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Motion planning on a class of 6-D Lie groups via a covering map

James D. Biggs Helen Henninger

Abstract—**This paper presents an approach to motion planning for left-invariant kinematic systems defined on the 6-D frame bundles of symmetric spaces of constant cross-sectional curvature. A covering map is used to convert the original differential equation into two coupled equations each evolving on a 3-D Lie group. These lower dimensional systems lend themselves to a minimal global representation that avoid singularities associated with the use of exponential coordinates. Open-loop and closed-loop kinematic control problems are addressed to demonstrate the use of this mapping for analytical and numerical based motion planning methods. The approach is applied to a spacecraft docking problem using two different types of actuation: (i) a fully-actuated continuous low-thrust propulsion system and (ii) an under-actuated single impulsive thruster and reaction wheel system.**

I. Introduction

A general motion planning problem consists of finding an admissible trajectory on an n-dimensional manifold G connecting two points $g(0) \in G$ and $g(T) \in G$. The trajectory is subject to a kinematic constraint defined by an affine control system:

$$\dot{g} = u_1 X_1(g) + \sum_{i=2}^{p} u_i X_i(g), \tag{1}$$

where $u_1, u_i \in \mathbb{R}$, $g \in G$. $X_1(g), X_i(g)$ are vector fields with distribution $\Delta = span(X_1, ..., X_p)$ of co-rank p, equipped with a bilinear form $\langle \cdot, \cdot \rangle$. The affine constraint (1) can also represent a kinematic system with drift if $u_1 = 1$, where $X_1(g)$ is the, uncontrolled, drift vector. In addition, there are usually further requirements imposed on the motion planning problem, such as obstacle avoidance [1], approximating a non-admissible curve with an admissible one [2], [3] and choosing a trajectory that minimises a prescribed cost function [3], [4], [5], [6], [7], [8], [9].

The kinematic constraint considered here is defined on the 6-dimensional manifold $G = SL_4(\mathbb{R})$, where $SL_4(\mathbb{R})$ is the group of 4×4 invertible matrices with real entries having determinant 1. The vector fields considered are left-invariant, such that, $X_1(g) = gB_1, X_i(g) = gB_i$, where B_1, B_i form a basis defined by $u_1 B_1 + \sum_{i=2}^{p} u_i B_i = J\Omega_{4\times4}$, with $\Omega_{4\times4} \in \mathfrak{so}(4)$, where $\mathfrak{so}(4)$ is the Lie algebra of the Special Orthogonal Group SO(4) and J is the 4×4 diagonal matrix

$$J = \text{diag}(K, 1, 1, 1), \tag{2}$$

with $K \in \mathbb{R}$ and where the inverse of J is defined by

$$J^{-1} = \begin{cases} \text{diag}(\ 1/K, & 1, & 1, & 1) & K \neq 0 \\ \text{diag}(\ 0, & 1, & 1, & 1) & K = 0. \end{cases} \tag{3}$$

This special case of the affine control system (1) can then be expressed compactly as:

$$\frac{dg}{dt} = gJ\Omega_{4\times4}, \tag{4}$$

where the bilinear form on Δ is the Killing form when $K \neq 0$ [5], [6], [7]. The controls u_i are equated to $u_i = v_i$ and $u_{i+3} = \omega_i$ to identify them with the components of translational and angular velocities respectively.

Kinematic motion planning problems for the affine control system (4) defined on $G = G(K)$ are relevant to a range of applications. For example, when $K = 1$ Eq. (4) is used to develop quantum control laws for coupled spin 1/2 particles [11], or to model loss-less electrical networks [12], [13], when $K = 0$ they are used to represent the kinematics of rigid-body systems such as underwater vehicles [14], [15], unmanned air vehicles [16] and formations of rigid-body systems [17]. In addition, when $K = 0$, Eq. 4 represents the static configuration of an elastic rod [10], [19], with applications to modelling DNA and cables [18]. When $K = -1$ Eq. (4) can represent the kinematics of relativistic particles [20].

In this paper two kinematic motion planning problems for (4) are considered: (i) An infinite-time problem, where the objective is to asymptotically approach $g(T)$ from a prescribed $g(0)$ using kinematic feedback, and (ii) a finite-time, open-loop, kinematic control problem with prescribed boundary conditions $g_0, g_d \in G(K)$. Defining the feedback functions $v_i = f_i(g), \omega_i = p_i(g)$, the *infinite-time kinematic control problem* considered is:

Problem Statement (PS) 1. *Compute a kinematic feedback control $v_i = f_i(g), \omega_i = p_i(g)$, with $i = 1, 2, 3$, where $g \in G(K)$ is such that the closed-loop system of*

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the differential equation (4) is almost globally asymptotically stable, that is, $g(t) \to I_{4\times4}$ as $t \to \infty$ for almost any $g(0) \in G(K)$.

Note that multiplying the right hand side of Eq.(4) by $g(0)^{-1}$ and using the same feedback controls $v_i = f_i(g), \omega_i = p_i(g)$ that solve PS 1, will drive the system towards $g(0)^{-1} \in G(K)$ from the identity. This class of problem has been addressed in [21] on the n-dimensional Special Orthogonal Group SO(n), where globally stabilizing kinematic feedback laws yield analytically defined solutions of the closed-loop system. Such kinematic feedback laws on SO(3) have previously been used to develop under-actuated attitude controls for spacecraft [23], [24], [22].

A *finite-time kinematic control problem* is also addressed. Defining basis functions for the velocities $v_i = r_i(t, \alpha_j), \omega_i = s_i(t, \alpha_k)$ where r_i, s_i are nonlinear functions of time and the free parameters α_j, α_k, then

Problem Statement (PS) 2. *Compute an open-loop control $v_i = r_i(t, \alpha_j), \omega_i = s_i(t, \alpha_k)$, such that $g(t) \in G(K)$ is a solution of the differential equations (4) subject to the prescribed boundary conditions $g(0) = g_0$ and $g(T) = g_T$.*

The basis functions $v_i = r_i(t, \alpha_j), \omega_i = s_i(t, \alpha_k)$ can be defined as analytic solutions to optimal kinematic control problems [6], [8], [5], [4], [7], [1], [27], [24], [10], [28]. For example, with respect to Eq. (4), a finite-time optimal kinematic control problem could be defined as one that minimizes a quadratic cost function of the form:

$$\xi(\omega_i, v_i) = \int_0^T \sum_{i=1}^{n} c_i v_i^2 + \sum_{i=1}^{m} c_{i+3}\omega_i^2 dt \tag{5}$$

where $n + m \leq 6$ and is subject to the boundary conditions $g(0) = g_0$ and $g(T) = g_T$. Here c_i, c_{i+3} are constant weights of the cost function. The Pontryagin Maximum principle associates to (4) and (5) an optimal Hamiltonian function on $T^*G(K) = G(K) \times \mathfrak{g}_K$ where $\mathfrak{g}_K$ is the Lie algebra of the Lie group $G(K)$ defined in (10). An optimal trajectory is a projection of an integral curve of this time-varying Hamiltonian vector field $(g(t), \lambda(t))$ that satisfies the boundary conditions $g(0) = g_0$ and $g(T) = g_T$. $g(t) \in G(K)$ is then a candidate optimal solution (satisfying the necessary conditions for optimality) of (4) and $\lambda_j \in \mathfrak{g}_K^*$ the extremal curve. The optimal kinematic feedback laws are then functions of the extremal curves, where $v_i = r_i(\lambda_j), \omega_i = s_i(\lambda_j)$ (see [4], [5], [6], [26], [10], [8], [9]). In some cases explicit expressions for the extremal curves can be obtained ([6], [8], [5], [4], [7], [1], [27], [24], [10], [28]), such that, the optimal feedback controls are $v_i = r_i(t, \lambda_j(0)), \omega_i = s_i(t, \lambda_j(0))$. The problem of matching the boundary conditions is then equivalent to PS 2.

