Optimization Based AIMD Saturated Algorithms for Public Charging of Electric Vehicles

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Abstract

The Additive Increase Multiplicative Decrease (AIMD) algorithm is an interesting approach in congestion control of communication networks, as it maintains the good features of a distributed strategy, without sacrificing the network stability and robustness. Recent applications of these algorithms also concern other industrial fields such as Electric Vehicles (EVs) based transportation systems, for which the introduction of an optimal charging policy is an important challenge for power systems operation. Moreover, saturation constraints on the resource allocated to each vehicle need to be taken into account in order to avoid peak power requirements and grid overloads. Optimization based AIMD algorithms with saturation constraints are proposed in this paper for public charging of EVs. Specifically, a new AIMD approach is presented in order to capture the main advantages of optimal algorithms which minimize either the sum of charging times or the operation time of each vehicle, giving rise to a *mixed* AIMD strategy. Simulation results illustrate the performance of the proposal, even in comparison to the corresponding centralized optimal solutions.

Keywords: AIMD, distributed control, electric vehicles, optimal scheduling, distributed management.

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This is the final version of the accepted paper submitted to European Journal of Control.

1. Introduction

The Additive-Increase Multiplicative-Decrease (AIMD) algorithm owes its success mainly to its capability of achieving the sharing of constrained resources among competing users, without requiring direct communication among them. The AIMD algorithm has been introduced for the first time for congestion avoidance in computer network in the seminal work by *Chiu and Jain* [1]. Internet congestion control, in fact, consists in allocating a fixed bandwidth to a time varying and unknown number of users, while preventing congestion collapse [2, 3]. Furthermore, AIMD algorithms turned out to be powerful tools capable of guaranteeing stability and robustness of large scale networks characterized by distributed high delays and a time-varying number of users connected to the infrastructure [4].

In its standard formulation, the AIMD algorithm consists of three key elements: the *capacity events* (CEs), the *additive-increase* (AI) phase and the *multiplicative-decrease* (MD) phase, [5]. The mathematical model is instead captured by linear systems, the behavior of which is worth to investigate from one CE to the next. Furthermore, in the literature, several approaches have been used to study AIMD algorithms, making also reference to Markovian models [6] or passivity theory [7]. Other versions refer to the so-called *additive-increase nonlinear multiplicative-decrease* (AINMD) methods, presented for instance in [8, 9], while optimization problems solved via AIMD under limited communication constraints have been discussed in [10]. Since state constraints are present in all practical implementations, mainly in the form of saturations, in order to ensure acceptable performance of the systems in their presence, an AINMD with upper and lower saturation constraints has been introduced in [11].

As for the applicability of AIMD methods, besides the internet case, AIMD algorithms have recently received interest in other fields of implementations such as smart grids and transportation systems. In fact, these systems resemble the internet infrastructure so that AIMD algorithms perfectly fit to solve the resource allocation problems typical of these contexts. Among the applications for which this analogy is exploited, particularly interesting is the case of electrical charging of vehicles [12]. Nowadays, the progressive placing on the market of *electric vehicles* (EVs) makes the introduction of distributed energy management systems worth of analysis [13, 14, 15, 16, 17, 18]. Differently from other distributed algorithms, AIMD methods do not require the estimation of the total available (possibly time-varying) power of the grid, but, when a CE occurs, the only information needed is a flag which indicates the time instants whenever the total demand exceeds the available power, with considerable advantages in terms of communication bandwidth and flexibility.

In this paper, having in mind the EVs charging and inspired by [19] and [20], optimization based AIMD algorithms are developed. More specifically, making reference to the classical problems of *minimizing the sum of job completion times* and *minimizing the operation time* (see [21] and the references therein for more details), the corresponding AIMD counterparts are presented, also taking into account the presence of constraints for each user connected to the infrastructure. Starting from these two methods, a *mixed* AIMD algorithm with saturation constraints is also presented and its convergence properties, in case of fixed number of agents, are discussed. Note that, since a distributed optimization is required, it is reasonable that, besides the CE notification, the involved users need to share some additional information on the desired amount of resource in order to be fully served. Finally, the public charging scenario of EVs is considered as case study in order to verify the proposed distributed algorithms in comparison with the optimal centralized solutions. More precisely, the main contributions of this work are the following: i) two different optimization based AIMD strategies are proposed taking into account the presence of saturation constraints on the user shares; ii) a *mixed* AIMD algorithm is proposed to capture the main features of the optimal solutions which minimize the sum of job completion times and the operation time; iii) deterministic and probabilistic solutions to the AIMD problem are provided; iv) to the best of the authors' knowledge, this is the first time that the proposed optimization based *mixed* AIMD algorithm is applied with satisfactory results in comparison with centralized solutions to public charging scenarios of EVs. Specifically, in the literature (see [19]), it is usually assumed that all the considered EVs are connected to the network, that is the number of the users is fixed, which does not seem reasonable in a public charging scenario where both energy requirements and number of vehicles are random. In [19], in fact, a centralized solution which minimizes the sum of charging times is compared with an AIMD algorithm but without taking into account the varying number of EVs connected to the grid. In [22], where AIMD is applied also to provide reactive power management with the grid, random arrivals are instead taken into account for specific scenarios where, for instance, clusters of EVs would like to finish the charging process at the same time disregarding the time of arrival or the required energy. The novelty of this paper with respect to the literature is the introduction of an AIMD method which adapts to the number of the currently connected EVs through the reallocation of the residual energy when one user exits the network. For all these reasons, the proposed distributed strategies are finally applied to more realistic scenarios with random arrival of vehicles.

The notation adopted in the paper is mostly standard. Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers and let $\mathbb{N}_{\geq 0}$ be the set of non-negative integers. Let $\boldsymbol{x}$ be a column vector with entries x_i, and its transpose is $\boldsymbol{x}^\top$. Let $\mathbf{1}$ be the column vector containing all ones with proper dimension, and let $w(s) : \mathbb{R} \to \mathbb{R}$ be a scalar function, with w' denoting its derivative with respect to s. Given a real number $r \in \mathbb{R}$, $\lceil r \rceil$ denotes the ceiling operation to the closest integer.

The paper is organized as follows. In Section 2, some preliminary issues on AIMD algorithms and the problem formulation are reported. In Section 3 two different optimization based AIMD algorithms are discussed and then the *mixed* AIMD is introduced, the convergence analysis of which is reported in Section 4. Section 5 reports the case study relying on the public charging scenario of EVs. Some conclusions are gathered in Section 6.

2. Preliminaries and Problem Formulation

Consider N users connected to a certain infrastructure and competing for a resource at the time instant $t \in \mathbb{R}_{\geq 0}$. Let $p_i(t)$ be the share of user i at the time instant t and define the total capacity of resource available to the entire system as P such that

$$\sum_{i=1}^{N} p_i(t) \leq P, \quad \forall t \geq 0 \, . \tag{1}$$

Furthermore, it is said that a CE occurs at the time instant $t = \tau_k$, with $k \in \mathbb{N}_{\geq 0}$, if

$$\sum_{i=1}^{N} p_i(\tau_k) = P \ . \tag{2}$$

2.1. Classical AIMD Algorithms

Classical AIMD consists of two phases: i) the *additive increase* (AI) phase, during which the increase share is defined by a *growth factor* α_i, $i = 1, \dots, N$; ii) the *multiplicative decrease* (MD) phase in which each user instantaneously reduces its resource share by a multiplicative *decrease factor* $\beta_i < 1$, $i = 1, \dots, N$. When the kth CE occurs, then the MD phase is mathematically modeled as

$$p_i(\tau_k^+) = \beta_i p_i(\tau_k) \ , \tag{3}$$

with τ_k^+ being the time instant immediately after the CE. After this, the user starts again with the AI phase until the next CE. So, classical AIMD models take the form

$$p_i(t) = \beta_i p_i(\tau_k) + \alpha_i(t - \tau_k) \ , \tag{4}$$

with $\tau_k < t \leq \tau_{k+1}$. Whenever CEs occur all the users instantaneously decrease their share of resources in the MD phase, so that this is referred to as *synchronized AIMD*. Specifically, in this case the whole system dynamics can be captured by the following linear system

$$\boldsymbol{p}(\tau_{k+1}) = \boldsymbol{A}\boldsymbol{p}(\tau_k) \ , \tag{5}$$

with $\boldsymbol{p} = [p_1, \dots, p_N]^\top$ being the shares vector, while $\boldsymbol{A}$ is the so-called AIMD matrix, given by

$$\boldsymbol{A} = \mathrm{diag}(\boldsymbol{\beta}) + (\mathbf{1}^\top \boldsymbol{\alpha})^{-1} \boldsymbol{\alpha} [\mathbf{1} - \boldsymbol{\beta}]^\top \ , \tag{6}$$

with $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_N]^\top$ and $\boldsymbol{\beta} = [\beta_1, \dots, \beta_N]^\top$. The dynamics between two CEs for a single user is instead given by

$$p_i(\tau_{k+1}) = \beta_i p_i(\tau_k) + \alpha_i T_k \ , \tag{7}$$

where $T_k = \tau_{k+1} - \tau_k$ is the so-called *intercapacity time*, i.e.,

$$T_k = (\mathbf{1}^\top \boldsymbol{\alpha}) [\mathbf{1} - \boldsymbol{\beta}]^\top \boldsymbol{p}(\tau_k) \ . \tag{8}$$

