

# ON THE CLASSIFICATION OF NON-EQUAL RANK AFFINE CONFORMAL EMBEDDINGS AND APPLICATIONS

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**ABSTRACT.** We complete the classification of conformal embeddings of a maximally reductive subalgebra  $\mathfrak{k}$  into a simple Lie algebra  $\mathfrak{g}$  at non-integrable non-critical levels  $k$  by dealing with the case when  $\mathfrak{k}$  has rank less than that of  $\mathfrak{g}$ . We describe some remarkable instances of decomposition of the vertex algebra  $V_k(\mathfrak{g})$  as a module for the vertex subalgebra generated by  $\mathfrak{k}$ . We discuss decompositions of conformal embeddings and constructions of new affine Howe dual pairs at negative levels. In particular, we study an example of conformal embeddings  $A_1 \times A_1 \hookrightarrow C_3$  at level  $k = -1/2$ , and obtain explicit branching rules by applying certain  $q$ -series identity. In the analysis of conformal embedding  $A_1 \times D_4 \hookrightarrow C_8$  at level  $k = -1/2$  we detect subsingular vectors which do not appear in the branching rules of the classical Howe dual pairs.

*To the memory of Bertram Kostant 5/24/1928–2/2/2017*

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a semisimple finite-dimensional complex Lie algebra and  $\mathfrak{k}$  a reductive subalgebra of  $\mathfrak{g}$ . The embedding  $\mathfrak{k} \hookrightarrow \mathfrak{g}$  is called conformal if the central charge of the Sugawara construction for the affinization  $\widehat{\mathfrak{g}}$ , acting on an integrable  $\widehat{\mathfrak{g}}$ -module of level  $k$ , equals that for  $\widehat{\mathfrak{k}}$ . Then necessarily  $k = 1$  [9]. Maximal conformal embeddings were classified in [38], [9], and related decompositions can be found in [25], [24], [15]. In the vertex algebra framework the definition can be rephrased as follows: the simple affine vertex algebras  $V_1(\mathfrak{g})$  and the vertex subalgebra generated by  $\{x_{(-1)}\mathbf{1} \mid x \in \mathfrak{k}\}$  have the same Sugawara conformal vector.

Let us denote by  $\widetilde{V}(k, \mathfrak{k})$  the vertex subalgebra of  $V_k(\mathfrak{g})$  generated by  $\{x_{(-1)}\mathbf{1} \mid x \in \mathfrak{k}\}$ . In [2] we generalized the previous situation to study when the simple affine vertex algebra  $V_k(\mathfrak{g})$  and its subalgebra  $\widetilde{V}(k, \mathfrak{k})$  have the same Sugawara conformal vector for some non-critical level  $k$ , not necessarily 1, assuming  $\mathfrak{k}$  to be an equal rank reductive subalgebra. We also considered the problem of providing the explicit decomposition of  $V_k(\mathfrak{g})$  regarded as a  $\widetilde{V}(k, \mathfrak{k})$ -module.

The present paper is divided into two parts. In the first part (Sections 3–5) we deal with the classification problem in the non-equal rank case: in particular, we give the complete classification of conformal embeddings when  $\mathfrak{k}$  is a maximal non-equal rank semisimple subalgebra of  $\mathfrak{g}$ . In the second part (Sections 7–11) we discuss some instances of the decomposition problem that have interesting applications, such as a representation theoretic interpretation of an  $\eta$ -function identity and the emergence of new Howe dual pairs. Section 6 combines the results from Sections 3–5 and those of [2] to obtain the classification of conformal embeddings

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of maximally reductive subalgebras (cf. Definition 2.1) in simple Lie algebras: see Theorem 6.1.

We should mention that the some of the affine vertex algebras occurring in our analysis of conformal embeddings, have appeared recently in various mathematics and physics papers on simple affine vertex algebras associated with the Deligne exceptional series at levels  $k = -h^\vee/6 - 1$  [10], [11], [12], [19].

**1.1. Classification of conformal embeddings.** We now discuss the methods employed in our solution of the classification problem. Our main tool is a criterion given in [1] for conformal embeddings (see Subsection 2.1), referred to in the following as the *AP-criterion*. As explained in Subsection 2.3, the classification of maximally reductive subalgebras reduces to Dynkin’s classification of maximal semisimple subalgebras of a simple Lie algebra. This classification splits these subalgebras in some classes. For each of these classes, we develop methods to enforce the AP-criterion.

In Section 3, we discuss the conformal embeddings of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(\mathfrak{g})$  with  $\mathfrak{g} = so(V), sp(V), sl(V)$ , and  $V$  irreducible as a representation of  $\mathfrak{k}$ . Each of these cases requires similar but not uniform approaches. In the case of  $\mathfrak{g} = so(V)$ , we use Kostant’s theory of pairs of Lie type [28]. Kostant found a condition in terms of the Clifford algebra  $Cl(V)$  for  $\mathfrak{k} \oplus V$  to have a Lie algebra structure with natural properties. Reformulating the Symmetric Space Theorem [20] in terms of pairs of Lie type, we find in Proposition 3.1 a very strong condition for the existence of conformal embeddings at level  $k \neq 1$ . It turns out that all but two cases are ruled out (cf. Proposition 3.3):

**Theorem 1.1.** *Assume that  $\tilde{V}(k, \mathfrak{k})$  embeds conformally in  $V_k(so(V))$  such that  $k \notin \mathbb{Z}_+$  and  $V$  is an irreducible  $\mathfrak{k}$ -module. Then  $k = -2$ , and we are in one of the following cases*

- $\mathfrak{k} = B_3$ ,  $V = L_{B_3}(\omega_3)$ , conformal embedding of  $V_{-2}(B_3)$  into  $V_{-2}(D_4) = V_{-2}(so(V))$ .
- $\mathfrak{k} = G_2$ ,  $V = L_{G_2}(\omega_1)$ , conformal embedding of  $V_{-2}(G_2)$  into  $V_{-2}(B_3) = V_{-2}(so(V))$ .

