, n \geq k$. Since the semigroup $\mathcal{S}(\partial\mathscr{C}(G))$ is cancellative, the last statement is equivalent to saying that uv^{-1} is a relation in $\mathcal{G}(\partial\mathscr{C}(G))$. We may suppose that among the relations of this form, uv^{-1} is minimal with respect to the length. Since (G) and $(G)^{-1}$ are invariant under the action of $\partial\mathscr{C}(G)$, we have from minimality that, for any $\mathbf{g} \in \mathbf{G}$ either $\mathbf{g} \cdot uv^{-1} = uv^{-1}$ or $\overline{\epsilon_{(e)}(\mathbf{g} \cdot uv^{-1})} = 1$. Otherwise $\epsilon_{(e)}(\mathbf{g} \cdot uv^{-1})$ would be a strictly shorter relation of the same form. First observe that $u_1 \cdots u_n v_k^{-1} \cdots v_1^{-1} = e$ (in G) and for every $i = 1, \ldots, n$, $j = 1, \ldots, k$ there exist elements $\mathbf{g}_i, \mathbf{h}_j$ such that $\mathbf{g}_i \circ (u_1) \cdots (u_i) = \mathbf{e}$ and $\mathbf{h}_j \circ (u_1) \cdots (u_n)(v_k)^{-1} \cdots (v_i)^{-1} = \mathbf{e}$. Let us consider the element $\mathbf{g}_{n-1}$ and apply it in order to get a new word $\overline{\epsilon_{(e)}(\mathbf{g}_{n-1} \cdot uv^{-1})}$ with $|\overline{\epsilon_{(e)}(\mathbf{g}_{n-1} \cdot uv^{-1})}| < |uv^{-1}|$, whence $\overline{\epsilon_{(e)}(\mathbf{g}_{n-1} \cdot uv^{-1})} = 1$ because of the minimality of $|uv^{-1}|$. On the other hand, by direct computation we have

$$(\mathbf{g}_{n-1} \circ (u_1) \cdots (u_{n-1})) \cdot (u_n) = (u_n)$$

and

$$(\mathbf{g}_{n-1} \circ (u_1) \cdots (u_n)) \cdot (v_k)^{-1} = (v_k)^{-1}$$

This means that there is no cancelation of occurrences $(x)(x)^{-1}$ but this is absurd. We have only to treat the case when $n = 1$ (resp. $k = 1$), in this case we may get a relation $\mathbf{g} \cdot uv^{-1}$ with no occurrences in (G) (resp. $(G)^{-1}$). By using a minimality argument similar to the one seen before one gets the assertion. □

The previous result immediately gives a statement on the growth of the generated group [8].

Corollary 1. *The group $\mathcal{G}(\partial\mathscr{C}(G))$ has exponential growth, for any non trivial group G.*

We stress once more the fact that the elements (g) and (g^-) of $\mathcal{G}(\partial\mathscr{C}(G))$ are not inverses to each other. On the other hand we have the following result.

Proposition 7. *For any non trivial group G, $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$ is not free. In particular, for all $g \in G$, one has $(g)^{-1} = (g^-)$.*

Proof. Firstly let us order the elements of G in such a way that e is the first element, g the second element and g^- the third one, under the condition that $g^2 \neq e$. By using this convention, the self-similar representation of the generators (g) and (g^-) of $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$ is

$$(g) = ((e), (g), (e), (g), \ldots, (g))\sigma$$

and

$$(g^-) = ((e), (e), (g^-), \ldots, (g^-))\sigma^{-1}$$

for some permutation $\sigma \in Sym(|\mathbf{G}|)$. By using the fact that

$$\mathbf{g}\xrightarrow{(g^-)|(e)}\mathbf{e} \quad \text{and} \quad \mathbf{g^-}\xrightarrow{(g)|(e)}\mathbf{e}$$

we get

$$(g)(g^-) = ((e), (g)(g^-), (e), (g)(g^-), \ldots, (g)(g^-))$$

and, similarly

$$(g^-)(g) = ((e), (e), (g^-)(g), \ldots, (g^-)(g))$$

It is an easy exercise to prove that $(g)(g^-)$ and $(g^-)(g)$ coincide with the trivial automorphism. If $g = g^-$ the proof works in the same way by showing that $(g)^2$ is trivial. □

We end this section with the following easy observation

Proposition 8. *The (finite) group G is a quotient of $\mathcal{G}(\partial\mathscr{C}(G))$ and $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$.*

Proof. It is enough to observe that if u is a relation in $\mathcal{G}(\partial\mathscr{C}(G))$ or $\mathcal{G}(\partial\widetilde{\mathscr{C}}(G))$, then u is a loop in the Cayley graph of G with set of generators G. This is equivalent to say that u is trivial in G (see Lemma 2). □

As an example, consider the 0-transition Cayley machine $\mathscr{C}(\mathbb{Z}_n) = (\mathbf{Z_n}, (\mathbb{Z}_n), \circ, \cdot)$, with $n \in \mathbb{N}$ (see Fig. 3). As usual, $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with operation $i + j = i + j \ mod\ n$ and for $i \neq 0$ we denote by $-i$ the unique element $n - i$ in $\mathbb{Z}_n$. In this case we export the additive notation to the automaton, in order to get the following transitions

- $\mathbf{k}\xrightarrow{(i)|(i)}\mathbf{k}+\mathbf{i}$ for all $k,i\in\mathbb{Z}_n$ such that $k\neq -i$;
- $\mathbf{k}\xrightarrow{(-k)|(0)}\mathbf{0}$ for all $k\in\mathbb{Z}_n$.