Remark 1. *The case where some agent i is not notified about the CE or, if notified, it chooses not to respond, this case is referred to as* nonsynchronized AIMD*, that means that the decrease factor is a function of time $\beta_i(\tau_k) \in \{\beta_i, 1\}$ allowed to be one if no decrease is operated. If users are allowed to vary both their increase and decrease parameters at CEs, that is each user i may select constant $\alpha_i(\tau_k)$ and $\beta_i(\tau_k)$ at the kth cycle, one has instead a* nonhomogeneous AIMD *model. For further details, see [5].*

2.2. IID AIMD Algorithms

Another class of AIMD algorithms is represented by *Independent and Identically Distributed* (IID) AIMD models in which CEs have some randomness [5]. More specifically, consider the case in which the growth rate and multiplicative decrease factors are randomly selected. Then, in correspondence of the kth CE the AIMD matrix $\boldsymbol{A}$ belongs to a finite set $\mathcal{A} = \{\boldsymbol{A}_1, \dots, \boldsymbol{A}_s\}$, and is selected according to some rule that indicates randomness. Analogously, a matrix index $\sigma(\tau_k)$ is chosen from the finite set $\mathcal{S} = \{1, \dots, s\}$. This system has a form captured by (5) with instead $\boldsymbol{A}(\tau_k) = \boldsymbol{A}_{\sigma(\tau_k)}$. Let $\boldsymbol{p}(\tau_0) = \boldsymbol{p}_0$ be the initial condition, then it holds

$$\boldsymbol{p}(\tau_k) = \boldsymbol{A}(\tau_{k-1})\boldsymbol{A}(\tau_{k-2}) \cdots \boldsymbol{A}(0)\boldsymbol{p}_0 \ . \tag{9}$$

Let randomness in $\sigma(\tau_k)$ be modeled by a random variable $\Sigma(\tau_k)$ and that of $\boldsymbol{p}(\tau_k)$ is given by $\boldsymbol{\Pi}(\tau_k)$, with $\boldsymbol{\Pi}(\tau_0) = \boldsymbol{\Pi}_0$. Therefore, the evolution of the random variable can be written as

$$\boldsymbol{\Pi}(\tau_{k+1}) = \boldsymbol{A}(\tau_k)\boldsymbol{\Pi}(\tau_k) \tag{10}$$

where $\boldsymbol{A}(\tau_k)$ is such that $\boldsymbol{A}(\tau_k) = \boldsymbol{A}_{\Sigma(\tau_k)}$. Analogously to (9), it holds

$$\boldsymbol{\Pi}(\tau_k) = \boldsymbol{A}(\tau_{k-1})\boldsymbol{A}(\tau_{k-2}) \cdots \boldsymbol{A}(0)\boldsymbol{\Pi}_0 \ , \tag{11}$$

where $\boldsymbol{\Pi}(\tau_k)$ is a random variable and the sequences $\{\boldsymbol{\Pi}(\tau_k)\}$ are random sequences. Suppose that the random events are IID, that is $\Sigma(\tau_i)$ and $\Sigma(\tau_j)$ are independent for $j \neq i$ and there is a fixed probability distribution ρ_σ on $\mathcal{S}$ so that

$$\mathbb{P}(\Sigma(\tau_k) = \sigma) = \rho_\sigma \ , \tag{12}$$

with $k \in \mathbb{N}_{\geq 0}$ and $\mathbb{P}(\Sigma(\tau_k) = \sigma)$ denoting the probability of $\Sigma(\tau_k)$ being equal to σ. Hence, it holds

$$\mathbb{P}(\boldsymbol{A}(\tau_k) = \boldsymbol{A}_\sigma) = \rho_\sigma \ . \tag{13}$$

Let $\bar{\boldsymbol{p}}(\tau_k) = \mathbb{E}(\boldsymbol{\Pi}(\tau_k))$ be the expected value of a random sequence $\boldsymbol{\Pi}(\tau_k)$. It is easy to show that, since $\boldsymbol{\Pi}(\tau_k)$ and $\boldsymbol{A}(\tau_k)$ are independent, it holds

$$\mathbb{E}[\boldsymbol{A}(\tau_k)\boldsymbol{\Pi}(\tau_k)] = \mathbb{E}[\boldsymbol{A}(\tau_k)]\mathbb{E}[\boldsymbol{\Pi}(\tau)] = \bar{\boldsymbol{A}}\bar{\boldsymbol{p}}(\tau_k) \tag{14}$$

where $\bar{\boldsymbol{A}} = \mathbb{E}[\boldsymbol{A}(\tau_k)]$ is the average AIMD matrix, also given by

$$\bar{\boldsymbol{A}} = \sum_{\sigma \in \mathcal{S}} \rho_\sigma \boldsymbol{A}_\sigma \ . \tag{15}$$

Analogously, with α_i fixed and only $\beta_i^{(\sigma)}$ random, the average value of β_i is given by

$$\bar{\beta}_i = \sum_{\sigma \in \mathcal{S}} \rho_\sigma \beta_i^{(\sigma)} \ . \tag{16}$$

Therefore, the expected value of the AIMD matrix is equivalent to that in (6) with growth rate factor α_i and the multiplicative decrease factor equal to $\bar{\beta}_i$, while the average time between two CEs is obtained as follows

$$\mathbb{E}[T_k] = (\mathbf{1}^\top \boldsymbol{\alpha})[\mathbf{1} - \bar{\boldsymbol{\beta}}]^\top \boldsymbol{p}(\tau_k) \ ,$$

with $\bar{\boldsymbol{\beta}} = \left[\bar{\beta}_1, \ \ldots, \ \bar{\beta}_N\right]^\top$.

Remark 2. *Note that IID AIMD model is a special case of Markov chain model [23] such that*

$$\boldsymbol{\eta}^{[i]}(\tau_{k+1}) = \boldsymbol{A}_i \sum_{j=1}^{M} \rho_{ji} \boldsymbol{\eta}^{[j]}(\tau_k) \ , \tag{17}$$

where $\boldsymbol{\eta}^{[i]}(\tau_k) = \mathbb{E}(\boldsymbol{p}(\tau_k) J_{\sigma(\tau_{k-1})=i})$, $i = 1, \ldots, M$ *is the conditional expectation of* $\boldsymbol{p}(\tau_k)$ *with* $J_{\sigma(\tau_k)}$ *being the indicator function equal to one if* $\sigma(\tau_k) = i$*, otherwise zero, while* ρ_{ji} *are elements of the transition matrix* $\boldsymbol{P} \in \mathbb{R}^{M \times M}$*, with* M *being the number of elements in* $\mathcal{S}$*. Indeed, for an IID model* $\rho_{ji} = \rho_i$*, moreover,* $\sum_{i=1}^{M} \boldsymbol{\eta}^{[i]}(\tau_k) = \mathbb{E}(\boldsymbol{p}(\tau_k))$*, so the Markov chain model reduces to an IID one.*

3. The Proposed AIMD Algorithms

Having in mind the public charging of EVs, in this paper the considered problem to solve is guaranteeing equal opportunity to share resources among the users, minimizing the length of queues, and taking into account saturation constraints, i.e.,

$$\boldsymbol{p} \in \mathcal{P} \,, \tag{18}$$

where $\mathcal{P}$ is a compact set

$$\mathcal{P} := \{\boldsymbol{p} \in \mathbb{R}^N \,:\, 0 \leq p_i \leq \bar{p}_i,\; i = 1, \ldots, N\} \,. \tag{19}$$

In the following subsections, first, two basic formulations to achieve the minimization of either the sum of job completion times or the operation time are recalled [21].

Remark 3. *In AIMD, a continuous-time framework is typically considered [5]. It can appear that this approach is only conceptual, because a numerical implementation would need a time discretization. For the solution of the considered optimal problems a discretization has to be introduced. Let $h \in \mathbb{N}_{\geq 0}$ be the generic discrete time step, while T is the sampling time such that*

$$p_i[h] = p_i(t), \quad \forall\, t \in [hT,\, (h+1)T) \,. \tag{20}$$

3.1. Minimizing the Sum of Job Completion Times

In the following both a centralized solution and an AIMD solution of the problem at hand will be presented. Let h_i be the time that the ith user takes to complete its job, and let E_i^* be the total amount of the desired resource to be supplied to user i. The optimal control problem to solve consists in minimizing with respect to $p_i[h]$ the sum of job completion times, i.e.,

$$\min_{p_i[h]} J_1(N) \tag{21}$$

with

$$J_1(N) = \sum_{i=1}^{N} h_i \tag{22}$$

subject to

$$\sum_{h=0}^{h_i-1} p_i[h] = \frac{E_i^*}{T} \tag{23}$$

$$\sum_{i=1}^{N} p_i[h] \leq P, \quad \forall\, h \geq 0 \tag{24}$$

$$0 \leq p_i[h] \leq \bar{p}_i, \quad \forall\, h \geq 0,\ \forall\, i\ , \tag{25}$$

that is, the constraint on the desired amount of shared resource for each user (23), that on the total capacity (24), and the saturation constraints (25), respectively.