The same ideas are employed in the case of embeddings in  $sp(V)$ , where pairs of Lie type are substituted by Lie superalgebras of Riemannian type [29] and the Clifford algebra of  $V$  is substituted by the symmetric algebra  $S(V)$ . The final outcome is contained in Proposition 3.6; note that in this case we have to use genuine “super” techniques, as well as Kac’s classification of simple Lie superalgebras. The embeddings in  $sl(V)$  are dealt with by adapting Kostant’s ideas to the algebra  $End(V)$  and the corresponding classification appears in Proposition 3.8.

In Section 4, we deal with conformal embeddings of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(\mathfrak{g})$  with  $\mathfrak{g}$  classical and  $\mathfrak{k}$  semisimple non-simple. In this case the verification of the conditions of the AP-criterion is performed by using Classical Invariant Theory (cf. Theorem 4.1). It is worthwhile to remark that, as a consequence of our analysis, we are able to provide examples of conformal embeddings  $\tilde{V}(k, \mathfrak{k}) \subset V_k(\mathfrak{g})$  with  $V_k(\mathfrak{g})$  non semisimple as a  $\tilde{V}(k, \mathfrak{k})$ -module: see Example 4.1 and Theorem 7.2.

Section 5 is devoted to a direct analysis of conformal embeddings in  $V_k(\mathfrak{g})$  with  $\mathfrak{g}$  of exceptional type.

**1.2. Decomposition of embeddings.** The decomposition problem for conformal embeddings was studied in our previous paper [2] for equal rank affine embeddings, and in [3] for embeddings of affine vertex algebras into some  $\mathcal{W}$ -algebras. Quite surprisingly, in handling decompositions of conformal embeddings for non-equal rank subalgebras we found completely new phenomena: hence new ideas for a general approach will be required; here we analyze

some special cases. Even in integrable cases such decompositions are related with some non-trivial results on symmetric spaces [15], the theory of simple-current extensions [27], and the affine tensor categories and rank-level duality [36]. In non-integrable cases, the decomposition problem is very difficult, since most of the tools used in integrable cases do not apply. One reason, as noted above, is that decompositions in non-integrable cases need not to be semisimple. In this paper we study in detail the decomposition of the Weyl vertex algebra  $M_{(m)}$  as a  $V^{-m/2}(sl(2)) \otimes V^{-2}(so(m))$ -module, which exhibits an infinite-dimensional generalization of classical results due to Howe [22, Section 4]. As a byproduct, we get the realization of the simple affine vertex algebra  $V_{-2}(A_3)$  (investigated in [12]) inside of the Weyl vertex algebra  $M_{(6)}$ . Theorem 7.2 implies the following result.

**Proposition 1.2.** *There exists a non-trivial homomorphism  $\Phi : V_{-2}(A_3) \rightarrow M_{(6)}$ .*

We hope that this realization can be used to verify the fusion-rules conjecture for  $V_{-2}(A_3)$ -modules from the category  $KL_{-2}$  presented in [12].

Moreover, in Section 7 we identify all singular vectors in  $M_{(m)}$  which correspond to singular vectors obtained using classical invariant theory. The special case  $m = 8$  is further studied in Section 10, where we show that  $M_{(8)}$  has subsingular vectors which do not appear in the classical case: see Proposition 10.2.

**1.3. A connection with Howe dual pairs.** The vertex subalgebra  $\tilde{V}(k, \mathfrak{k})$  of  $V_k(\mathfrak{g})$  usually has the form  $\tilde{V}(k, \mathfrak{k}_1) \otimes \tilde{V}(k, \mathfrak{k}_2)$  for certain simple Lie algebras  $\mathfrak{k}_i$ , and it is natural to consider the commutant (=coset) vertex algebra  $\text{Com}(\tilde{V}(k, \mathfrak{k}_i), V_k(\mathfrak{g}))$ . The determination of commutants is slightly easier than the problem of finding explicit decompositions, nevertheless it gives relevant information on the structure of embeddings. If

$$\text{Com}(\tilde{V}(k, \mathfrak{k}_1), V_k(\mathfrak{g})) = \tilde{V}(k, \mathfrak{k}_2), \quad \text{Com}(\tilde{V}(k, \mathfrak{k}_2), V_k(\mathfrak{g})) = \tilde{V}(k, \mathfrak{k}_1),$$

we say that  $\tilde{V}(k, \mathfrak{k}_1)$  and  $\tilde{V}(k, \mathfrak{k}_2)$  are an affine Howe dual pair inside of  $V_k(\mathfrak{g})$ .

In the vertex algebra setting the commutant problem and Howe dual pairs were extensively studied by A. Linshaw and collaborators (see [30], [31] and reference therein). We have noticed that the results from [31] can be applied to conformal embeddings from our paper. In Proposition 8.1 we prove the following.

**Theorem 1.3.** *For  $m \geq 3$ , we have*

$$\text{Com}(V_{-m}(sl(2)), M_{(2m)}) = \tilde{V}_{-2}(so(2m)),$$

where  $\tilde{V}_{-2}(so(2m))$  is a certain quotient of  $V^{-2}(so(2m))$ .

The vertex algebra  $V^{-2}(so(2m))$  is simple if and only if  $m = 3$ .

Then, by combining methods from [31] and [2], [3] we are able to construct new examples of affine Howe dual pairs in Corollary 8.2.

**Theorem 1.4.** *The following pairs of vertex algebras are affine Howe dual pairs:*

- (1)  $V_{-m}(sl(2))$  and  $\tilde{V}_{-2}(D_m)$  inside  $V_{-1/2}(C_{2m})$  for  $m \geq 3$ .
- (2)  $V_{-m}(sl(2))$  and  $\overline{V}_{-2}(A_{m-1})$  inside  $V_{-1}(A_{2m-1})$  for  $m \geq 5$ .

The case  $m = 4$  is related with a recent physics conjecture of D. Gaiotto [19] (cf. Remark 8.1).

**1.4. Examples of branching rules.** Even in the cases when the decomposition is semi-simple,  $V_k(\mathfrak{g})$  is not a simple current extension of  $\widetilde{V}(k, \mathfrak{k})$ , and studying such extensions is a very difficult problem in vertex algebra theory. Nevertheless, in some cases, explicit decompositions can be obtained by using combinatorial/number theoretic methods and applying certain  $q$ -series identities. In the present paper we present a decomposition of the vertex algebra  $M_{(3)}$  as a  $V_{-4}(sl(2)) \otimes V_{-3/2}(sl(2))$ -module. We identify all singular vectors in  $M_{(3)}$  and the characters of certain irreducible  $V_{-4}(sl(2)) \otimes V_{-3/2}(sl(2))$ -modules. Then, the decomposition of  $M_{(3)}$  is obtained in Theorem 9.2 as a consequence of the  $q$ -series identity from [26, Example 5.2].