In addition, interpolation problems can be framed in the context of PS2 with the use of basis functions (such as polynomials). Basis functions can be used to represent the path of a motion, whereby the parameters of the basis function are chosen to match the boundary conditions and/or minimize a cost function. For interpolation on SO(3), normalized polynomials [29] and exponentials of polynomials [30] have been used as basis functions to design efficient motions for spacecraft. However, for left (respectively right) invariant system, the basis functions maybe better utilized to represent the kinematic controls at the level of the Lie algebra, avoiding the complication of constraining them to the structure of the group or using exponential coordinates. The boundary conditions can then be matched using a numerical shooting method [32] to find the α_j, α_k such that $g(0) = g_0, g(T) = g_T$. To give an example for Eq. (4), assume a constraint $v_1 = 1, v_2 = 0, v_3 = 0$ (a kinematic system with drift, for example, [10], [15], [18], [19], [20], [28]), then one could choose the basis functions for the angular velocity to be cubic splines (assuming for simplicity that $t \in [0, 1]$) of the form:

$$\begin{aligned} \omega_i = \omega_i(0) + \alpha_{i,1}t + \alpha_{i,2}t^2 + \\ (\omega_i(1) - \omega_i(0) - \alpha_{i,1} - \alpha_{i,2})t^3, \end{aligned} \tag{6}$$

with $i = 1, 2, 3$. The basis functions (6) automatically satisfy the boundary conditions on the angular velocities and the parameters $\alpha_{i,1}, \alpha_{i,2}$ are chosen, such that, given $g(0) = g_0$ the following cost function is minimized:

$$\xi(g) = tr(I - g(T)g_T^{-1}). \tag{7}$$

where $tr(\cdot)$ denotes the trace. A shooting method can be implemented to minimize (7) which requires the numerical integration of (4), equivalent to the integration of 16 coupled scalar differential equations. One possibility to simplify the shooting method would be to use exponential coordinates that would reduce the numerical task to integrating only 6 coupled scalar differential equations. For example, local co-ordinate representations of a Lie group using the Wie-Norman representation were used in [14] to enable the application of classical averaging. However, these exponential coordinates are not globally defined for $g \in G(K)$, thus not ideally suited for planning global motions.

In this paper the kinematic system (4) is converted into two coupled kinematic equations using a covering map. This decomposition enables PS 1 to be re-defined as a simpler infinite-time problem on a lower-dimensional space. Furthermore, PS 2 can be re-defined in terms of quaternions, a minimal set of globally defined coordinates.

The paper is presented as follows: Section II defines the decomposition of the kinematic equations on $G(K)$ into two coupled equations which are isomorphic to a set of global coordinates. Section III re-defines PS 1 and 2 using a co-ordinate transformation, such that, PS 1 reduces to a lower-dimensional problem and PS 2 is re-defined in terms of global coordinates on $G(K)$. Section IV provides an example application to spacecraft docking. Two cases are considered which incorporate different types of propulsion technology: (i) an infinite time, closed-loop control problem, where the spacecraft is fully-actuated, with continuous thrust for translation control and reaction wheels for attitude control. (ii) a finite-time, open-loop control problem, where the spacecraft can only thrust impulsively in a single direction and which uses reaction wheels for steering the spacecraft to the target.

II. The kinematic equations expressed in global coordinates

In this section a covering map is derived for $G(K)$, which is used to convert the original kinematic equations into two coupled equations each evolving on a 2×2 complex matrix Lie group. Here we begin by presenting some geometric properties related to the structure of $G(K)$.

A. Geometric properties of $g \in G(K)$

By differentiating the following equation with respect to time

$$g^T(J^{-1})g - J^{-1} = C, \tag{8}$$

it can be seen that C is constant along the flow of (4). Since $g \in SL_4(\mathbb{R})$ is the connected component of the group of 4×4 invertible matrices with real entries through the identity, then $C = 0$ and Eq. (8) can be expressed as:

$$g^T J^{-1} g = J^{-1}. \tag{9}$$

It follows that $g \in SL_4(\mathbb{R})$ subject the kinematic constraint (4) is always contained in its sub-group:

$$G(K) = \{g \in SL_4(\mathbb{R}) : \ g^T J^{-1} g = J^{-1}\}. \tag{10}$$

For $K \neq 0$ the group $G(K)$ is a semi-simple Lie group, with non-degenerate trace form, while for $K = 0$ the trace form is degenerate [6]. For $K = 0$ an element $g \in G(K)$ has the form

$$g = \begin{pmatrix} 1 & 0 \\ \boldsymbol{\gamma} & R \end{pmatrix}, \tag{11}$$

such that $G(K) =$ SE(3) where SE(3) is the Special Euclidean Group of Motions ([4], [5], [6], [7], [10]) with $\boldsymbol{\gamma} \in \mathbb{R}^3$ and $R \in$ SO(3) (where the condition (9) degenerates to $R^T R = I_{3\times3}$). Other classical sub-groups of $G(K)$ include $K = 1$ where $G(K) =$ SO(4) and $K = -1$ where $G(K) =$ SO(1,3) where SO(1,3) is the Lorentz group.

A Lie group defined by (10) is the frame bundle of simply-connected surfaces of constant cross-sectional curvature K [6], where the simply-connected surface is defined explicitly by $M_K = G(K)\boldsymbol{e}_1$, where $\boldsymbol{e}_1 = [1, 0, 0, 0]^T$. For $\boldsymbol{a} = [a_0, a_1, a_2, a_3]^T$, $\boldsymbol{b} = [b_0, b_1, b_2, b_3]^T \in M_K \subseteq \mathbb{R}^4$ a bilinear form on M_K is given by

$$\boldsymbol{a} \odot \boldsymbol{b} = \boldsymbol{a}^T(J^{-1})\boldsymbol{b}. \tag{12}$$

The classical non-Euclidean geometries can be recognized as $M_1 = \mathbb{S}^3$, where $\mathbb{S}^3$ is the 3-dimensional sphere isomorphic to the unit quaternions and $M_{-1} = \mathbb{H}^3$, where $\mathbb{H}^3$ is 3-dimensional Hyperbolic space. For the special case $K = 0$ it is necessary to set $a_0 = b_0 = 1$ as $g \in$ SE(3) acts on a projective space, where $\begin{bmatrix} 1 & \gamma \end{bmatrix}^T \in M_0$. It follows from (9) and (12) that

$$\begin{aligned} g\boldsymbol{a} \odot g\boldsymbol{b} &= (g\boldsymbol{a})^T(J^{-1})g\boldsymbol{b} = \boldsymbol{a}^T g^T (J^{-1}) g\boldsymbol{b} \\ &= \boldsymbol{a}^T(J^{-1})\boldsymbol{b} = \boldsymbol{a} \odot \boldsymbol{b}. \end{aligned} \tag{13}$$

and therefore $G(K)$ is also the isometry group of M_K.

B. Constructing a group homomorphism on $G(K)$

In this sub-section the kinematic equation (4) defined on $G(K)$ is converted to an equivalent system defined on a subgroup of $GU(2) \times GU(2)$, where $GU(2)$ is the General Unitary group defined by

$$GU(2) = \{g \in GL_2(\mathbb{C}) : gg^* = \alpha I_{2\times2}\}, \tag{14}$$

where $GL_2(\mathbb{C})$ is the group of 2×2 invertible matrices with complex entries, α a scalar and g^* the transpose conjugate of g. The original kinematic equations (4) are converted to a set of equations evolving on their covering group.