3.1.1. Centralized Solution

The optimal centralized solution consists in allocating the maximum share capacity first to those users which have lower requirements and in serving one user at time. Specifically, assume that there exist job time intervals $[h_{i-1},\, h_i - 1]$ during which each user is competing for the available commodity and the share is constant for any $h \in [h_{i-1},\, h_i - 1]$ and $h_0 = 0$. Furthermore, without any loss of generality, assume that $E_1^* \leq E_2^* \leq \cdots \leq E_N^*$. If constraint (25) is relaxed, the optimal solution is achieved when the job intervals $[h_{i-1},\, h_i - 1]$ are disjoint for any ith user, according to the following policy,

$$p_i[h] = \begin{cases} P, & h_{i-1} \leq h \leq h_i - 1 \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

such that the time to complete the job is

$$h_i = h_{i-1} + \left\lceil \frac{E_i^*}{T} \frac{1}{P} \right\rceil , \tag{27}$$

while the optimal cost function is

$$J_1^{\text{o}} = \frac{1}{P} \sum_{i=1}^{N} \frac{E_i^*}{T} (N - i + 1)\ . \tag{28}$$

In this case the total available power is provided to one vehicle at a time and, as discussed before, constraints are not considered. In fact, the only purpose of analyzing this method is to introduce a benchmark as a term of comparison with other solutions.

Since in practical cases, due to available power in charging stations, one has to take into account the presence of constraints (25), assume now to provide first the maximum share $\bar{p}_i$ to the user with lower requirement. In this case, when the first user is fully served, the centralized solution to the problem at hand is achieved by allocating the resources from the residual capacity to the next user in ascending order of share requirements. Specifically, the latter computes the difference between the available capacity and the share already allocated to the previous user, and, if the residual capacity is greater than $\bar{p}_i$, the algorithm allocates $\bar{p}_i$, otherwise it allocates the residual capacity.

Remark 4. *Note that, although the solution of the considered problem is computed taking into account a constant value N, when the number of users connected to the network changes, N is updated and the optimization problem is solved again.*

3.1.2. AIMD Solution

It is now possible to introduce the AIMD solution to the problem previously discussed. So, consider that the growth factors α_i are fixed and at the kth CE a desired share p_i^* is computed as

$$p_i^*(\tau_k) = \min\left\{ p_i(\tau_k) + \sum_{j=1,\, j\neq i}^{N} (E_j^* - E_i^*),\ \bar{p}_i \right\} . \tag{29}$$

For the sake of simplicity define the value

$$c_{1_i} := \sum_{j=1,\, j\neq i}^{N} (E_j^* - E_i^*) . \tag{30}$$

If in (29) the difference c_{1_i} is positive, the resource demand of user i is relatively low, hence, it should be assigned higher share. Vice versa, if c_{1_i} is negative, the resource demand of ith user is relatively high, hence, it should be assigned lower share. This strategy requires for each user the knowledge of its own amount of desired resource E_i^* and the sum of other users' demands.

Assume that each user can either reduce its share by a large or a small amount, according to two possible decrease factors, nominally $\beta^{(1)}$ and $\beta^{(2)}$, with $\beta^{(1)} < \beta^{(2)}$. Therefore, at the CE, these multiplicative factors can be assigned in a deterministic way as follows

$$\beta_i(\tau_k) = \begin{cases} \beta^{(1)}, & p_i^* < p_i \\ \beta^{(2)}, & p_i^* \geq p_i \ . \end{cases} \tag{31}$$

Remark 5. *Note that the selection of $\beta_i(\tau_k)$ could be also defined by adapting the probabilities assigned to the choice of $\beta^{(1)}$ and $\beta^{(2)}$, for instance, as discussed in [19].*

3.2. Minimizing the Operation Time

Assume now to allocate a fair share of resources to the users so as to minimize the time at which they have completed their jobs, that is to minimize the so-called operation time. Let $p_i[h] = p_i$, $\forall\, h$ and let $f(E_i^*, p_i)$ be the predefined utility function representing the operation time, i.e.,

$$f_i(E_i^*, p_i) = \frac{E_i^*}{p_i} \ . \tag{32}$$

Hence, the optimal problem to solve consists in minimizing with respect to p_i the following cost function

$$J_2(E_i^*, p_i) = \max_i f_i(E_i^*, p_i) \tag{33}$$

subject to (24) and (25).

3.2.1. Centralized Solution

The optimal centralized solution to minimize the operation time is to allocate the resources in the same amount of time for each user, i.e.,

$$\frac{E_1^*}{p_1} = \frac{E_2^*}{p_2} = \cdots = \frac{E_N^*}{p_N} \ . \tag{34}$$

Since it holds that

$$\begin{aligned} P =& p_1 + \cdots + p_N \\ =& \left(\frac{E_1^*}{E_i^*}\right) p_i + \cdots + \left(\frac{E_N^*}{E_i^*}\right) p_i \ , \end{aligned} \tag{35}$$

then all the resources can be allocated according to the following policy

$$p_i = \min\left\{\frac{E_i^* P}{\sum_{j=1}^{N} E_j^*},\ \bar{p}_i\right\}\ . \tag{36}$$

Assuming that the allocated resources are less than $\bar{p}_i$, the optimal operation time is equal to

$$f^{\text{o}} = \frac{1}{P}\sum_{j=1}^{N} E_j^*\ . \tag{37}$$

3.2.2. AIMD Solution

Starting from the previous algorithm, in the corresponding AIMD distributed solution the cost function (33) is instead minimized at the CEs with respect to $p_i(\tau_k)$. In this case, it is assumed that each user computes its utility function and knows the sum of those of other users. The desired share is then calculated as follows

$$\begin{aligned} p_i^*(\tau_k) = \min\{&p_i(\tau_k)+ \\ -\eta_1 &\textstyle\sum_{j=1,\,j\neq i}^{N}(f_j(E_j^*, p_j(\tau_k)) - f_i(E_i^*, p_i(\tau_k))),\ \bar{p}_i\Big\}\ , \end{aligned} \tag{38}$$

where each user has the knowledge of its own utility function f_i and the sum of the other ones $f_j,\ j \neq i$ and $\eta_1 > 0$ is a scalar design parameter. For the sake of simplicity define the value

$$c_{2_i} := \sum_{j=1,\,j\neq i}^{N} (f_j(E_j^*, p_j(\tau_k)) - f_i(E_i^*, p_i(\tau_k)))\ . \tag{39}$$

If c_{2_i} is positive, it means that the share has to be decreased in order to equalize the utility functions, otherwise, if c_{2_i} is negative, the share has to be increased. The decrease factors $\beta_i(\tau_k)$ can be adjusted again as in (31) or as outlined in Remark 5.

3.3. Mixed AIMD Algorithm

In the previous subsections two AIMD algorithms for the minimization of job completion times and minimization of the operation time have been discussed. The algorithm in Section 3.1 requires that users with higher energy demands have to wait till the others with lower demand are served. However, having in mind the public charging of EVs, this method could appear less fair as well as the operation time could result considerably higher than the minimum achieved by the centralized algorithm presented in Section 3.2. On the other hand, the algorithm in Section 3.2 takes the same time to serve the users, so as to imply that the sum of completion time increases, thus leading to bigger queues.

Therefore, the aim is now to design an AIMD algorithm in which a fair share of the available commodity is allocated to each connected user capturing the main advantages of both the previously described approaches. Assume that all the users know their required share E_i^*, while the new utility function is given by

$$f_i(E_i^*, \gamma_i, P) = \frac{E_i^*}{\gamma_i P} \ , \tag{40}$$

where $0 < \gamma_i \leq 1$ is a factor such that $p_i(\tau_k) = \gamma_i P$. The optimal control problem consists in minimizing with respect to $p_i(\tau_k)$ the following cost function

$$J_3(E_i^*, \gamma_i, P, N) = \sum_{i=1}^{N} \frac{E_i^*}{\gamma_i P} \ , \tag{41}$$

subject to

$$\sum_{i=1}^{N} \gamma_i = 1 \ , \tag{42}$$

and (24), (25). Notice that the cost function (41) corresponds to the sum of the job completion times as in Section 3.1 but with the additional constraints that the shares γ_i are constant over time. Two different solutions are now discussed.

3.3.1. Non Adaptive Solution

Consider now the non adaptive centralized case, as presented in [19]. If the number of users is constant, one could compute the optimal value for γ_i of the problem at hand as

$$\gamma_i = \frac{\sqrt{E_i^*}}{\sum_{j=1}^{N} \sqrt{E_j^*}} \ . \tag{43}$$

However, this value of γ_i does not take into account that N changes, for instance in case of possible disconnections of the users when they are completely served.