In what follows we see which conditions should be satisfied by a minimal relation in this special case. Given any word $u\in\widetilde{(\mathbb{Z}_n)}^*$, this is given by a sequence $u=(u_1)\cdots(u_m)$ where $(u_i)\in\widetilde{(\mathbb{Z}_n)}$ or, equivalently, by the word $u=(v_1)^{e_1}\cdots(v_m)^{e_m}$, where $(v_i)\in(\mathbb{Z}_n)$ and $e_i\in\{-1,+1\}$. By Lemmata 2 and 3, u is a relation of minimal length if and only if

(1) $\sum e_iv_i=0 \bmod n$
(2) for $\mathbf{h}\in\mathbf{Z_n}$ either $\mathbf{h}\cdot u=u$ or $\overline{\epsilon_{(e)}(\mathbf{h}\cdot u)}=1$.

The first condition ensures that u is a relation in $\mathbb{Z}_n$, the second one takes into account the fact that if we process u, and some letter (u_i) is changed, then it is replaced by (e) (or by $(e)^{-1}$). With any $u\in\widetilde{(\mathbb{Z}_n)}^*$ one may associate a (finite) set $S(u)$ of elements of $\mathbb{Z}_n$ defined by $S(u):=\{s_i(u):\ i=1,\ldots,|u|\}$, where $s_j(u)=\sum_{i=1}^{j}e_iv_i \bmod n$. Given u, we consider the map $\phi_u:\{1,2,\ldots,|u|\}\to\mathbb{Z}_n$, such that $\phi_u(j)=s_j(u)$. Let $s_j(u)=q\in\mathbb{Z}_n$, and let $\mathbf{p}$ be the state of $\mathscr{C}(\mathbb{Z}_n)$ corresponding to the element $-q$, then for every $k\in\phi_u^{-1}(q)$ one gets one of the following four cases

- if $e_k=e_{k+1}=1$, then
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_{k-1})^{e_{k-1}})\cdot(v_k)=(e)$$
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_k))\cdot(v_{k+1})=(v_{k+1})$$
- if $e_k=1$ and $e_{k+1}=-1$, then
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_{k-1})^{e_{k-1}})\cdot(v_k)=(e)$$
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_k))\cdot(v_{k+1})^{-1}=(e)$$
- if $e_k=-1$ and $e_{k+1}=1$, then
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_{k-1})^{e_{k-1}})\cdot(v_k)^{-1}=(v_k)^{-1}$$
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_k)^{-1})\cdot(v_{k+1})=(v_{k+1})$$
- if $e_k=e_{k+1}=-1$, then
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_{k-1})^{e_{k-1}})\cdot(v_k)^{-1}=(v_k)^{-1}$$
$$(\mathbf{p}\circ(v_1)^{e_1}\cdots(v_k)^{-1})\cdot(v_{k+1})^{-1}=(e)$$

If u is a relation of minimal length (so $\overline{\epsilon_{(e)}(\mathbf{p}\cdot u)}$ is trivial), the cancellation of occurrences $(x)(x^{-1})$ is performed around at least one index $k\in\phi_u^{-1}(q)$, for some $q\in S(u)$. The only possibility may occur either for $e_k=1, e_{k+1}=-1$ or $e_k=-1, e_{k+1}=1$. Despite the easy combinatorial properties of the relations of minimal length, it seems to be hard to get an explicit description and a simple geometrical interpretation of such relations. From the computational point of view, by using the GAP package AutomGrp developed by Y. Muntyan and D. Savchuk [11] we have not been able to find non-trivial relations for such examples of groups. This suggests that, if there are relations, they are very long with respect to the size of the generating set.

4. Open Problems

We give a list of open problems.

Problem 1. *Is there a characterization of strongly fragile words? Does the geometric argument described in Section 3.1 extend to give a full description of strongly fragile words? Give a tight upper and lower bound for the shortest strongly fragile word depending on the cardinality m of the alphabet of the letters occurring in such word.*

Problem 2. *The groups generated by the dual of the 0-transition Cayley machines have exponential growth (see Section 3.2). What can be said about the amenability of such groups? More generally, is it possible to find a suitable output-coloring of such transducers in order to get free groups or free products of groups? This question can be specialized for the example of Cayley machines, where $G = \mathbb{Z}_n$. Are the groups generated by dual of 0-transition Cayley machine $\mathscr{C}(\mathbb{Z}_n)$ free? Are the groups generated by dual of 0-transition Cayley machine $\widetilde{\mathscr{C}}(\mathbb{Z}_n)$ free products? In any case, does there exist a simple combinatorial description of the relations?*

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