Definition 1. *[25] Let G be a connected Lie group, then a covering of G is a simply-connected Lie group H together with a Lie group homomorphism $\Phi : G \to H$, such that, the associated Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism.*

Explicit covering maps have been computed for specific cases when $K = 1$ and $K = -1$ [33]. For $K = 1$ and where the entries of $G(K)$ are complex an explicit covering by the 2×2 complex Special Linear Group reveals a connection between complex quaternions and the complexification of SO(4) [4]. We define a covering group of $G(K)$ by constructing a Lie group homomorphism $\Phi : Q_1 \times Q_2 \to G(K)$, where Q_1, Q_2 are 2×2 matrix Lie groups with complex entries and $\phi : \mathfrak{q}_1 \times \mathfrak{q}_2 \to \mathfrak{g}(K)$ is a Lie algebra isomorphism where $\mathfrak{q}_i$ is the Lie algebra of Q_i. The direct product $Q_1 \times Q_2$ is equipped with the standard matrix product

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$(g_1,g_2)\cdot(h_1,h_2)=(g_1h_1,g_2h_2)$ for $(g_1,h_1),(g_2,h_2)\in Q_1\times Q_2$. Firstly, we define $(g_1,g_2)\in Q_1\times Q_2$ such that for $\Phi(g_1,g_2)\in SL_4(\mathbb{R})$ and $\boldsymbol{x},\boldsymbol{w}\in M_K\subseteq\mathbb{R}^4$, $\boldsymbol{x}=[x_0,x_1,x_2,x_3]^T$, $\boldsymbol{w}=[w_0,w_1,w_2,w_3]^T$, then

$$\Phi(g_1,g_2)\boldsymbol{x}=\boldsymbol{w} \quad \text{if} \quad g_1Xg_2^{-1}=W \tag{15}$$

with the isomorphism $(\cdot)^\sharp: X\to\boldsymbol{x}$ defined by

$$\begin{array}{l} X=x_0I_{2\times 2}+2\sqrt{K}(x_1A_1+x_2A_2+x_3A_3) \\ W=w_0I_{2\times 2}+2\sqrt{K}(w_1A_1+w_2A_2+w_3A_3) \end{array} \tag{16}$$

with $I_{2\times 2}$ the identity and A_i defined by

$$\begin{array}{l} A_1=\frac{1}{2}\left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right), A_2=\frac{1}{2}\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \\ A_3=\frac{1}{2}\left(\begin{array}{cc} 0 & i \\ i & 0 \end{array}\right). \end{array} \tag{17}$$

Note that in the case $K=0$, $g\in$ SE(3) acts on a projective space where X,Y are defined by $X=\sum_{i=1}^{3}2\sqrt{K}x_iA_i$ and $W=\sum_{i=1}^{3}2\sqrt{K}w_iA_i$. Furthermore, the case $g\in$ SE(3) is distinguished from the other cases as $K=0$, but for X,Y to be defined $\sqrt{K}\neq 0$, meaning that it must be a dual number (dual numbers are used to define dual-quaternions [34], [35], [36], [37], [38], [39]). A dual number is a number ϵ defined by the property that $\epsilon\neq 0$ and $\epsilon^2=0$. Dual numbers can be thought of in an analogous way to imaginary numbers, that is, an imaginary number $\epsilon\neq 0$ would be defined by the property $\epsilon^2<0$. The properties of the dual numbers are detailed in [37]. Here $\sqrt{K}$ is defined as an extension of a dual number where

$$\sqrt{K}=\left\{\begin{array}{ll} \sqrt{K}\in\mathbb{R}, & K>0 \\ \sqrt{K}\neq 0, & K=0. \end{array}\right. \tag{18}$$

A simple computation shows that Φ is a Lie group homomorphism, preserving the group product:

$$\begin{array}{l} \Phi((g_1,g_2)\cdot(h_1,h_2))\boldsymbol{z}=g_1h_1\,Z\,h_2^{-1}g_2^{-1}\,, \\ \Phi(g_1,g_2)\Phi(h_1,h_2)\boldsymbol{z}=g_1h_1\,Z\,h_2^{-1}g_2^{-1}. \end{array}$$

Using the definition of the bilinear form (12) and the isomorphism (16), it is straightforward to show that

$$XW^*+WX^*=2K(\boldsymbol{x}\odot\boldsymbol{w})I_{2\times 2} \tag{19}$$

then defining the matrices $X_1=X\det(X), W_1=W\det(W)$ then $X_1^{-1}=X^*, W_1^{-1}=W^*$, it follows from Eq. (19) that

$$XW_1^{-1}+WX_1^{-1}=2K(\boldsymbol{x}\odot\boldsymbol{w})I_{2\times 2}. \tag{20}$$

As the action of $g\in G(K)$ on $\boldsymbol{x}$ in Eq. (15) preserves the product $\odot$ then $\boldsymbol{x}\odot\boldsymbol{x}=\boldsymbol{w}\odot\boldsymbol{w}$ which implies that $\det(X)=\det(W)$. Therefore, $g_1Xg_2{}^{-1}=W$ is equivalent to $g_1X_1g_2{}^{-1}=W_1$ then

$$\begin{array}{l} 2K(g\boldsymbol{x}\odot g\boldsymbol{w})I_{2\times 2} \\ =(g_1Xg_2{}^{-1})(g_1W_1g_2{}^{-1})^{-1} \\ +(g_1Wg_2{}^{-1})(g_1X_1g_2{}^{-1})^{-1} \\ =g_1Xg_2{}^{-1}(g_2{}^{-1})^{-1}W_1^{-1}g_1^{-1} \\ +g_1Wg_2{}^{-1}(g_2{}^{-1})^{-1}X_1^{-1}g_1{}^{-1} \\ =g_1XW_1^{-1}g_1{}^{-1}+g_1WX_1^{-1}g_1{}^{-1} \\ =2K(\boldsymbol{x}\odot\boldsymbol{w})I_{2\times 2}. \end{array}$$

and therefore $g\boldsymbol{x}\odot g\boldsymbol{w}=\boldsymbol{x}\odot\boldsymbol{w}$.

Thus, $\Phi(g_1,g_2)=g$ preserves the product $\odot$, and so the range of $\Phi(\cdot,\cdot)$ is $G(K)$ with $\ker(\Phi)=\{(I,I),(-I,-I)\}$ and $\Phi: Q_1\times Q_2\to G(K)$ is a Lie group homomorphism. Having established that (15) implicitly defines a homomorphism it is used to construct an explicit relation:

Lemma 1. *The homomorphism $\Phi: Q_1\times Q_2\to G(K)$ with $(g_1,g_2)\in Q_1\times Q_2$ and $g\in G(K)$ is given by the relation*

$$g=[\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3 \quad \boldsymbol{x}_4] \tag{21}$$

with

$$\begin{array}{l} \boldsymbol{x}_1=(g_1g_2^{-1})^\sharp, \boldsymbol{x}_2=(2\sqrt{K}g_1A_1g_2^{-1})^\sharp \\ \boldsymbol{x}_3=(2\sqrt{K}g_1A_2g_2^{-1})^\sharp, \boldsymbol{x}_4=(2\sqrt{K}g_1A_3g_2^{-1})^\sharp \end{array} \tag{22}$$

Proof. From (16) note that $\boldsymbol{x}=[1\quad 0\quad 0\quad 0]^T$ corresponds to $X=I_{2\times 2}$ and substituting these into (15) gives $(g_1g_2^{-1})=W$. Therefore, the first column of $g\in G(K)$ is $g[1\quad 0\quad 0\quad 0]^T=(g_1g_2^{-1})^\sharp$. The other columns of g are computed in an analogous way to give (21) and (22).