3.3.2. The Adaptive Solution

An alternative solution to (43) is an adaptive algorithm. It consists in recomputing the total capacity of the network whenever the number of users connected to the infrastructure changes, and redistributing the portion of share corresponding to the disconnected agents. Let $\bar{h}$ be the time instant when the number of users changes from N to $\bar{N}$. In the centralized solution the updated share requirement is computed as

$$\gamma_i[\bar{h}] = \frac{\sqrt{E_i^* - E_i[\bar{h}-1]}}{\sum_{j=1}^{\bar{N}} \sqrt{E_j^* - E_j[\bar{h}-1]}} \ , \tag{44}$$

where $E_i[h]$ is the effective energy supplied to user i up to h. As anticipated, it means that the values of γ_i are not updated constantly but only when the number of users connected to the infrastructure changes.

Now, the so-called *mixed* AIMD algorithm is introduced. Given the values of the current γ_i updated as in (44) at the $(k+1)$th CE with respect to $E_i(\tau_k)$, in order to minimize the cost function (41), the slope of the utility functions (40) has to be equalized. Assume that each user is capable to compute the slope of its own utility function, and has information about the sum of the utility function slopes of other users. The desired shares are then computed as follows

$$\begin{aligned} &p_i^*(\tau_k) = \min \{ p_i(\tau_k) + \\ &+\eta_2 \textstyle\sum_{j=1,\, j\neq i}^{N} (f_j^{'}(E_j^*, p_j(\tau_k)) - f_i^{'}(E_i^*, p_i(\tau_k))), \bar{p}_i \Big\} \ , \end{aligned} \tag{45}$$

where $\eta_2 > 0$ is a scalar adaptation parameter [24] and

$$f_i^{'}(E_i^*, p_i(\tau_k)) = \frac{\partial}{\partial p_i} \frac{E_i^*}{p_i(\tau_k)} = -\frac{E_i^*}{p_i^2(\tau_k)} \ . \tag{46}$$

The desired share can be explicitly written for the sake of simplicity as

$$p_i^*(\tau_k) = \min \{ p_i(\tau_k) + \eta_2 c_{3_i}, \bar{p}_i \} \ , \tag{47}$$

where

$$c_{3_i} := (N-1)\frac{E_i^*}{p_i^2(\tau_k)} - \sum_{j=1, j\neq i}^{N} \frac{E_j^*}{p_j^2(\tau_k)} \ . \tag{48}$$

If c_{3_i} is positive, this means that the share has to be increased in order to equalize the utility function and the desired share p_i^* is greater than the actual one p_i. Vice versa, when c_{3_i} is negative, the share has to be decreased. Again, the decrease factors β_i can be adjusted as in a deterministic way (31) or defining the probabilities to select $\beta^{(1)}$, $\beta^{(2)}$ with $\beta^{(1)} < \beta^{(2)}$.

Specifically, we consider the probability ρ_i of the user i of selecting $\beta^{(1)}$. In accordance to this strategy, inspired by [19], the desired share p_i^* is as in (45), and the probabilities are adapted as follows

$$\rho_i(\tau_k) = \rho_i(\tau_{k-1}) - \eta_3(p_i^*(\tau_k) - p_i(\tau_k)) \tag{49}$$

where $\eta_3 > 0$ is another adaptation factor.

4. Convergence of the Mixed AIMD Algorithm

In this section the theoretical analysis related to the convergence property of the *mixed* AIMD algorithm is reported. In the following it is assumed that the number N of agents is kept constant, that is, in the considered case study, the vehicles are charged indefinitely. This assumption is supported by the fact that examining the convergence of the proposed AIMD algorithm with varying number of agents has in practice little significance. Indeed, as also claimed in [10], when users join or leave the network at any time, the network automatically readjusts the share allocation strategy according to the new set of demands E_i^*.

4.1. Convergence Analysis

We begin by recalling the stochastic nature of the *mixed* AIMD introduced in Section 3.3. This is in fact a stochastic process with state-dependent transition probabilities for which existence conditions of an attractive steady-state distribution have been investigated by *Corless et al.* in [5] starting from the results by *Barnsley et al.* in [25].

Specifically, consider the state-dependent AIMD model as in (10), where $\mathbf{\Pi}(\tau_k)$ is the random vector of shares at the capacity events, with values in the simplex S_N, and $\boldsymbol{A}(\tau_k) = \boldsymbol{A}_{\Sigma(\tau_k)}$ is a random matrix belonging to the finite set $\mathcal{A} = \{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_s\}$ according to the random extraction of the sample $\sigma(\tau_k) \in \mathcal{S}$, with $\mathcal{S} = \{1, \ldots, s\}$. The probability distribution of $\Sigma(\tau_k)$ depends on the share vector $\boldsymbol{p}(\tau_k)$ and the conditional probability $q_\sigma(\boldsymbol{p})$ that the matrix $\boldsymbol{A}_\sigma$ is selected is given by

$$\mathbb{P}\left(\boldsymbol{A}(\tau_k) = \boldsymbol{A}_\sigma \,|\, \mathbf{\Pi}(\tau_k) = \boldsymbol{p}\right) = q_\sigma(\boldsymbol{p}) \ . \tag{50}$$

One can observe that the *mixed* AIMD model (45)-(49) perfectly fits the model (10)-(50), with $\sigma(\tau_k)$ related to all possible combinations of $\beta^{(1)}$ and $\beta^{(2)}$ in the vector of decrease factors $\boldsymbol{\beta}$. Moreover, the probability of extracting $\beta^{(1)}$ for each agent is a function of the current share $\boldsymbol{p}$.

Making reference to [5, Theorem 7.4], the *mixed* AIMD model has a unique steady-state probability distribution, and this distribution is attractive if each agent has a non null probability of reducing its share at the capacity events and the conditional probability $q_\sigma(\boldsymbol{p})$ is strictly positive and Lipschitz continuous. Since in the considered AIMD algorithm the parameters β_i are less than 1 for all $i = 1, \ldots, N$, for each agent i the first condition is satisfied. What is left to show is that $q_\sigma(\boldsymbol{p})$ is a Lipschitz continuous function of $\boldsymbol{p}$. To this purpose, notice that, neglecting saturation for the sake of simplicity, each agent i uses $\beta_i = \beta^{(1)}$ with probability $\rho_i(\tau_k)$ computed from (45)-(48). This probability is easily shown to be Lipschitz continuous with respect to $\boldsymbol{p}(\tau_k)$, provided that $p_i(\tau_k) > \varepsilon > 0$, $\forall\, i,\, k$. On the other hand this last condition is always satisfied whenever $p_i(0) > \varepsilon$, thanks to irreducibility of all the matrices in $\mathcal{A}$. Then, a first conclusion is that $\rho_i(\boldsymbol{p})$ is Lipschitz continuous. Finally, let $\mathcal{I}$ be the set of indices of agents which select $\beta^{(1)}$, then, in view of the independent extraction of the factors β_i, the conditional probability of matrix $\boldsymbol{A}_\sigma$ given $\boldsymbol{p}$ is

$$q_\sigma(\boldsymbol{p}) = \prod_{i \in \mathcal{I}} \rho_i(\boldsymbol{p}) \prod_{i \notin \mathcal{I}} (1 - \rho_i(\boldsymbol{p})) \tag{51}$$

which is Lipschitz continuous with respect to $\boldsymbol{p}$ in view of Lipschitz continuity and boundedness of $\rho_i(\boldsymbol{p})$, $\forall\, i$. Hence, one can conclude that the random vector $\boldsymbol{\Pi}(\tau_k)$ tends, for $k \to \infty$, to a unique steady-state probability distribution $\overline{\boldsymbol{\Pi}}$.

One might conjecture that the expected value of the asymptotic distribution $\overline{\boldsymbol{\Pi}}$ coincides with the vector of optimal shares $p_i^{\mathrm{o}} = \gamma_i P$ achieved via the corresponding centralized algorithm. This theoretical issue is left open to further investigation. However, the following simple example shows that from a practical point of view the *mixed* AIMD algorithm can be seen as an approximation of the centralized optimal strategy.