**Theorem 1.5.**  *$M_{(3)}$  is a completely reducible  $V_{-3/2}(sl(2)) \otimes V_{-4}(sl(2))$ -module and the following decomposition holds*

$$(1.1) \quad M_{(3)} = \bigoplus_{\ell=0}^{\infty} \left( L_{\widehat{sl(2)}}\left(-\left(\frac{3}{2} + \ell\right)\Lambda_0 + \ell\Lambda_1\right) \bigotimes L_{\widehat{sl(2)}}\left(-(4 + 2\ell)\Lambda_0 + 2\ell\Lambda_1\right) \right).$$

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## 2. SETUP

**2.1. AP-criterion.** Let  $\mathfrak{g}$  be a simple Lie algebra. Let  $\mathfrak{h}$  be a Cartan subalgebra,  $\Delta$  the  $(\mathfrak{g}, \mathfrak{h})$ -root system,  $\Delta^+$  a set of positive roots and  $\rho$  the corresponding Weyl vector. Let  $(\cdot, \cdot)$  denote the normalized bilinear invariant form (i.e.,  $(\alpha, \alpha) = 2$  for any long root). The dual Coxeter number is denoted by  $h^\vee$ . This is half the eigenvalue of the Casimir element corresponding to  $(\cdot, \cdot)$  when acting on  $\mathfrak{g}$ . We shall write  $L_{\widehat{\mathfrak{g}}}(\eta)$  to denote the irreducible highest weight  $\mathfrak{g}$ -module of highest weight  $\eta$ ; when clear from the context, we write simply  $L(\eta)$ . We let  $V^k(\mathfrak{g})$  be the universal affine vertex algebra of level  $k$  and, if  $k + h^\vee \neq 0$ , we denote by  $V_k(\mathfrak{g})$  its simple quotient. We let  $L_{\widehat{\mathfrak{g}}}(\Lambda)$  denote the irreducible highest weight  $V^k(\mathfrak{g})$ -module as well as, when the action pushes down, the corresponding  $V_k(\mathfrak{g})$ -module. The notation  $\widetilde{L}_{\widehat{\mathfrak{g}}}(\Lambda)$  usually denotes a highest weight  $V^k(\mathfrak{g})$ -module, not necessary simple, with highest weight  $\Lambda$ . Similarly  $\widetilde{V}_k(\mathfrak{g})$  or  $\overline{V}_k(\mathfrak{g})$  usually denote a quotient of  $V^k(\mathfrak{g})$  which are possibly different than  $V_k(\mathfrak{g})$ .

Assume that  $\mathfrak{k}$  is a semisimple subalgebra of  $\mathfrak{g}$ . Then  $\mathfrak{k}$  decomposes as

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_t.$$

where  $\mathfrak{k}_1, \dots, \mathfrak{k}_t$  are the simple ideals of  $\mathfrak{k}$ . Let  $\mathfrak{p}$  be the orthocomplement of  $\mathfrak{k}$  w.r.t to  $(\cdot, \cdot)$  and let

$$\mathfrak{p} = \bigoplus_{i=1}^s \bigotimes_{j=1}^t L_{\mathfrak{k}}(\mu_i^j)$$

be its decomposition as a  $\mathfrak{k}$ -module.

Let  $(\cdot, \cdot)_j$  denote the normalized invariant bilinear form on  $\mathfrak{k}_j$ . We denote by  $\widetilde{V}(k, \mathfrak{k})$  the vertex subalgebra of  $V_k(\mathfrak{g})$  generated by  $\{x_{(-1)}\mathbf{1} \mid x \in \mathfrak{k}\}$ . Note that  $\widetilde{V}(k, \mathfrak{g})$  is an affine vertex

algebra, more precisely it is a quotient of  $\otimes V^{k_j}(\mathfrak{k}_j)$ , with the levels  $k_j$  determined by  $k$  and the ratio between  $(\cdot, \cdot)$  and  $(\cdot, \cdot)_j$ .

**Theorem.** (AP-criterion) [1]  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(\mathfrak{g})$  if and only if

$$(2.1) \quad \sum_{j=0}^t \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_j}{2(k_j + h_j^\vee)} = 1$$

for any  $i = 1, \dots, s$ .

**Remark 2.1.** We note that the AP-criterion allows conformal embeddings in non-simple vertex algebras. Let  $\omega_{\mathfrak{g}}$  (resp.  $\omega_{\mathfrak{k}}$ ) be the Sugawara Virasoro vector in  $V^k(\mathfrak{g})$  (resp.  $V^k(\mathfrak{k})$ ). It was proved in [1] that (2.1) implies that  $\omega_{\mathfrak{k}} - \omega_{\mathfrak{g}}$  belongs to the maximal ideal in  $V^k(\mathfrak{g})$ . So we automatically have conformal embedding of  $\mathcal{V}_k(\mathfrak{k})$  (= certain quotient of  $\tilde{V}(k, \mathfrak{k})$ ) in the vertex algebra

$$\frac{V^k(\mathfrak{g})}{V^k(\mathfrak{g}) \cdot (\omega_{\mathfrak{k}} - \omega_{\mathfrak{g}})},$$

which doesn't need to be simple. In present paper we identify a family of conformal embeddings in non-simple vertex algebras (see Corollaries 7.4 and 10.5).

We reformulate the criterion highlighting the dependence from the choice of the form  $(\cdot, \cdot)$ . As invariant symmetric form on  $\mathfrak{k}$  we choose  $(\cdot, \cdot)_{|\mathfrak{k} \times \mathfrak{k}}$ . Fix an orthonormal basis  $\{X_i\}$  of  $\mathfrak{k}_j$  and let  $C_{\mathfrak{k}_j} = \sum_i X_i^2$  be the corresponding Casimir operator. Let  $2g_j$  be the eigenvalue for the action of  $C_{\mathfrak{k}_j}$  on  $\mathfrak{k}_j$  and  $\gamma_i^j$  the eigenvalue of the action of  $C_{\mathfrak{k}_j}$  on  $L(\mu_i^j)$ . Then  $\tilde{V}_k(\mathfrak{k})$  is conformally embedded in  $V_k(\mathfrak{g})$  if and only if

$$(2.2) \quad \sum_{j=0}^t \frac{\gamma_i^j}{2(k + g_j)} = 1$$

for any  $i = 1, \dots, s$ .