In addition, from Eq. (15) we also have the condition

$$\begin{array}{l} \det(g_1)\det(X)\det(g_2^{-1})=\det(W) \\ \det(g_1)\det(g_2^{-1})=1 \\ \det(g_1g_2^{-1})=1. \end{array} \tag{23}$$

C. A covering group of $G(K)$

To show that $Q_1\times Q_2$ is a covering group we need to show, in addition to Lemma 1, that its Lie algebra $\mathfrak{q}_1\times\mathfrak{q}_2$ is isomorphic to $\mathfrak{g}_K$. Firstly, a basis for $\mathfrak{g}_K$ is explicitly defined:

$$B_1=\left(\begin{array}{cccc} 0 & -K & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), B_2=\left(\begin{array}{cccc} 0 & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$B_3=\left(\begin{array}{cccc} 0 & 0 & 0 & -K \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), B_4=\left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$B_5=\left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), B_6=\left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

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with Lie bracket $[X, Y] = XY - YX$ and corresponding commutator table:

$[\cdot,\cdot]$	B_1	B_2	B_3	B_4	B_5	B_6
B_1	0	KB_6	$-KB_5$	0	B_3	$-B_2$
B_2	$-KB_6$	0	KB_4	$-B_3$	0	B_1
B_3	KB_5	$-KB_4$	0	B_2	$-B_1$	0
B_4	0	B_3	$-B_2$	0	B_6	B_5
B_5	$-B_3$	0	B_1	$-B_6$	0	B_4
B_6	B_2	$-B_1$	0	B_5	$-B_4$	0

(24)

Then defining $(U, V) \in \mathfrak{q}_1 \times \mathfrak{q}_2$ where

$$U = \sum_{i=1}^{3} u_i \sqrt{K} A_i, V = \sum_{i=1}^{3} s_i A_i, \tag{25}$$

with $u_i, s_i \in \mathbb{R}$, with the basis elements defined by $(\sqrt{K}A_i, 0), (0, A_i) \in \mathfrak{q}_1 \times \mathfrak{q}_2$ for $i = 1, 2, 3$. The Lie bracket is defined by $[X_i, X_j] = X_iX_j - X_jX_i$, which are mapped from the elements $(X_i, 0), (X_j, 0), (0, X_i), (0, X_j) \in \mathfrak{q}_1 \times \mathfrak{q}_2$, then the Lie bracket commutator table is:

$[\cdot,\cdot]$	$\sqrt{K}A_1$	$\sqrt{K}A_2$	$\sqrt{K}A_3$	A_1	A_2	A_3
$\sqrt{K}A_1$	0	KA_3	$-KA_2$	0	$\sqrt{K}A_3$	$-\sqrt{K}A_2$
$\sqrt{K}A_2$	$-KA_3$	0	KA_1	$-\sqrt{K}A_3$	0	$\sqrt{K}A_1$
$\sqrt{K}A_3$	KA_2	$-KA_1$	0	$\sqrt{K}A_2$	$-\sqrt{K}A_1$	0
A_1	0	$\sqrt{K}A_3$	$-\sqrt{K}A_2$	0	A_3	A_2
A_2	$-\sqrt{K}A_3$	0	$\sqrt{K}A_1$	$-A_3$	0	A_1
A_3	$\sqrt{K}A_2$	$-\sqrt{K}A_1$	0	A_2	$-A_1$	0

(26)

Note that the commutator tables in (26) and (24) are equivalent by equating $B_1 = \sqrt{K}A_1, B_2 = \sqrt{K}A_2, B_3 = \sqrt{K}A_3, B_4 = A_1, B_5 = A_2, B_6 = A_3$. In addition, the mapping is one-to-one so the Lie algebras are isomorphic. Note that the isomorphism is only defined for $K = 0$ if $\sqrt{K}$ is a dual number. It follows that:

Corollary 2. *The group $Q_1 \times Q_2$ is a covering group of $G(K)$.*

To define an explicit mapping between an element of the Lie algebra $(U, V) \in \mathfrak{q}_1 \times \mathfrak{q}_2$ and $S \in \mathfrak{g}_K$ we substitute the one parameter sub-groups $g = \exp(St)$ and $g_1 = \exp(Ut)$, $g_2 = \exp(Vt)$ into (15), where S, U, V are constant matrices. Then for any constant vector $\boldsymbol{x} \in M_K \subseteq \mathbb{R}^4$ and on differentiating (15), the following necessary condition for $Q_1 \times Q_2$ to be a covering group of $G(K)$ (denoting $\boldsymbol{z} = \dot{\boldsymbol{w}}(0)$ and $Z = \dot{W}(0)$), is given by the equivalence of the relation:

$$S\boldsymbol{x} = \boldsymbol{z} \tag{27}$$

to

$$UX - XV = Z \tag{28}$$

Theorem 3. *The solution $g \in G(K)$ to the kinematic equations (4) can be expressed in the form (21), (22) where*

$$\begin{aligned} \dot{g}_1 &= g_1 \sum_{i=1}^{3} (\omega_i + v_i\sqrt{K})A_i \\ \dot{g}_2 &= g_2 \sum_{i=1}^{3} (\omega_i - v_i\sqrt{K})A_i \end{aligned} \tag{29}$$

where $(g_1, g_2) \in Q_1 \times Q_2$ is defined by

$$\begin{aligned} &Q_1 \times Q_2 = \\ &\left\{(g_1, g_2) \in GU(2) \times GU(2) : \det(g_1 g_2^{-1}) = 1\right\} \end{aligned} \tag{30}$$

Proof. Substitute $S = B_1v_1 + B_2v_2 + B_3v_3 + B_4\omega_1 + B_5\omega_2 + B_6\omega_3 \in \mathfrak{g}_K$ into (27) and (25) into (28). Equating $\boldsymbol{z}$ with Z through the mapping (16) then gives

$$U = \sum_{i=1}^{3} (\omega_i + v_i\sqrt{K})A_i, V = \sum_{i=1}^{3} (\omega_i - v_i\sqrt{K})A_i \tag{31}$$

which yields (29). As $\mathfrak{q}_1 \times \mathfrak{q}_2$ is isomorphic to $\mathfrak{g}_K$ and $\mathfrak{q}_1, \mathfrak{q}_2$ are traceless, skew-Hermitian matrices, then the covering group $Q_1 \times Q_2$ inherits the properties of the Unitary Group. In its most general form we can define the covering group as $GU(2) \times GU(2)$. Moreover, consider the exponential map of an element $g_i \in U(2)$ such that

$$g_i = \exp(Xt) = \sum_{k=0}^{\infty} \frac{(Xt)^k}{k!} = I_{2\times 2} + Xt + o(t^2) \tag{32}$$

then an element of $GU(2)$ is a scalar multiple of an element of $g_i \in U(2)$:

$$\begin{aligned} g &= \sqrt{\alpha} g_i = \sqrt{\alpha} \sum_{k=0}^{\infty} \frac{(Xt)^k}{k!} \\ &= \sqrt{\alpha} I_{2\times 2} + \sqrt{\alpha} Xt + o(t^2) \end{aligned} \tag{33}$$

where $\sqrt{\alpha} \in \mathbb{R}$ is a scalar such that

$$gg^* = \alpha I_{2\times 2} + \alpha(X + X^*)t + o(t^2), \tag{34}$$

then $gg^* = \alpha I_{2\times 2}$ if and only if $X^* = -X$, that is, if X are traceless, skew-Hermitian, matrices (14). Then along with (23) we have the covering group defined by (30) □.

In the case of $g \in$ SE(3) the equations (29) are not suitable for direct numerical implementation as $\sqrt{K}$ is a dual number. In the next section we define a convenient form of the kinematic equations which allow numerical implementation for all $K \geq 0$.