4.2. Illustrative Example

In this subsection an illustrative example of the *mixed* AIMD with three agents

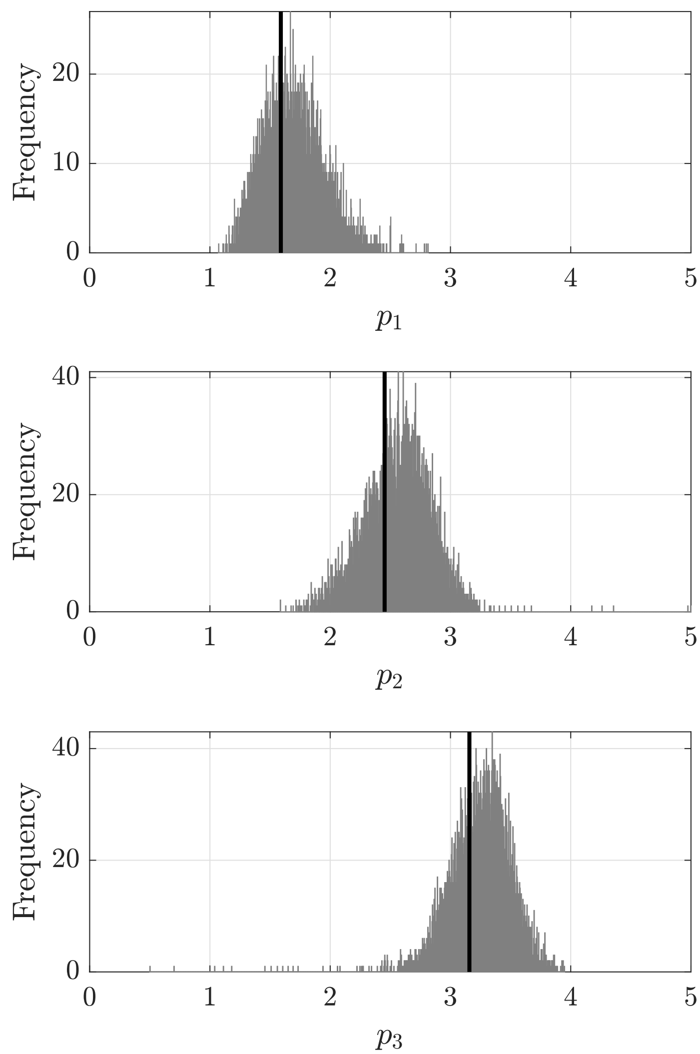

Figure 1: Estimate of the steady-state distribution of Π_1, Π_2 and Π_3

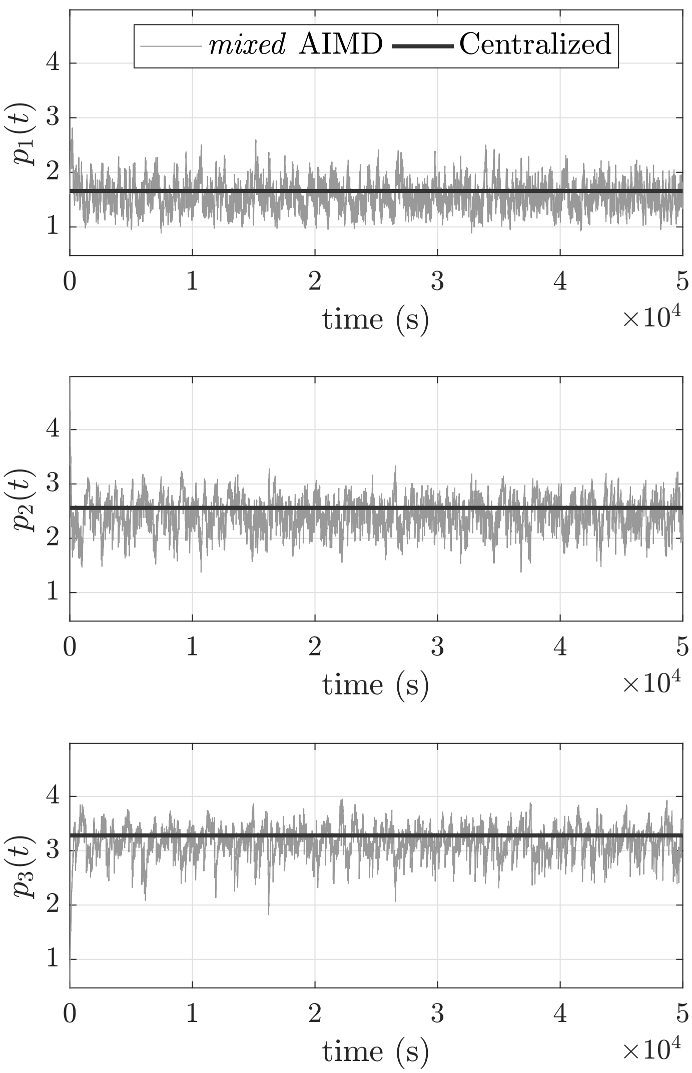

Figure 2: Time evolution of the shares $p_1(t)$, $p_2(t)$ and $p_3(t)$

Table 1: Comparison between the *mixed* AIMD and the corresponding centralized solution

	$i = 1$	$i = 2$	$i = 3$
p_i^{o}	1.658	2.559	3.282
$\widetilde{p}_i$	1.684	2.566	3.253

($N = 3$), having random amount of demands $\boldsymbol{E}^* = [2.19,\, 5.22,\, 8.58]'$, is reported in order to clarify the previous convergence arguments. The initial conditions are $\boldsymbol{p}(0) = [2,\, 5,\, 0.5]'$, $\beta^{(1)} = 0.8$ and $\beta^{(2)} = 0.95$, $\alpha_i = 0.02$ and $\rho_i(0) = 0.06$, for $i = 1,\, 2,\, 3$, while the capacity is $P = 7.5$. In order to examine the process, 50000 iterations of the system are performed and ergodicity is exploited. Histograms of the values p_1, p_2 and p_3 at the capacity events are then constructed. These histograms represent an estimate of the steady-state distribution of the random variables Π_1, Π_2 and Π_3, respectively. The resulting histograms are displayed in Figure 1 together with the sample averages $\widetilde{p}_i$. Figure 2 shows instead the time evolution of the shares $p_1(t)$, $p_2(t)$ and $p_3(t)$ and the values of the optimal shares $p_i^{\text{o}} = \gamma_i P$ achieved via the centralized solution. The sample averages of the shares obtained via the proposed AIMD, and those of the optimal centralized solution are reported in Table 1. As expected, for all the agents the values of $\widetilde{p}_i$ result to be very close to the corresponding values of p_i^{o}.

5. Case Study: Public Charging of Electric Vehicles

In this section simulation results carried out relying on four EVs in a public charging scenario are reported in order to validate the proposed optimization based AIMD approaches. Figure 3 illustrates a schematic representation of the considered AIMD infrastructure. However, note that the proposed scheme has general validity, even for a greater number of EVs. The considered charging rate model is given by equations (4) where the growth factors α_i, $i = 1, \ldots, 4$ are the same for all the EVs. As for the decrease factors β_i, each vehicle can suitably select them between two values $\beta^{(1)}$ and $\beta^{(2)}$, as listed in Table 2, where the probability ρ_i associated to $\beta^{(1)}$ is also reported. Furthermore, assume that each vehicle has a random energy need such that

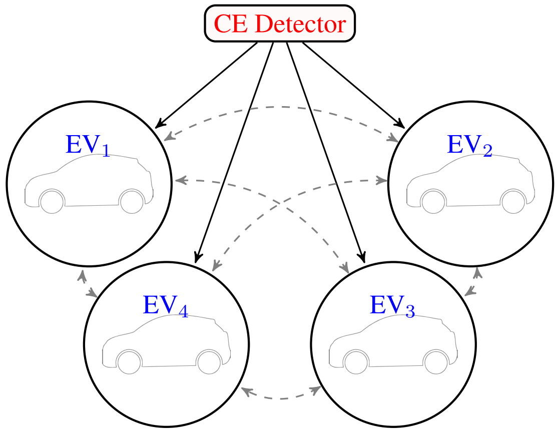

Figure 3: Scheme of the considered AIMD based grid with four EVs

Table 2: AIMD parameters of the EVs

Parameters	Value
α_i	$0.02\,\mathrm{kW\,h^{-1}}$
$\beta^{(1)}$	0.7
$\beta^{(2)}$	0.98
ρ_i	0.06

$\boldsymbol{E}^* = [9.09\,, 11.17\,, 16.82\,, 24.79]^\top\mathrm{kW\,h}$ and a rate limitation $\bar{p}_i = 4\,\mathrm{kW}$ (chosen as in [19]), so that the decrease factors β_i are adjusted online, depending on the requirements. The total available power is $P = 10\,\mathrm{kW}$, while the sampling time has been set equal to $1\,\mathrm{s}$.

5.1. Fixed Arrival Scenario

In the following, first, the results obtained via the AIMD solutions discussed in Sections 3.1 and 3.2 are illustrated.