**Corollary 2.1.** Assume  $\mathfrak{k}$  is simple, so that  $\mathfrak{k} = \mathfrak{k}_1$ . Then there is  $k \in \mathbb{C}$  such that  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(\mathfrak{g})$  if and only if  $C_{\mathfrak{k}}$  acts scalarly on  $\mathfrak{p}$ .

*Proof.* If  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(\mathfrak{g})$  then, by (2.2),  $\gamma_i^1 = 2(k + g_1)$  is independent of  $i$ . If  $C_{\mathfrak{k}}$  acts scalarly on  $\mathfrak{p}$ , then, solving (2.2) for  $k$ , one finds a level where, by AP-criterion, conformal embedding occurs.  $\square$

**2.2. Dynkin index.** Recall the notion of Dynkin index of a representation. Let  $\mathfrak{g}$  be a simple Lie algebra and  $V$  a finite dimensional  $\mathfrak{g}$ -module. Let  $tr_V$  be the trace form of  $V$ . It defines a nondegenerate bilinear symmetric invariant form on  $\mathfrak{g}$ , hence it is a multiple of the Killing form  $\kappa$ . The Dynkin index  $Ind_{\mathfrak{g}}(V)$  of  $V$  is the multiplicative factor between the two forms; more precisely

$$tr_V x^2 = Ind_{\mathfrak{g}}(V) \kappa(x, x), \quad x \in \mathfrak{g}.$$

A theorem of Dynkin states that if  $V$  is an irreducible  $\mathfrak{g}$ -module of highest weight  $\mu$  then

$$(2.3) \quad Ind_{\mathfrak{g}}(V) = \frac{\dim V}{\dim \mathfrak{g}} \frac{(\mu, \mu + 2\rho)}{(\theta, \theta + 2\rho)},$$

where  $(\cdot, \cdot)$  is any nondegenerate bilinear symmetric invariant form on  $\mathfrak{g}$  and  $\theta$  is the highest root of  $\mathfrak{g}$ . Let  $C_{\mathfrak{g}}^{(\cdot, \cdot)} = \sum_{i=1}^{\dim \mathfrak{g}} X_i^2$  be the Casimir element corresponding to the form  $(\cdot, \cdot)$ .

Let  $\lambda, \lambda'$  be the eigenvalues of  $C_{\mathfrak{g}}^{(\cdot, \cdot)}$  acting on  $V, \mathfrak{g}$ , respectively. Then  $Ind_{\mathfrak{g}}(V) = \frac{\dim V}{\dim \mathfrak{g}} \frac{\lambda}{\lambda'}$ . In particular, choosing  $(\cdot, \cdot) = tr_V$ , we find that  $\lambda = \frac{\dim \mathfrak{g}}{\dim V}$ . Hence with obvious notation

$$(2.4) \quad Ind_{\mathfrak{g}}(V) = \frac{1}{(\lambda')^{tr_V}}.$$

Let  $V$  be an irreducible  $\mathfrak{g}$ -module; denote by  $s(V)$  either  $sl(V)$ , or  $so(V)$  if  $V$  admits a nondegenerate bilinear symmetric invariant form.

**Lemma 2.2.** *Assume that  $C_{\mathfrak{g}}^{tr_V}$  acts with at most two eigenvalues  $\lambda_1^{tr_V}, \lambda_2^{tr_V}$  on  $s(V)$  according to the eigenspace decomposition  $s(V) = \mathfrak{g} \oplus \mathfrak{p}$ . Assume also that*

$$(2.5) \quad \lambda_1^{tr_V} > \lambda_2^{tr_V} + 1$$

*Then  $Ind_{\mathfrak{g}}(V) < 1$ .*

*Proof.* By (2.4)

$$Ind_{\mathfrak{g}}(V) = \frac{1}{\lambda_1^{tr_V}} < \frac{\lambda_1^{tr_V} - \lambda_2^{tr_V}}{\lambda_1^{tr_V}} = 1 - \frac{\lambda_2^{tr_V}}{\lambda_1^{tr_V}} < 1.$$

**2.3. Classification of maximal reductive subalgebras of a simple Lie algebra.** Recall that a Lie subalgebra  $\mathfrak{k}$  is said to be reductive in a Lie algebra  $\mathfrak{g}$  if the adjoint action of  $\mathfrak{k}$  on  $\mathfrak{g}$  is completely reducible.

**Definition 2.1.** Let  $\mathfrak{g}$  be a simple Lie algebra. We call a subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$  *maximally reductive* if it is maximal among subalgebras reductive in  $\mathfrak{g}$ .

Note that a maximally reductive algebra need not be a maximal subalgebra.

The next lemma shows that the classification of maximally reductive subalgebras can be reduced to Dynkin classification of maximal semisimple subalgebras (i.e., semisimple subalgebras which are maximal among all subalgebras) and the classification of maximally reductive equal rank subalgebras.