D. Global co-ordinates using quaternions

In this section the kinematic equations (29) are expressed in a new set of coordinates $\hat{q}_R = [q_1 \quad q_2 \quad q_3 \quad q_4]^T$ and $\hat{q}_D = [q_5 \quad q_6 \quad q_7 \quad q_8]^T$, where $\hat{q}_R, \hat{q}_D \in \mathbb{R}^4$ using the co-ordinate transformation for $(g_1, g_2) \in Q_1 \times Q_2$ where:

$$\begin{aligned} g_1 &= g_R + \sqrt{K} g_D \\ g_2 &= g_R - \sqrt{K} g_D, \end{aligned} \tag{35}$$

with the conjugate defined by

$$\begin{aligned} {g_1}^* &= {g_R}^* + \sqrt{K} g_D^* \\ {g_2}^* &= {g_R}^* - \sqrt{K} g_D^*, \end{aligned} \tag{36}$$

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where

$$g_R = \begin{pmatrix} q_4 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_4 - iq_1 \end{pmatrix} \tag{37}$$

and

$$g_D = \begin{pmatrix} q_8 + iq_5 & q_6 + iq_7 \\ -q_6 + iq_7 & q_8 - iq_5 \end{pmatrix}. \tag{38}$$

Substituting (35) and (36) into $g_1 g_1^* = \alpha I_{2\times 2}$ yields:

$$\alpha I_{2\times 2} = (1+K)I_{2\times 2} + \sqrt{K}(g_D {g_R}^* + g_R g_D^*) \tag{39}$$

Therefore, g_R and g_D are Unitary matrices ($g_R g_R^* = I_{2\times 2}$ and $g_D g_D^* = I_{2\times 2}$) if $\alpha = (1+K)$ and

$$g_D g_R^* + g_R g_D^* = 0. \tag{40}$$

It follows that if $g_R, g_D \in$ U(2) then $\hat{q}_R, \hat{q}_D$ are quaternions (in general these are not unit quaternions). Furthermore, the inverse of $g_1, g_2 \in GU(2)$ can now be defined as:

$$g_1^{-1} = \frac{g_1^*}{(1+K)}, g_2^{-1} = \frac{g_2^*}{(1+K)}. \tag{41}$$

Substituting (35) into (29) (either equation can be used here and yield the same result) gives

$$\begin{aligned} &\dot{g}_R + \sqrt{K}\dot{g}_D = \\ &g_R \sum_{i=1}^{3} \omega_i A_i + K g_D \sum_{i=1}^{3} v_i A_i \\ &+\sqrt{K} g_D \sum_{i=1}^{3} \omega_i A_i + \sqrt{K} g_R \sum_{i=1}^{3} v_i A_i \\ &= g_R \sum_{i=1}^{3} (\omega_i + \sqrt{K} v_i) A_i \\ &+\sqrt{K} g_D \sum_{i=1}^{3} (\omega_i + \sqrt{K} v_i) A_i. \end{aligned} \tag{42}$$

Equating the coefficients of $\sqrt{K}$ on both sides of equation (42) yields

$$\begin{aligned} \dot{g}_R &= g_R \sum_{i=1}^{3} \omega_i A_i + K g_D \sum_{i=1}^{3} v_i A_i \\ \dot{g}_D &= g_D \sum_{i=1}^{3} \omega_i A_i + g_R \sum_{i=1}^{3} v_i A_i. \end{aligned} \tag{43}$$

The form of equations (43) naturally incorporate the Euclidean case when $K = 0$ as the term $\sqrt{K} \neq 0$ does not appear in the equations. It follows from (37), (38) and (43) that the kinematics can be expressed in the global co-ordinate form

$$\frac{d\hat{q}_R}{dt} = \sum_{i=1}^{3} \omega_i E_i \hat{q}_R + K \sum_{i=1}^{3} v_i E_i \hat{q}_D \tag{44}$$

$$\frac{d\hat{q}_D}{dt} = \sum_{i=1}^{3} \omega_i E_i \hat{q}_D + \sum_{i=1}^{3} v_i E_i \hat{q}_R \tag{45}$$

where

$$\begin{aligned} E_1 &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ E_2 &= \tfrac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ E_3 &= \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \end{aligned} \tag{46}$$

Note that when $K = 0$ (44, 45) can be recognized as the dual-quaternion representation of rigid-body kinematics.

E. Converting boundary conditions on $G(K)$ to the quaternions $\hat{q}_R$ and $\hat{q}_D$

In order to convert PS 1 and 2 to problems on U(2) $\times$ U(2) it is necessary to define boundary conditions equivalently on $g(0), g(T) \in G(K)$ and $\hat{q}_R(0), \hat{q}_D(0), \hat{q}_R(T), \hat{q}_D(T)$. The projection onto the homogeneous space $g\boldsymbol{e}_1 \in M_K$ where $\boldsymbol{e}_1 = [1 \quad 0 \quad 0 \quad 0]^T$ is given by (16) and (41) such that:

$$\begin{aligned} &(1+K)g\boldsymbol{e}_1 = ((g_R + \sqrt{K}g_D)(g_R - \sqrt{K}g_D)^*)^\sharp = \\ &((g_R + \sqrt{K}g_D)(g_R^* - \sqrt{K}g_D^*))^\sharp \\ &= ((1-K)I_{2\times 2} + \sqrt{K}(g_D g_R^* - g_R g_D^*))^\sharp \end{aligned}. \tag{47}$$

Similar expressions follow for $\boldsymbol{x}_2 = g\boldsymbol{e}_2, \boldsymbol{x}_3 = g\boldsymbol{e}_3, \boldsymbol{x}_4 = g\boldsymbol{e}_4$. Then converting to quaternions using (37) and (38) we have the mapping from quaternions to $g \in G(K)$ given by $g = \frac{1}{(1+K)}[\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3 \quad \boldsymbol{x}_4]$:

$$\boldsymbol{x}_1 = \begin{bmatrix} ||\hat{q}_R||^2 - K||\hat{q}_D||^2 \\ 2(q_4q_5 - q_3q_6 + q_2q_7 - q_1q_8) \\ 2(q_3q_5 + q_4q_6 - q_1q_7 - q_2q_8) \\ 2(q_1q_6 + q_4q_7 - q_2q_5 - q_3q_8 \end{bmatrix}, \tag{48}$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 2K(q_2q_7 + q_1q_8 - q_4q_5 - q_3q_6) \\ q_1^2 - q_2^2 - q_3^2 + q_4^2 + K(q_6^2 + q_7^2 - q_5^2 - q_8^2) \\ 2(q_1q_2 + q_3q_4 - K(q_5q_6 + q_7q_8)) \\ 2(q_1q_3 - q_2q_4 + K(q_6q_8 - q_5q_7)) \end{bmatrix}$$

$$\boldsymbol{x}_3 = \begin{bmatrix} 2K(q_3q_5 - q_4q_6 - q_1q_7 + q_2q_8) \\ 2(q_1q_2 - q_3q_4 + K(q_7q_8 - q_5q_6)) \\ q_2^2 - q_3^2 - q_1^2 + q_4^2 + K(q_5^2 - q_6^2 + q_7^2 - q_8^2) \\ 2(q_1q_4 + q_2q_3 - K(q_6q_7 + q_5q_8)) \end{bmatrix}$$

$$\boldsymbol{x}_4 = \begin{bmatrix} 2K(q_1q_6 - q_2q_5 - q_4q_7 + q_3q_8) \\ 2(q_1q_3 + q_2q_4 + K(-q_6q_8 - q_5q_7)) \\ 2(q_2q_3 - q_1q_4 + K(q_5q_8 - q_6q_7)) \\ q_3^2 + q_4^2 - q_1^2 - q_2^2 + K(q_5^2 + q_6^2 - q_7^2 - q_8^2) \end{bmatrix}$$

We express this mapping as $\mathcal{F} : \widetilde{Q} \to SL_4(\mathbb{R})$ where $(\hat{q}_R, \hat{q}_D) \in \widetilde{Q}$ is isomorphic to $(g_R, g_D) \in$ U(2) $\times$ U(2) with $\mathcal{F}(\hat{q}_R, \hat{q}_D) = [\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4]$. Each component of this mapping is a quadratic function and it follows from the mapping between U(2) and $\mathbb{R}^4$ that $\mathcal{F}$ is surjective.