Figure 4 shows the evolution of charge rates in the centralized solution when the sum of charging times is minimized without taking into account the presence of constraints on each vehicle. The algorithm, according to the policy (26), assigns a charge

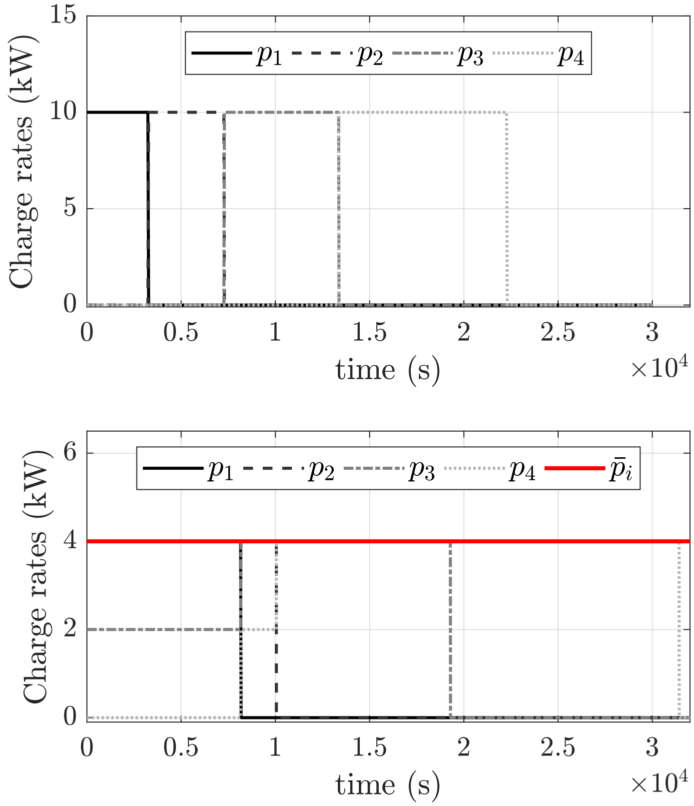

Figure 4: Charge rates scheduled by a centralized optimal algorithm without saturation constraints and by the centralized counterpart with constraints when the sum of charging times is minimized

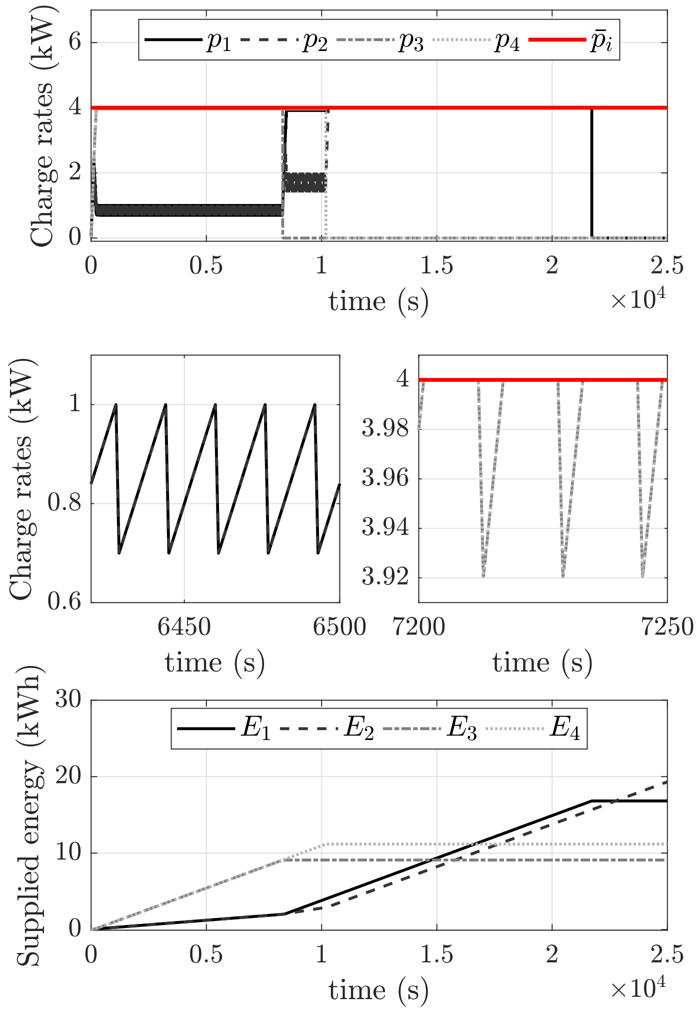

Figure 5: Charge rates and supplied energy scheduled by the AIMD algorithm with saturation constraints when the sum of charging times is minimized

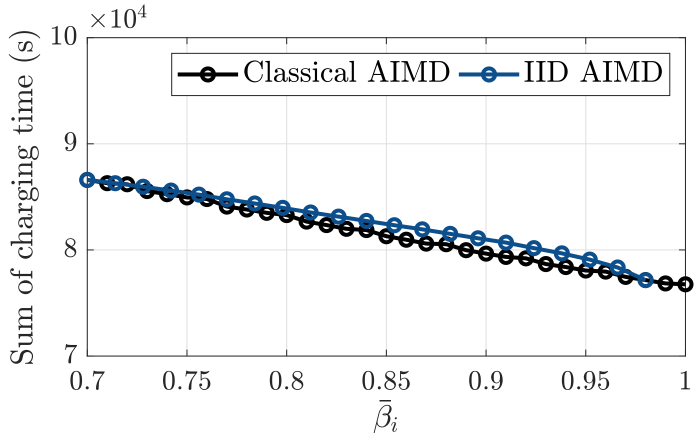

Figure 6: Sum of charging times when a classical AIMD solution is used with $\beta_i \in \{0.7, 1\}$ and the solution achieved by IID AIMD with $\bar{\beta}_i$ computed as in (16)

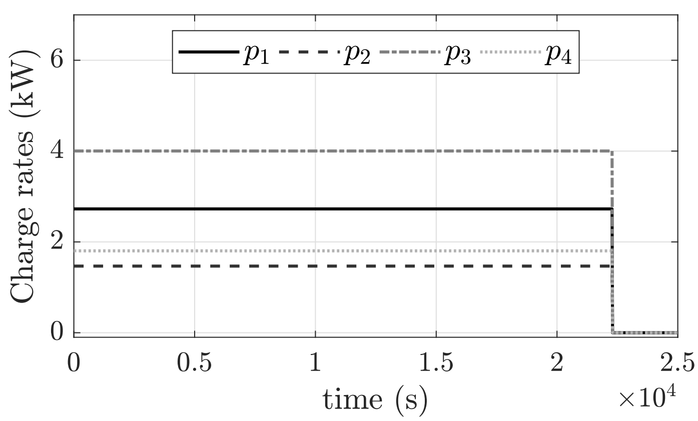

Figure 7: Charge rates scheduled by the centralized algorithm with saturation constraints when the operation time is minimized and the allocated power is less than the maximum charging rate $\bar{p}_i$

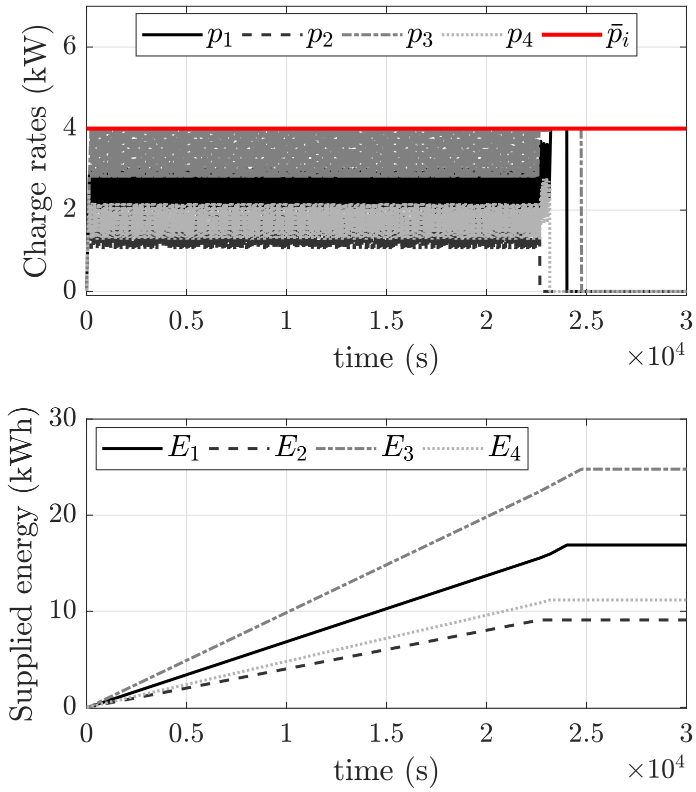

Figure 8: Charge rates and supplied energy scheduled by the AIMD algorithm with saturation constraints when the operation time is minimized

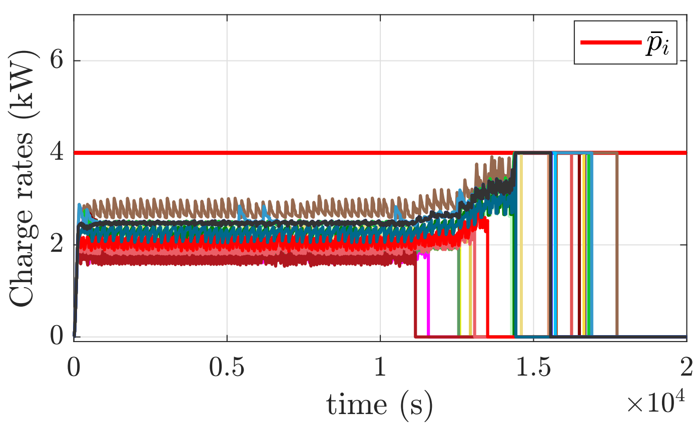

Figure 9: Time evolution of charge rates scheduled by the *mixed* AIMD algorithm with saturation constraints when the selection of the decrease factors β_i is deterministic