**Lemma 2.3.** (1). *Suppose that  $\mathfrak{k}$  is semisimple and maximally reductive in  $\mathfrak{g}$ . Then  $\mathfrak{k}$  is a maximal subalgebra of  $\mathfrak{g}$ .*

(2). *Suppose that  $\mathfrak{k}$  is maximally reductive in  $\mathfrak{g}$  and that it is not semisimple. Then  $\mathfrak{k}$  is an equal rank subalgebra.*

*Proof.* (1). If  $\mathfrak{k}$  is maximally reductive but it is not maximal, then it is contained in a non-semisimple maximal subalgebra  $\mathfrak{r}$  of  $\mathfrak{g}$ . By a theorem of Morozov (see e.g. [34]),  $\mathfrak{r}$  is a parabolic subalgebra. By [14, I, §6.8, Corollaire 2],  $\mathfrak{k}$  is a Levi component (=maximal semisimple subalgebra)  $\mathfrak{m}$  of  $\mathfrak{r}$ . Write  $\mathfrak{r} = \mathfrak{l} \oplus \mathfrak{n}$  with  $\mathfrak{l}$  reductive in  $\mathfrak{g}$  and  $\mathfrak{n}$  the nilradical of  $\mathfrak{r}$ . Then  $\mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}$ , with  $\mathfrak{z}$  the center of  $\mathfrak{l}$ . Since  $[\mathfrak{l}, \mathfrak{l}]$  is a Levi component for  $\mathfrak{r}$ , by the Levi-Malcev theorem,  $[\mathfrak{l}, \mathfrak{l}]$  and  $\mathfrak{k}$  are conjugated by an inner automorphism  $e^{ad(x)}, x \in \mathfrak{r}$ , hence  $\mathfrak{k} \subset \mathfrak{k} \oplus e^{ad(x)}(\mathfrak{z})$  and  $\mathfrak{k} \oplus e^{ad(x)}(\mathfrak{z})$  is reductive in  $\mathfrak{g}$ . A contradiction.

(2). Let  $\mathfrak{z} \neq \{0\}$  be the center of  $\mathfrak{k}$ . Then  $\mathfrak{z}$  acts semisimply on  $\mathfrak{g}$ , hence it is contained in a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . It follows that the centralizer of  $\mathfrak{z}$  in  $\mathfrak{g}$  is  $\mathfrak{l} + \mathfrak{h}$ ,  $\mathfrak{l}$  being the Levi component of a parabolic subalgebra of  $\mathfrak{g}$ . Since  $\mathfrak{k}$  is maximal reductive and  $\mathfrak{k} \subset \mathfrak{l} + \mathfrak{h}$ , we have  $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$ , hence it is an equal rank subalgebra.  $\square$



Combining Lemma 2.3 with Dynkin's classification of maximal semisimple subalgebras of a simple Lie algebra  $\mathfrak{g}$  (see [17], [18]), we obtain the following description of maximally reductive subalgebras.

Let  $\mathfrak{k}$  be a maximally reductive subalgebra of a simple Lie algebra  $\mathfrak{g}$ . Then, either  $\mathfrak{k}$  is equal rank subalgebra or, up to an inner automorphism of  $\mathfrak{g}$ , it falls in one of the following classes:

- (1) if  $\mathfrak{k}$  is simple, then either  $\mathfrak{k} = so(2n-1) \subset \mathfrak{g} = so(2n)$  or  $\mathfrak{k} \subset \mathfrak{g} = so(V), sp(V), sl(V)$  with  $V$  an irreducible  $\mathfrak{k}$ -module except for the cases listed in [17, Table 1];
- (2) if  $\mathfrak{g}$  is of classical type and  $\mathfrak{k}$  is non-simple then  $\mathfrak{k}$  is one of the subalgebras  $\mathfrak{k}$  in Table 2, in Section 4;
- (3) if  $\mathfrak{g}$  is of exceptional type, then  $\mathfrak{k}$  is one of the algebras in [18, Theorem 14.1] (see also [32]).

### 3. CONFORMAL EMBEDDINGS OF $\tilde{V}(k, \mathfrak{k})$ IN $V_k(\mathfrak{g})$ WITH $\mathfrak{k}$ SIMPLE AND $\mathfrak{g}$ OF CLASSICAL TYPE.

**3.1. Conformal embeddings of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(so(V))$  with  $\mathfrak{k}$  simple.** In this section we discuss the conformal embeddings of a simple Lie algebra  $\mathfrak{k}$  in  $so(V)$ . More specifically we consider an irreducible finite dimensional representation  $V$  of  $\mathfrak{k}$  admitting a  $\mathfrak{k}$ -invariant nondegenerate symmetric form  $\langle \cdot, \cdot \rangle$ .

From now on we will denote  $so(V, \langle \cdot, \cdot \rangle)$  simply as  $so(V)$ .

The conformal embeddings in  $V_k(so(V))$  with  $k \in \mathbb{Z}_+$  and  $V$  a finite dimensional representation are the subject of the symmetric space theorem. We recall this theorem in a vertex algebra formulation.

**Theorem (Symmetric space theorem, [20]).** *Assume that a compact group  $U$  with complexified Lie algebra  $\mathfrak{u}$  acts faithfully on a finite dimensional complex space  $V$  admitting a  $U$ -invariant symmetric nondegenerate form.*

*Then there is a conformal embedding of  $\tilde{V}(k, \mathfrak{u})$  in  $V_k(so(V))$  with  $k \in \mathbb{Z}_+$  if and only if  $k = 1$  and there is a Lie algebra structure on  $\mathfrak{r} = \mathfrak{u} \oplus V$  making  $\mathfrak{r}$  semisimple,  $(\mathfrak{r}, \mathfrak{u})$  is a symmetric pair, and a nondegenerate invariant form on  $\mathfrak{r}$  is given by the direct sum of a invariant form on  $\mathfrak{u}$  with the chosen  $U$ -invariant form on  $V$ .*

Note that the symmetric space theorem applies to a more general setting than ours as  $\mathfrak{u}$  need not be simple and  $V$  is not necessarily irreducible. Following [28], we let  $\nu : \mathfrak{k} \rightarrow so(V)$  be the representation map. We identify  $\mathfrak{k}$  and  $\nu(\mathfrak{k})$ . Let also  $\tau : \bigwedge^2 V \rightarrow so(V)$  be the  $\mathfrak{k}$ -equivariant isomorphism such that  $\tau(u)(v) = -2i(v)(u)$ , where  $i$  is the contraction map, extended to  $\bigwedge V$  as an odd derivation. More explicitly  $\tau^{-1}(X) = \frac{1}{4} \sum_i X(v_i) \wedge v_i$ , where  $\{v_i\}$  is an orthonormal basis of  $V$ .