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Thus $\widetilde{Q}$ is defined as the set of all $\hat{q}_R, \hat{q}_D$ satisfying the properties

$$\begin{array}{l}\det(\mathcal{F}(\hat{q}_R, \hat{q}_D)) = 1 \\ \mathcal{F}(\hat{q}_R, \hat{q}_D)^T J^{-1} \mathcal{F}(\hat{q}_R, \hat{q}_D) = J^{-1}\end{array} \tag{49}$$

then

$$\begin{array}{l}\mathcal{F}(\widetilde{Q}) = \{g \in SL_2(\mathbb{R}) : \ g^T J^{-1} g = J^{-1}\} \\ = G(K) \mid_{K>0}\end{array} . \tag{50}$$

Thus, $\mathcal{F} : \widetilde{Q} \to G(K) \mid_{K>0}$ is a surjective function (since the co-domain is equal to the range). Therefore, there exists a right-inverse, that is, given $g \in G(K)$ there exists a function $\mathcal{H} : G(K) \to \widetilde{Q}$ such that $\mathcal{F}(\mathcal{H}(g)) = g$. However, the representation of $g \in G(K)$ using the global co-ordinates $\hat{q}_R, \hat{q}_D$ is not unique and could lead to the problem of unwinding [40]. For example, when using quaternions to represent rotation, $q_4 = \pm 1$ corresponds to a single physical rotation. However, mathematically, it is possible for $q_4 = 1$ to be a stable equilibrium point, while $q_4 = -1$ is unstable. Thus, if the desired position is $q_4 = \pm 1$ and the system begins close to the unstable equilibrium $q_4 = -1$, it will perform an unnecessarily large rotation to get to the desired orientation.

One possibility to construct the right-inverse for $K > 0$ given an arbitrary constant matrix $g_c \in G(K)$ would be to use a numerical optimization method to select the quaternions $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8$ that minimise an appropriate cost function, for example:

$$\mathcal{J} = \mathrm{tr}(I_{4\times4} - \mathcal{F}(q_R, q_D) g_c^{-1})^2, \tag{51}$$

subject to the conditions (50). Since the function (51) is nonlinear, local minima exist which makes it necessary to use a constrained global optimizer similarly to the approach in [1]. In the case of SE(3) exact equations exist for the conversion between boundary conditions. Moreover, by equating the first column of (11) with (48) when $K = 0$, it can be seen that $\|\hat{q}_R\| = 1$ and the inverse of the mapping $\hat{q}_R, \hat{q}_D$ to $g \in$ SE(3) is:

$$q_R = \begin{bmatrix} (R_{23} - R_{32})/4q_4 \\ (R_{31} - R_{13})/4q_4 \\ (R_{12} - R_{21})/4q_4 \\ q_4 \end{bmatrix} \tag{52}$$

where $q_4 = \pm\frac{1}{2}\sqrt{1 + R_{11} + R_{22} + R_{33}}$ with R_{ij} the components of the matrix $R \in$ SO(3). The path $\boldsymbol{\gamma} = [x_1 \quad x_2 \quad x_3] \in \mathbb{R}^3$ is then given by equation (48). Moreover, we have $\boldsymbol{x}_1 = [1 \quad \boldsymbol{\gamma}]^T$ where:

$$\boldsymbol{\gamma} = [\begin{array}{c} 2(q_4 q_5 - q_3 q_6 + q_2 q_7 - q_1 q_8) \\ 2(q_3 q_5 + q_4 q_6 - q_1 q_7 - q_2 q_8) \\ 2(q_1 q_6 + q_4 q_7 - q_2 q_5 - q_3 q_8) \end{array}]. \tag{53}$$

Recalling the normalized image space coordinates conditions $\hat{q}_R \cdot \hat{q}_R = 1, \quad \hat{q}_R \cdot \hat{q}_D = 0$, thus we can write the solution to (53) including the constraint $\hat{q}_R \cdot \hat{q}_D = 0$ as [36]:

$$\begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} q_5 & q_6 & q_7 & q_8 \\ -q_8 & q_7 & -q_6 & q_5 \\ -q_7 & -q_8 & q_5 & q_6 \\ q_6 & -q_5 & -q_8 & q_7 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{54}$$

and as the matrix in (54) is orthogonal it is easily shown that:

$$\begin{bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -x_3 & x_2 & x_1 \\ x_3 & 0 & -x_1 & x_2 \\ -x_2 & x_1 & 0 & x_3 \\ -x_1 & -x_2 & -x_3 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}. \tag{55}$$

Therefore, given $R(0), R(T) \in$ SO(3) we can compute $\hat{q}_R(0), \hat{q}_R(T)$, then given $\boldsymbol{\gamma}(0), \boldsymbol{\gamma}(T) \in \mathbb{R}^3$ we can compute $\hat{q}_D(0), \hat{q}_D(T)$ using (55).

III. Motion planning problems on the covering groups and global co-ordinates

From (35) it is clear that if $g_1 \to I_{2\times2}$ then $g_2 \to I_{2\times2}$. It follows from (21), (22) and the mapping (16) that if $g_1 \to I_{2\times2}$ and (4) is controllable then $g \to I_{4\times4}$. This means that by designing a kinematic feedback-control that drives $g_1 \in GU(2)$ to the identity then the same control will also drive $g \in G(K)$ to the identity. This implies that PS 1 can be defined equivalently on $GU(2)$:

Re-defined PS 1. *Compute a feedback control $v_i = f_i(g_1), \omega_i = p_i(g_1)$ where the closed-loop system of the differential equation (29) is almost globally asymptotically stable, that is, $g_1(t) \to I_{2\times2}$ as $t \to \infty$ for almost any $g_1(0) \in$ GU(2).*

This PS can be solved in a number of ways using kinematic-feedback controls analogously to those developed on SO(3), such as [21], as their Lie algebras are isomorphic. An example feedback control is given in the following Lemma:

Lemma 4. *The trajectory $g_1 = g_R + \sqrt{K} g_D \in GU(2)$ of the coupled equations (43) subject to the kinematic feedback controls defined by*

$$\begin{array}{l}\sum_{i=1}^{3} \omega_j A_i = \frac{g_R^*}{1+K} P - P g_R \\ \sum_{i=1}^{3} v_i A_i = \frac{g_D^*}{1+K} P - P g_D\end{array} \tag{56}$$

where P is a positive semi-definite gain matrix yields the following closed form solution

$$\begin{array}{l}g_1 = (\sinh(Pt) + \cosh(Pt) g_1(0)) \\ \cdot(\cosh(Pt) + \sinh(Pt) g_1(0))^{-1}\end{array} \tag{57}$$

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Proof. Replacing the transpose in the control law [21] with the inverse (41) we can define a feedback law for the reduced system (29) as:

$$\sum_{i=1}^{3}(\omega_i + v_i\sqrt{K})A_i = (\frac{g_1^*}{1+K}P - Pg_1) \quad (58)$$