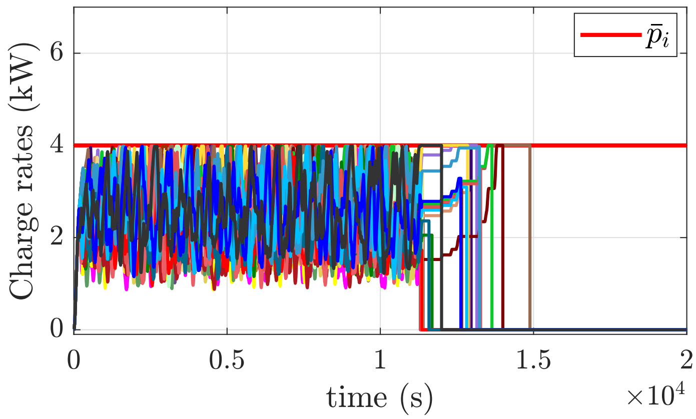

Figure 10: Time evolution of charge rates scheduled by the *mixed* AIMD algorithm with saturation constraints when the selection of the decrease factors β_i is probabilistic

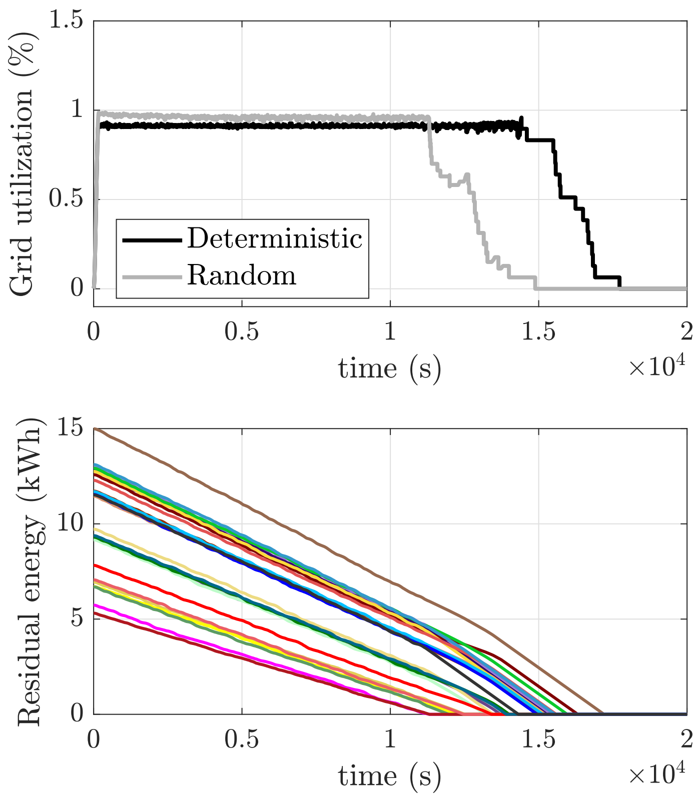

Figure 11: Ratio between the available power and allocated power in case of the *mixed* AIMD algorithm when $\beta^{(1)}$ and $\beta^{(2)}$ are chosen as in (31) or (49), and evolution of residual energy requested by each vehicle

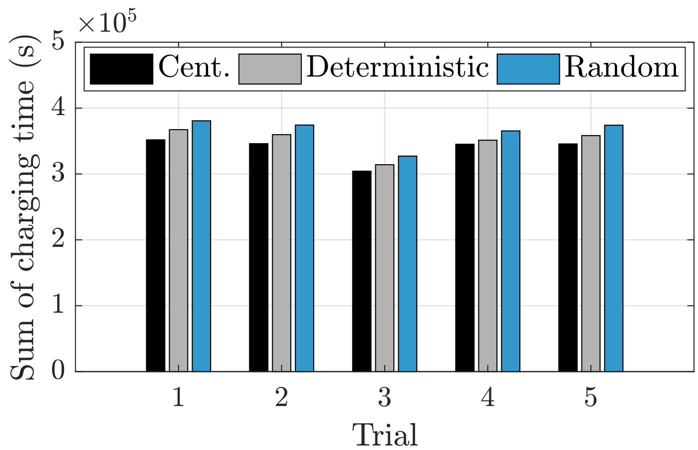

Figure 12: Comparision between the sum of charging times required by centralized and *mixed* AIMD solutions in (31) or (49)

rate equal to P to EV_1, which has the lowest energy demand, and then continues charging in ascending order of charge requirements. The charge rates allocated by the centralized algorithm with saturation constraints are also illustrated. As it can be observed, the difference of this algorithm with respect to the optimal one is that multiple vehicle are charged at a time, and always in ascending order of energy requests.

When the corresponding AIMD solution with saturation constraints in (29) is applied, the results are those illustrated in Figure 5. The charge rates are assigned according to the AIMD deterministic rule (31), and the constraints are always fulfilled as observed by zooming in the share signals between the time instants $7200\,\mathrm{s}$ and $7250\,\mathrm{s}$. The energy supplied to the vehicles is also illustrated. In this case the values of β_i remain constant until a vehicle is disconnected from the grid and thanks to this strategy more than one vehicle can be served contemporaneously. Figure 6 shows instead that if the probabilistic AIMD is used, the average value of the sum of charging times corresponds to the value achieved when an average $\bar{\beta}_i$ computed as in (16) is applied.

Consider now the case when the operation time is minimized. Figure 7 depicts the charge rates allocated by the centralized solution such that all the vehicles finish charging at the same time, according to (36). In this case, it is assumed that the allocated power is less than the maximum charging rate $\bar{p}_i$ for each vehicle. Note that

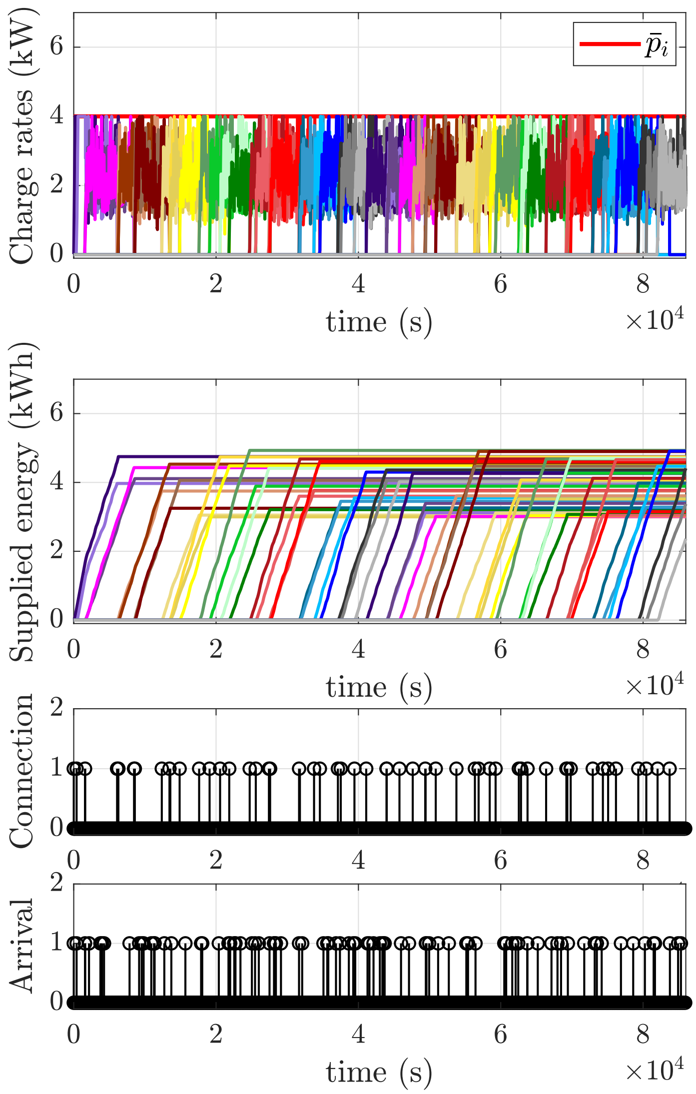

Figure 13: Time evolution of the charge rates and supplied energy in a random arrival scenario when the *mixed* AIMD algorithm is used

in general, if constraints on each user are applied, in the centralized case the vehicles are fully charged at the same time. Figure 8 shows instead the time evolution of the charge rates and the supplied energy when the corresponding AIMD algorithm is used. Note that, in this case, the vehicles do not exactly finish charging at the same time, thus implying an increase in the operation times with respect to the centralized solution. However, the AIMD solution results again more advantageous from the communication viewpoint. In fact, only the CE notification and the utility functions f_i of all the vehicles are required, while the total available power, charge rates and number of the connected vehicles are needed in the centralized case. In order to make the scenario more realistic, the same simulations have been performed in case of 25 vehicles. The basic AIMD algorithm with constant decrease factors $\beta_i = 0.8$ and the AIMD methods tuned via both a deterministic and a probabilistic procedure have been also considered and compared, in terms of charging time, with the optimal centralized solution (36). It has been observed that the probabilistic AIMD algorithm requires a 6% percentage greater than the centralized method, while the deterministic AIMD algorithm and the classical AIMD result 11% and 20% worse, respectively.