Let  $(\cdot, \cdot)$  be the normalized invariant form on  $so(V)$ . Recall that  $(X, Y) = \frac{1}{2} tr_V(XY)$ . As invariant symmetric form on  $\mathfrak{k}$  we choose  $(\cdot, \cdot)|_{\mathfrak{k} \times \mathfrak{k}}$ . With notation as in Subsection 2.1, let  $g_1 = \frac{1}{2} \lambda_1^{(\cdot, \cdot)|_{\mathfrak{k} \times \mathfrak{k}}} = \frac{1}{2} \lambda_1^{\frac{1}{2} tr_V} = \lambda_1^{tr_V}$  be the eigenvalue for the action of  $C_{\mathfrak{k}}$  on  $\mathfrak{k}$ . Let  $\mathfrak{p}$  be the orthogonal complement of  $\mathfrak{k}$  in  $so(V)$ . Let  $p_{\mathfrak{k}}, p_{\mathfrak{p}}$  be the orthogonal projections of  $\bigwedge^2 V$  onto  $\mathfrak{k}$  and  $\mathfrak{p}$  respectively.

We would like to classify all irreducible representations  $V$  of  $\mathfrak{k}$  such that there is  $k \in \mathbb{C}$  such that  $\tilde{V}(k, \mathfrak{k})$  embeds conformally in  $V_k(so(V))$ . According to Corollary 2.1,  $C_{\mathfrak{k}}$  must act scalarly on  $\mathfrak{p}$ . Let  $\gamma = 2\lambda_2^{tr_V}$  be the eigenvalue for the action of  $C_{\mathfrak{k}}$  on  $\mathfrak{p}$ .

To study conformal embeddings at non-integrable levels, we reformulate the symmetric space theorem using a criterion due to Kostant. Recall the following notation from [28]: let

$\nu_* : \mathfrak{k} \rightarrow \bigwedge^2 V$  the unique Lie algebra homomorphism such that  $\tau \circ \nu_* = \nu$ . Let  $Cl(V)$  denote the Clifford algebra of  $(V, \langle \cdot, \cdot \rangle)$ . Extend  $\nu_*$  to a Lie algebra homomorphism  $\nu_* : \mathfrak{k} \rightarrow Cl(V)$ , hence to a homomorphism of associative algebras  $\nu_* : U(\mathfrak{k}) \rightarrow Cl(V)$ . Consider  $\nu_*(C_{\mathfrak{k}}) = \sum_i \nu_*(X_i)^2$ , where  $\{X_i\}$  is an orthogonal basis of  $\mathfrak{k}$ . Also recall that a pair  $(\mathfrak{k}, \nu)$  consisting of a Lie algebra  $\mathfrak{k}$  with a bilinear symmetric invariant form  $(\cdot, \cdot)_{\mathfrak{k}}$  and of a representation  $\nu : \mathfrak{k} \rightarrow so(V)$  is said to be *of Lie type* if there is a Lie algebra structure on  $\mathfrak{r} = \mathfrak{k} \oplus V$ , extending that of  $\mathfrak{k}$ , such that 1)  $[x, y] = \nu(x)y$ ,  $x \in \mathfrak{k}$ ,  $y \in V$  and 2) the bilinear form  $B_{\mathfrak{r}} = (\cdot, \cdot)_{\mathfrak{k}} \oplus \langle \cdot, \cdot \rangle$  is  $ad_{\mathfrak{r}}$ -invariant. In [28, Theorem 1.50] Kostant proved that  $(\mathfrak{k}, \nu)$  is a pair of Lie type if and only if there exists  $v \in (\bigwedge^3 V)^{\mathfrak{k}}$  such that

$$(3.1) \quad \nu_*(C_{\mathfrak{k}}) + v^2 \in \mathbb{C}.$$

Moreover he proved in [28, Theorem 1.59] that  $v$  can be taken to be 0 if and only if  $(\mathfrak{k} \oplus V, \mathfrak{k})$  is a symmetric pair.

We now come to the reformulation of the Symmetric Space Theorem.

**Proposition 3.1.** *Let  $\mathfrak{k}$  be a simple Lie algebra and  $V$  an irreducible finite-dimensional representation of  $\mathfrak{k}$  admitting a nondegenerate symmetric  $\mathfrak{k}$ -invariant form.*

*Then there is  $k \in \mathbb{Z}_+$  such that  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(so(V))$  if and only if*

$$(3.2) \quad \sum_i \tau^{-1}(X_i) \wedge \tau^{-1}(X_i) = 0,$$

where  $\{X_i\}$  is an orthogonal basis of  $\mathfrak{k}$ .

*Proof.* In [28, Proposition 1.37] it is shown that  $\nu_*(C_{\mathfrak{k}})$  might have nonzero components only in degrees 0, 4 w.r.t. the standard grading of  $\bigwedge V \cong Cl(V)$ . Recall that, if  $y \in V$  and  $w \in Cl(V)$ , then  $y \cdot w = y \wedge w + i(y)w$ , hence  $\nu_*(C_{\mathfrak{k}}) = \sum_i \tau^{-1}(X_i) \wedge \tau^{-1}(X_i) + a$  with  $a \in \bigwedge^0 V = \mathbb{C}$ . Thus, if (3.2) holds, then,  $\nu_*(C_{\mathfrak{k}})$  is a constant in  $Cl(V)$ . Thus equation (3.1) holds with  $v = 0$  and by Kostant's theorem quoted above ([28, Theorem 1.59]) we can conclude that  $(\mathfrak{k} \oplus V, \mathfrak{k})$  is a symmetric pair. Clearly we can assume that  $V$  is not the trivial one-dimensional  $\mathfrak{k}$ -module. Since  $V$  is irreducible, we see that  $V^{\mathfrak{k}} = \{0\}$ . Thus the hypothesis of [28, Theorem 1.61] are satisfied and  $\mathfrak{k} \oplus V$  is semisimple. By the symmetric space theorem, we know that  $\tilde{V}(1, \mathfrak{k})$  embeds conformally in  $V_1(so(V))$ .

Conversely, if  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(so(V))$  for some  $k \in \mathbb{Z}_+$ , then, by the symmetric space theorem, there is a Lie algebra structure on  $\mathfrak{r} = \mathfrak{k} \oplus V$  making  $\mathfrak{r}$  semisimple,  $(\mathfrak{r}, \mathfrak{k})$  is a symmetric pair, and a nondegenerate invariant form on  $\mathfrak{r}$  is given by the direct sum  $(\cdot, \cdot)_{|\mathfrak{k} \times \mathfrak{k}} \oplus \langle \cdot, \cdot \rangle$ . It follows from [28, Theorem 1.59] that (3.2) holds.  $\square$

**Remark 3.1.** Let us provide an interpretation of Proposition 3.1 and Kostant's results in the context of the representation theory of affine vertex algebras.