Substituting Eq. (35) into Eq. (58) gives

$$\begin{aligned}&\sum_{i=1}^{3}(\omega_i + v_i\sqrt{K})A_i \\ &= (\tfrac{g_R^*}{1+K}P - Pg_R) + \sqrt{K}(\tfrac{g_D^*}{1+K}P - Pg_D)\end{aligned} \quad (59)$$

and (56) follows by equating the real and the dual components $\sqrt{K}$. Multiplying (58) by g_1 on both sides yields

$$\dot{g}_1 = P - g_1 P g_1 \quad (60)$$

and following an analogous procedure to [21] let $X, Y \in GL(2,\mathbb{C})$ satisfy

$$\dot{X} = PY, \quad \dot{Y} = PX \quad (61)$$

with initial conditions $X(0) = g_1(0)$ and $Y(0) = I_{2\times 2}$. It follows that $g_1 = XY^{-1}$ is in $GU(2)$ as

$$\begin{aligned}\dot{g}_1 &= \dot{X}Y^{-1} + X\dot{Y}^{-1} \\ &= P + X(-Y^{-1}\dot{Y}Y^{-1}) = P - g_1 P g_1\end{aligned} \quad (62)$$

which is equivalent to equation (60). Noting that (61) is linear with transition matrix

$$\exp(\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} t) = \begin{bmatrix} \cosh Pt & \sinh Pt \\ \sinh Pt & \cosh Pt \end{bmatrix} \quad (63)$$

yields (57). □

As $\sinh Pt \rightarrow (1/2)\text{exp}(Pt)$ and $\cosh(Pt) \rightarrow (1/2)\text{exp}(Pt)$ when $t \rightarrow \infty$ it follows from (57) that $g_1 \rightarrow I_{2\times 2}$ as $t \rightarrow \infty$. A rigorous proof of the global stability for the case of SO(3) is given in [21] which naturally extends to $GU(2)$. Note that $g \in G(K)$ can be reconstructed from (21) and (16). Therefore, $g \rightarrow I_{4\times 4}$ under the feedback law (56).

Re-defined PS 2. *Compute an open-loop control $v_i = r_i(t, \alpha_j), \omega_i = s_i(t, \alpha_k)$ such that $(\hat{q}_R, \hat{q}_D) \in \mathbb{R}^8$ is a solution of the differential equations (44) and (45) subject to the prescribed boundary conditions $(\hat{q}_R(0), \hat{q}_D(0))$ and $(\hat{q}_R(T), \hat{q}_D(T))$.*

Given the basis functions r_i, s_i then a numerical shooting method [32] can be implemented such that the cost function:

$$\xi_{\hat{q}_R,\hat{q}_D} = \sum_{i=1}^{n=8}(q_i - q_i(T))^2 \quad (64)$$

is minimized where $q_i(T)$ are the prescribed boundary condition at the final time T. In this set of coordinates the shooting method only requires the integration of 8 coupled scalar differential equations (44) and (45) (as opposed to 16 coupled scalar differential equations in the original problem). These approaches are demonstrated with an application to a spacecraft docking problem.

IV. An application to Spacecraft Docking

Spacecraft docking specifically refers to the joining of two separate free-flying space vehicles; a target and a chaser spacecraft. Current applications of spacecraft docking include the docking of small spacecraft with the International Space Station for crew transfer. Future applications include the possibility of in-orbit refuelling and maintenance. Of recent interest is the prospect of using nano-spacecraft (1-50kg spacecraft) for autonomous docking operations to provide support to larger space assets. In this section we consider the possibility of using a 12 U CubeSat spacecraft (with mass of 22 kg and dimensions 20cm $\times$ 20cm $\times$ 30cm) for deep space docking. Guidance methods have been developed for spacecraft docking in the planar case without a gravitational field (deep space docking) using inverse dynamics and nonlinear programming [41] and in Earth orbit [43] using a potential function approach. In [42] the approach direction is constrained along the target docking axis and uses a feedback control. However, none of these papers consider the extension to the spatial case or the possibility of undertaking docking with an under-actuated spacecraft. In this section we consider an extension of previous work to the spatial docking problem with a 12 U nano-spacecraft in deep-space where the gravitational influence of other bodies is negligible such that the dynamics can be described by the equations [41]:

$$\begin{aligned}m\dot{\boldsymbol{v}} &= \boldsymbol{F} \\ I\dot{\boldsymbol{\omega}} &= I\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{T}\end{aligned} \quad (65)$$

where $\boldsymbol{v} = [v_1 \quad v_2 \quad v_3]^T, \boldsymbol{\omega} = [\omega_1 \quad \omega_2 \quad \omega_3]^T$ and I the inertia matrix with zero cross terms and principal moments of inertia equal to $I_1 = 0.1656, I_2 = 0.2671, I_3 = 0.2643$ kg m^2 with 3 reactions wheels each with a maximum torque $||\boldsymbol{T}|| = 2$ mNm and applied force $\boldsymbol{F}$ whose maximum is dependent on the thrusters used. The data for this spacecraft is taken from the preliminary design of the LUMIO 12 U Cubesat described in [44], [45]. The kinematics of the nano-spacecraft can be described by (4) with $K = 0$. The initial position relative to the target is $\boldsymbol{\gamma}(0) = [9.4 \quad 6 \quad 4]^T\ m$ and initial attitude:

$$R(0) = \begin{bmatrix} -0.782 & 0 & 0.6233 \\ -0.6233 & 0 & -0.782 \\ 0 & -1.000 & 0 \end{bmatrix} \quad (66)$$

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with final conditions $\gamma(T) = [0 \quad 0 \quad 0]^T$ and $R(T) = I_{4\times4}$. Converting this to boundary conditions on q_R and q_D using equations (54) and (55) yields

$$\begin{aligned} \hat{q}_R(0) &= [0.233 \quad -0.667 \quad 0.667 \quad 0.233]^T \\ \hat{q}_D(0) &= [4.435 \quad -1.970 \quad -3.371 \quad -0.430]^T \\ \hat{q}_R(T) &= [0 \quad 0 \quad 0 \quad \pm 1]^T \\ \hat{q}_D(T) &= [0 \quad 0 \quad 0 \quad 0]^T. \end{aligned} \tag{67}$$

We consider two types of propulsion systems: Case 1: 6 micro-thrusters for translational control (fully-actuated) which can be throttled continuously up to a maximum of 90 μ N typical of Nano-spacecraft plasma thrusters. Case 2: a cold-gas propulsion system which can provide an impulsive thrust of 0.1 N. In this case the impulse is provided only at the beginning of the motion to induce a constant translational velocity and the reaction wheels then used to steer the spacecraft to the target.

A. *Fully-actuated continuous thrust spacecraft docking*

In this example we use Lemma 4 to design a control law. For simplicity of exposition we choose the positive definite matrix $P = k_1 I_{2\times2}$ and define $k = 4k_1$. Then converting the feedback-controls (56) to quaternions using the equations (37) and (38) yields the simple feedback law $\omega_i = -kq_i, v_i = -kq_{i+4}$. From Lemma 4 it is known that the closed loop system is almost globally asymptotically stable such that $g \to I_{4\times4}$ as $t \to \infty$. This control law is implemented on the kinematic equations (44) and (45) for $K = 0$. The path that the spacecraft traces in Euclidean space is given by Equation (53). The parameter k is tuned experimentally (k =0.00025 in this example) such that the time of convergence is minimized, while the thrust remains within the limits of the propulsion system (which is checked using equation (65)). The distance from the target over time and the corresponding thrust is shown in Figure 1. The resulting rotation is displayed in Figure 2 in unit quaternions over time, along with the corresponding required reaction wheel torques.

This kinematic feedback control provides convergence to the desired position and attitude within the physical limits of a nano-spacecraft low-thrust propulsion device such as a pulsed-plasma thruster.