Now the results obtained by using the proposed *mixed* AIMD algorithm are illustrated. In this case 25 vehicles have been considered. Figure 9 and Figure 10 show the charge rates when $\beta^{(1)}$ and $\beta^{(2)}$ are selected according to (31) or (49), respectively. A moving average filter for signal processing with window length of 50 samples has been used for the sake of clarity. Figure 11 depicts instead the ratio between the available power and the allocated power from the EVs in the corresponding cases illustrated in Figure 9 and Figure 10. Note that, when all the vehicles are connected, the utilization is high in both cases but slightly better in the case of AIMD with probabilistic tuning of β_i. The same figure shows the residual energy requested by the EVs. Computing in this case the sum of charging times, the obtained results are comparable to those achieved via the centralized solution, as shown in Figure 12. Five trials for centralized, deterministic AIMD and probabilistic AIMD, with random energy need, have been performed. In both AIMD cases a 10% increase is observed with respect to the centralized case.

Table 3: Comparison among the AIMD algorithms in terms of served EVs in a random arrival scenario

Algorithm	#EVs	%EVs	#CEsh	AoCT (h)	AoWT (h)
AIMD min sum of charging times	53.07	73.34	197.002	4.02	5.71
Mixed AIMD	50.57	70.18	137.57	4.42	6.37
AIMD min operation time	50.68	70.30	137.89	4.37	6.34
Classical AIMD ($\beta_i = 0.7$)	48.15	66.85	85.63	4.76	7.03
Classical AIMD ($\beta_i = 0.98$)	55.43	76.95	912.69	3.56	4.9

5.2. *Random Arrival Scenario*

Finally, in a public charging scenario one needs to account the possibility of random arrival of vehicles. Simulation tests have been carried out assuming that each vehicle has random energy demand and the total available power is equal to $P = 2.5N_{\mathrm{s}}$ kW, with $N_{\mathrm{s}} = 4$ being the total number of available charging spots. The random arrival has been modeled relying on the average arrival rate of the vehicles, selected equal to 3 veh h^{-1}, with queuing, that is, if a vehicle arrives and there is not an available charging spot, it waits for it. Figure 13 shows the charge rates and the energy supplied by the *mixed* AIMD algorithm in random arrival scenario when (31) is used. Flag signals equal to 1 indicate the connection or the arrival time instants.

5.3. *Comparison and Discussion*

The daily performace obtained by applying the proposed optimization based AIMD algorithms and classical AIMD algorithm with fixed $\beta_i \in \{0.7, 0.98\}$ are hereafter compared in terms of the following indices: number of served EVs (#EVs), percentage of served EVs with respect to the total arrived vehicles (%EVs), number of CEs per hour (#CEsh), average of charging times (AoCT) expressed in hours and including the waiting times when no spots are available, maximum waiting time (WT). Table 3 summarizes all the average results achieved by performing 1000 simulation tests with the last column representing the average of the maximum waiting times (AoWT) expressed in hours. As expected, the AIMD solution which minimizes the sum of charging times outperforms the other ones in terms of number of served EVs, AoCT and AoWT. When classical AIMD with $\beta_i = 0.98$ is used, if on one hand the number of served vehicles is improved with respect to the *mixed* AIMD and the minumum operation time based AIMD, on the other hand the number of CEs is increased, which means more frequent notifications over the network. Finally, the AIMD strategy which minimizes the sum of charging times smooths the drawbacks of the corresponding centralized solution at the only prize of a number of CEs slightly higher with respect to the other two optimization based approaches.

6. Conclusions

In this paper a new policy for charging of electrical vehicles has been designed relying on the so-called AIMD resource allocation algorithms, widely used in internet networks for congestion control. Due to their successes in this massively large scale decentralized setting, AIMD has been preferred to other distributed solutions. After having introduced two different optimal solutions to minimize the sum of job completion times and the operation time, a new *mixed* AIMD solution with saturation constraints has been proposed and its convergence properties have been discussed. The proposed approaches have been developed having in mind a public charging scenario for EVs, where a fair energy share and the minimization of queues are required. Several simulation tests have been performed even in comparison with optimal centralized solutions and taking into account also a random arrival of vehicles. It has been observed that AIMD solutions always guarantee performance comparable to those of the centralized counterparts, with considerable advantages in terms of flexibility, scalability and communication bandwidth.

References

[1] D.-M. Chiu, R. Jain, Analysis of increase/decrease algorithms for congestion avoidance in computer networks," digital equipment corporation, Tech. rep., Technical Report TR-509, August 1987, Computer Networks and ISDN Systems.

[2] S. Floyd, K. Fall, Promoting the use of end-to-end congestion control in the internet, IEEE/ACM Transactions on Networking (ToN) 7 (4) (1999) 458–472.

[3] S. Keshav, C. Rosenberg, How internet concepts and technologies can help green and smarten the electrical grid, ACM SIGCOMM Computer Communication Review 41 (1) (2011) 109–114.

[4] R. Srikant, The mathematics of Internet congestion control, Springer Science & Business Media, 2012.

[5] M. Corless, C. King, R. Shorten, F. Wirth, AIMD dynamics and distributed resource allocation, Vol. 29, SIAM, 2016.

[6] V. Dumas, F. Guillemin, P. Robert, A Markovian analysis of additive-increase multiplicative-decrease algorithms, Advances in Applied Probability 34 (1) (2002) 85–111.

[7] J. T. Wen, M. Arcak, A unifying passivity framework for network flow control, IEEE Transactions on Automatic Control 49 (2) (2004) 162–174.

[8] S. Ha, I. Rhee, L. Xu, CUBIC: A new TCP-friendly high-speed TCP variant, ACM SIGOPS Operating Systems Review 42 (5) (2008) 64–74.

[9] G. Armitage, L. Stewart, M. Welzl, J. Healy, An independent H-TCP implementation under freebsd 7.0: Description and observed behaviour, ACM SIGCOMM Computer Communication Review 38 (3) (2008) 27–38.

[10] F. Wirth, S. Stuedli, J. Y. Yu, M. Corless, R. Shorten, Nonhomogeneous place-dependent Markov chains, unsynchronised AIMD, and network utility maximization, arXiv preprint arXiv:1404.5064.

[11] S. Stuedli, M. Corless, R. Middleton, R. Shorten, On the AIMD algorithm under saturation constraints, IEEE Transactions on Automatic Control 62 (12) (2017) 6392–6398.

[12] E. Crisostomi, R. Shorten, S. Studli, F. Wirth, Electric and Plug-in Hybrid Vehicle Networks: Optimization and Control, CRC Press, 2017.

[13] K. Schneider, C. Gerkensmeyer, M. Kintner-Meyer, R. Fletcher, Impact assessment of plug-in hybrid vehicles on pacific northwest distribution systems, in:

Power and Energy Society General Meeting-Conversion and Delivery of Electrical Energy in the 21st Century, 2008 IEEE, IEEE, 2008, pp. 1–6.

[14] P. Papadopoulos, S. Skarvelis-Kazakos, I. Grau, L. M. Cipcigan, N. Jenkins, Electric vehicles' impact on british distribution networks, IET Electrical Systems in Transportation 2 (3) (2012) 91–102.

[15] J. P. Lopes, F. J. Soares, P. R. Almeida, Identifying management procedures to deal with connection of electric vehicles in the grid, in: IEEE Bucharest Powertech, IEEE, 2009, pp. 1–8.

[16] S. Hajforoosh, M. A. Masoum, S. M. Islam, Real-time charging coordination of plug-in electric vehicles based on hybrid fuzzy discrete particle swarm optimization, Electric Power Systems Research 128 (2015) 19–29.

[17] K. Qian, C. Zhou, M. Allan, Y. Yuan, Modeling of load demand due to ev battery charging in distribution systems, IEEE Transactions on Power Systems 26 (2) (2011) 802–810.

[18] G. Putrus, P. Suwanapingkarl, D. Johnston, E. Bentley, M. Narayana, Impact of electric vehicles on power distribution networks, in: IEEE Vehicle Power and Propulsion Conference, 2009 (VPPC'09), 2009, pp. 827–831.

[19] S. Stüdli, E. Crisostomi, R. Middleton, R. Shorten, A flexible distributed framework for realising electric and plug-in hybrid vehicle charging policies, International Journal of Control 85 (8) (2012) 1130–1145.

[20] S. Stüdli, E. Crisostomi, R. Middleton, J. Braslavsky, R. Shorten, Distributed load management using additive increase multiplicative decrease based techniques, in: Plug In Electric Vehicles in Smart Grids, Springer, 2015, pp. 173–202.

[21] M. L. Pinedo, Scheduling: Theory, Algorithms and Systems, Springer-Verlag New York, 2008.

[22] S. Studli, E. Crisostomi, R. Middleton, R. Shorten, Optimal real-time distributed V2G and G2V management of electric vehicles, International Journal of Control 87 (6) (2014) 1153–1162.

[23] P. Bolzern, P. Colaneri, Positive Markov jump linear systems, Foundations and Trends® in Systems and Control 2 (3-4) (2015) 275–427.

[24] R. Stanojevic, R. Shorten, Distributed dynamic speed scaling, in: INFOCOM Proceedings IEEE, 2010, pp. 1–5.

[25] M. F. Barnsley, S. G. Demko, J. H. Elton, J. S. Geronimo, Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities, Annales de l'I.H.P. Probabilités et statistiques 24 (3) (1988) 367–394.