Recall that spin modules for the Lie algebra of type  $D_n$  are irreducible highest weight modules with highest weights  $\omega_n$  and  $\omega_{n-1}$  (resp.  $\omega_n$  if algebra is of type  $B_n$ ). Assume that  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(so(V))$  and that the spin  $so(V)$ -modules are modules for Zhu's algebra  $A(V_k(so(V)))$  for  $V_k(so(V))$ . Then one can show that there is a non-trivial homomorphism from  $A(V_k(so(V)))$  to the Clifford algebra  $Cl(V)$ . By using the Kostant criterion, we see that then (3.2) holds. Applying Proposition 3.1, we get  $k \in \mathbb{Z}_+$ .

So the only non-integrable candidates for realization of conformal embeddings are vertex algebras  $V_k(so(V))$  which do not admit embeddings of Zhu's algebra to the Clifford algebra, and therefore do not admit spin modules. Examples of such vertex algebras are  $V_{-2}(B_3)$  and



$V_{-2}(D_4)$  which provide models for realizations of non-integrable conformal embeddings (cf. [1], [37]).

Conformal embeddings at non-integrable levels do occur. Consider the following cases (cf. [1, Section 5]):

- $(B_3)_{\omega_3}$  :  $\mathfrak{k} = B_3$ ,  $V = L_{B_3}(\omega_3)$  be the irreducible 8-dimensional  $B_3$ -module. Then we have conformal embedding of  $V_{-2}(B_3)$  into  $V_{-2}(D_4) = V_{-2}(so(V))$ .  
 $(G_2)_{\omega_1}$  :  $\mathfrak{k} = G_2$ ,  $V = L_{G_2}(\omega_1)$  be the irreducible 7-dimensional  $G_2$ -module. Then we have conformal embedding of  $V_{-2}(G_2)$  into  $V_{-2}(B_3) = V_{-2}(so(V))$ .

**Lemma 3.2.** *Assume that  $\tilde{V}(k, \mathfrak{k})$  embeds conformally in  $V_k(so(V))$  with  $k \notin \mathbb{Z}_+$ . Then  $k = -2$  and*

$$\gamma = 2 \frac{(\dim V - 4) \dim \mathfrak{k}}{\dim so(V)}, \quad g_1 = \frac{1}{2}(\gamma + 4).$$

*Proof.* Denote by  $\lambda$  the eigenvalue of the action of  $C_{\mathfrak{k}}$  on  $V$ . Then

$$\begin{aligned} \sum_i \tau^{-1}(X_i) \wedge \tau^{-1}(X_i) &= \frac{1}{16} \sum_{i,j,r} X_i(v_j) \wedge v_j \wedge X_i(v_r) \wedge v_r \\ &= -\frac{1}{16} \sum_{i,j,r} X_i(v_j) \wedge X_i(v_r) \wedge v_j \wedge v_r = -\frac{1}{32} \sum_{i,j,r} X_i^2(v_j \wedge v_r) \wedge v_j \wedge v_r \\ &\quad + \frac{1}{32} \sum_{i,j,r} X_i^2(v_j) \wedge v_r \wedge v_j \wedge v_r + \frac{1}{32} \sum_{i,j,r} v_j \wedge X_i^2(v_r) \wedge v_j \wedge v_r \\ &= -\frac{1}{32} \sum_{i,j,r} X_i^2(v_j \wedge v_r) \wedge v_j \wedge v_r + \frac{\lambda}{16} \sum_{j,r} v_j \wedge v_r \wedge v_j \wedge v_r \\ &= -\frac{1}{32} \sum_{i,j,r} X_i^2(v_j \wedge v_r) \wedge v_j \wedge v_r. \end{aligned}$$

Write now

$$\begin{aligned} \sum_{i,j,r} X_i^2(v_j \wedge v_r) \wedge v_j \wedge v_r &= \sum_{i,j,r} X_i^2(p_{\mathfrak{k}}(v_j \wedge v_r)) \wedge v_j \wedge v_r + \sum_{i,j,r} X_i^2(p_{\mathfrak{p}}(v_j \wedge v_r)) \wedge v_j \wedge v_r \\ &= 2g_1 \sum_{j,r} p_{\mathfrak{k}}(v_j \wedge v_r) \wedge v_j \wedge v_r + \gamma \sum_{j,r} p_{\mathfrak{p}}(v_j \wedge v_r) \wedge v_j \wedge v_r \\ &= (2g_1 - \gamma) \sum_{j,r} p_{\mathfrak{k}}(v_j \wedge v_r) \wedge v_j \wedge v_r + \gamma \sum_{j,r} v_j \wedge v_r \wedge v_j \wedge v_r \\ &= (2g_1 - \gamma) \sum_{j,r} p_{\mathfrak{k}}(v_j \wedge v_r) \wedge v_j \wedge v_r. \end{aligned}$$

We now compute  $p_{\mathfrak{k}}(v_j \wedge v_r)$  explicitly. We extend  $\langle \cdot, \cdot \rangle$  to  $\bigwedge^2 V$  by determinants. Set  $u_i = \tau^{-1}(X_i)$  and note that

$$\begin{aligned} \langle u_i, u_j \rangle &= \frac{1}{16} \sum_{r,s} \det \begin{pmatrix} \langle X_i(v_r), X_j(v_s) \rangle & \langle X_i(v_r), v_s \rangle \\ \langle v_r, X_j(v_s) \rangle & \langle v_r, v_s \rangle \end{pmatrix} \\ &= \frac{1}{16} \sum_r (\langle X_i(v_r), X_j(v_r) \rangle - \langle v_r, X_j(X_i(v_r)) \rangle) \\ &= -\frac{1}{8} \sum_r \langle X_j(X_i(v_r)), v_r \rangle = -\frac{1}{8} \text{tr}(X_j X_i). \end{aligned}$$