B. *Under-actuated impulsive thrust spacecraft docking with reaction wheel steering*

In this application the spacecraft is now assumed to only have one main thruster fixed in the body frame for translational control and reaction wheels that can steer the orientation of the spacecraft in all three axis continuously. The thruster is assumed to be impulsive

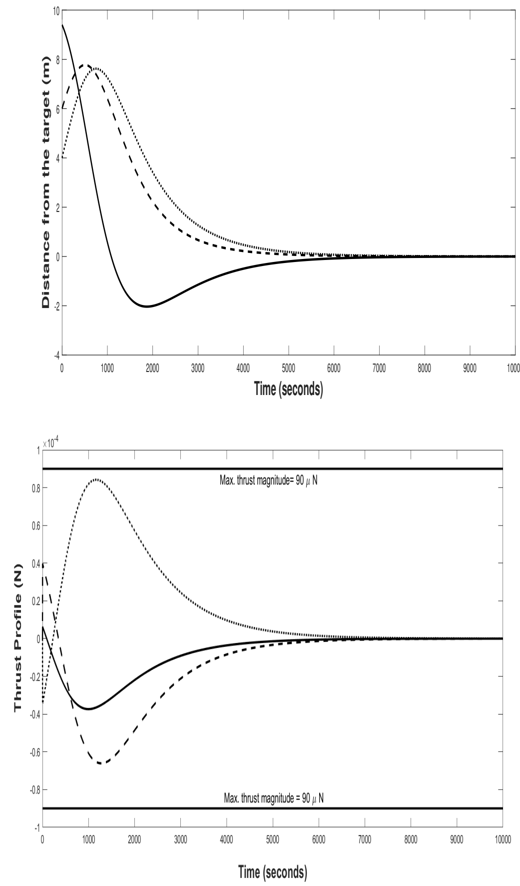

Fig. 1. The translation γ converging to the desired position and the thrust required to undertake the translation

typical of a chemical propulsion thruster. The spacecraft is assumed to be at rest and then to provide a single impulse that induces a velocity in the forward direction such that $v_1 = v$ where v is a constant velocity. Since there are no other thrusters the velocities in the other axis are zero such that $v_2 = 0$ and $v_3 = 0$ (equivalent to a nonholonomic sliding constraint). The objective is then to steer the spacecraft so that it reaches the target position and attitude within a finite-time. As suggested in [31] sinusoidal functions are used as basis functions for the angular velocity:

$$\omega_1 = D, \omega_2 = r_1 \sin(a_1 t + \beta_1), \omega_3 = r_2 \cos(a_2 t + \beta_2). \tag{68}$$

where v, D, r_i, a_i, β_i are free parameters that are computed such that the boundary condition (67) are matched on a virtual time domain $t \in [0, 1]$ using a standard numerical shooting method which yields the numerical values: $D = 0.155, r_1 = 79.524, r_2 = 1.1893 \times 10^4, a_1 = 0.0775, a_2 = 3.523 \times 10^{-4}, b_1 = -0.0595, b_2 = 1.57, v = 16.6$. Note that on the virtual time domain $t \in [0, 1]$ the resulting motion is dynamically infeasible

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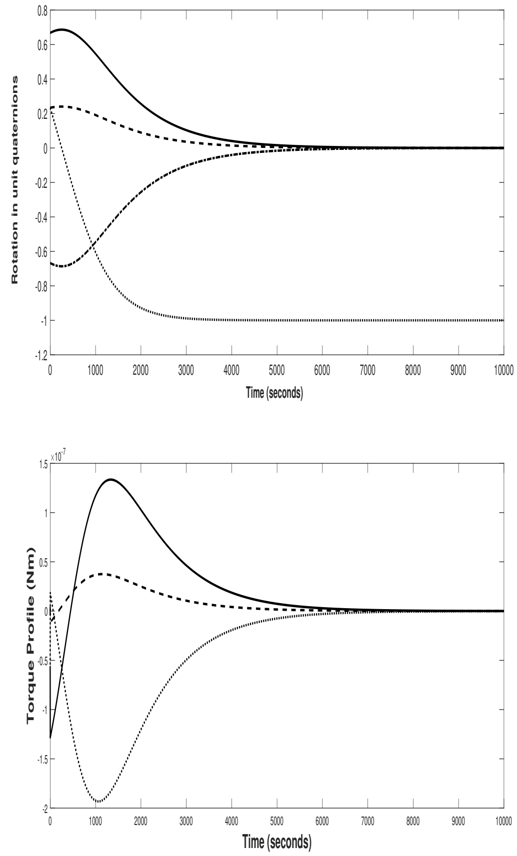

Fig. 2. The rotation in quaternion form converging to the desired rotation and the torque required to perform the rotation

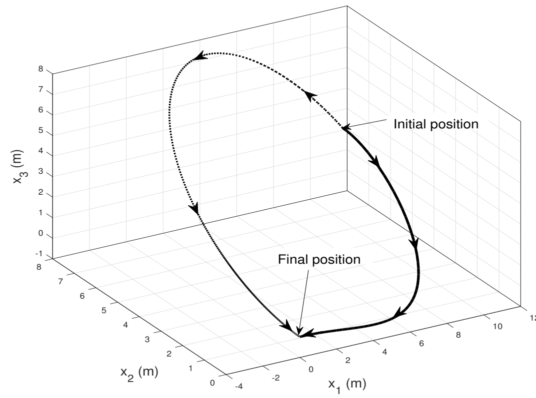

Fig. 3. The translational motion of the under-actuated spacecraft compared to the fully actuated case: The dotted line corresponds to the path traced by the spacecraft given the closed-loop (fully-actuated) control and the solid line the one traced with the open-loop (under-actuated) control.

which can be checked by substituting the velocities and their derivatives into (65). However, as the velocities (68) are defined analytically as a function of time the accelerations, torque and force can also be expressed analytically as a function of time t using equation (65). Therefore, to ensure dynamic feasibility t can be re-parametrized [29], [1] to reduce the required force and torques using the equation (65). For a direct comparison with the previous example the time is re-parametrized by the final time $\tau = t/T_f$ where $T_f = 10000$ secs. Converting $\omega(t) \rightarrow \omega(\tau/(T_f))/(T_f)$ it can be seen that to induce the required velocity the 0.1 N thruster must fire for 0.0166 secs to reach the required constant translational velocity of 0.00166 ms^{-1}. In Figure 3 the path the vehicle traces in Euclidean space is shown with the second plot displaying the comparison with the path generated using the closed-loop feedback with full actuation. In the open-loop case the torque profile can be computed exactly by substituting the basis functions (68) into (65) and solving for $\boldsymbol{T}$ which shows the required torques are feasible with nano-spacecraft reaction wheels.

V. Conclusion

This paper derives an explicit expression to decompose left-invariant differential equations defined on a class of 6-D frame bundles, into two coupled differential equations, that each evolve on a 2×2 complex matrix Lie group. The de-composition is shown to be useful for two types of kinematic control problems. In the case of designing closed-loop controls it is shown that the approach can simplify control design, by considering the problem defined on a lower-dimensional group. In addition, algorithms for finite-time, open-loop controls, are developed that exploit a global coordinate representation. In this case the problem of integrating the original kinematic system defined on a 6-D Lie group (equivalent to integrating 16 coupled scalar differential equations), is reduced to the integration of only 8 differential equations. In addition, a connection between kinematic systems defined on the frame bundles of symmetric spaces and those defined on dual quaternions is demonstrated via a covering map and a simple coordinate change.

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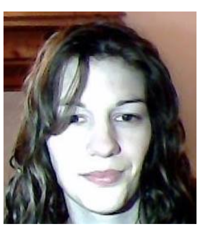

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