Recall that  $(X, Y) = \frac{1}{2} \text{tr}_V(XY)$ , hence  $\text{tr}_V(X_j X_i) = 2(X_j, X_i)$ , so that  $\langle u_i, u_j \rangle = -\frac{1}{4} \delta_{ij}$ ; therefore

$$\begin{aligned} p_{\mathfrak{k}}(v_j \wedge v_r) &= -4 \sum_t \langle v_j \wedge v_r, u_t \rangle u_t = -\sum_{t,k} \langle v_j \wedge v_r, X_t(v_k) \wedge v_k \rangle u_t \\ &= -\sum_{t,k} \det \begin{pmatrix} \langle v_j, X_t(v_k) \rangle & \langle v_j, v_k \rangle \\ \langle v_r, X_t(v_k) \rangle & \langle v_r, v_k \rangle \end{pmatrix} u_t = -2 \sum_t \langle v_j, X_t(v_r) \rangle u_t. \end{aligned}$$

Substituting we find that

$$\begin{aligned} \sum_i \tau^{-1}(X_i) \wedge \tau^{-1}(X_i) &= \frac{2g_1 - \gamma}{16} \sum_{j,r,t} \langle v_j, X_t(v_r) \rangle u_t \wedge v_j \wedge v_r \\ &= \frac{2g_1 - \gamma}{64} \sum_{j,r,t,k} \langle v_j, X_t(v_r) \rangle X_t(v_k) \wedge v_k \wedge v_j \wedge v_r \\ &= \frac{2g_1 - \gamma}{64} \sum_{r,t,k} X_t(v_k) \wedge v_k \wedge X_t(v_r) \wedge v_r \\ &= \frac{2g_1 - \gamma}{4} \sum_t \tau^{-1}(X_t) \wedge \tau^{-1}(X_t). \end{aligned}$$

By Proposition 3.1, since  $k \notin \mathbb{Z}_+$ , we have  $\sum_t \tau^{-1}(X_t) \wedge \tau^{-1}(X_t) \neq 0$ , thus

$$(3.3) \quad 2g_1 - \gamma = 4$$

Recall that, by AP-criterion,  $\frac{\gamma}{2(k+g_1)} = 1$  so  $k = \frac{\gamma - 2g_1}{2} = -2$ .

Finally, as central charges must be equal, we obtain

$$\frac{\dim \mathfrak{so}(V)}{\dim V - 4} = \frac{\dim \mathfrak{k}}{g_1 - 2},$$

so

$$(3.4) \quad g_1 = \frac{(\dim V - 4) \dim \mathfrak{k}}{\dim \mathfrak{so}(V)} + 2, \quad \gamma = 2 \frac{(\dim V - 4) \dim \mathfrak{k}}{\dim \mathfrak{so}(V)}.$$

**Proposition 3.3.** *The conformal embeddings  $(B3)_{\omega_3}, (G2)_{\omega_1}$  are the unique conformal embeddings of a simple Lie algebra in  $V_{-2}(\mathfrak{so}(V))$  with  $V$  irreducible.*

*Proof.* We will proceed to inspecting all cases; it will eventually turn out that cases  $(B3)_{\omega_3}, (G2)_{\omega_1}$  are the only occurring. Let  $\mu$  be a highest weight occurring in  $\mathfrak{p}$ . First observe  $\mu$  belongs to the root lattice. Indeed, recall that  $V$  is irreducible and let  $\lambda$  be its highest weight for some choice of a positive set of roots. Then, since  $\bigwedge^2 V$  is a submodule of  $V \otimes V$ , the

weights occurring in  $\bigwedge^2 V$  are of type  $2\lambda + \eta$  with  $\eta$  in the root lattice. Since  $\theta$  appears in the decomposition of  $\bigwedge^2 V$  as a  $\mathfrak{k}$ -module, we have that there is  $\xi$  in the root lattice such that  $2\lambda + \xi = \theta$ , thus  $2\lambda$  is in the root lattice. Thus any weight of  $\bigwedge^2 V$  is in the root lattice.

Now, by (3.4), we have

$$\lambda_1^{tr_V} - \lambda_2^{tr_V} = \frac{1}{2}(2g_1 - \gamma) = 2.$$

In turn, by Lemma 2.4 we have  $\text{Ind}_{\mathfrak{k}} V_{\mu} < 1$  for any irreducible component  $V_{\mu}$  of  $\mathfrak{p}$ . Inspecting the list of irreducible representations with Dynkin index less than one given in [8, Table 1] and selecting the weights in the root lattice we have that  $\mu$  is necessarily a fundamental weight. More precisely, using Bourbaki's notation for Dynkin diagrams, we have :

$\mathfrak{k}$	$B_n$	$C_n$	$F_4$	$G_2$
$\mu$	$\omega_1$	$\omega_2$	$\omega_4$	$\omega_1$

Table 1

It follows that every irreducible component of  $\mathfrak{p}$  must be of the form  $L(\mu)$  with  $\mu$  as in Table 1. Hence, as  $\mathfrak{k}$ -modules

$$(3.5) \quad so(V) = \mathfrak{k} \oplus pL(\mu), \quad p \geq 1.$$

Let  $(\cdot, \cdot)_n$  be the normalized form of  $\mathfrak{k}$ ; setting  $v = \dim V$ , one has

$$(3.6) \quad \lambda_2^{(\cdot, \cdot)_n} = \frac{2 \dim \mathfrak{k} h^{\vee}(v-4)}{\dim \mathfrak{k}(v-4) + v(v-1)}.$$

Let us discuss type  $G_2$ . Taking dimensions in (3.5) one has  $v(v-1)/2 = 14 + 7p$ . Solving for  $v$ , the only positive solution is  $v = (1 + \sqrt{113 + 56p})/2$ . Now we find, using (3.6), the values of  $p$  such that  $\lambda_2^{(\cdot, \cdot)_n} = (\omega_1, \omega_1 + 2\rho) = 4$ . One gets  $p = 1$  and  $p = 2$ . In the former case one obtains  $v = 7$ , which corresponds to  $(G_2)_{\omega_1}$ ; in the latter case one gets  $v = 8$ , which is excluded since no irreducible representation of  $G_2$  has dimension 8. Type  $F_4$  is treated similarly: one gets a quadratic equation in  $p$  which has no integral solution.

In type  $B_n$  it is convenient to use a slightly different strategy. One checks, by direct computation, that  $\lambda_1^{(\cdot, \cdot)_n} = 4n - 2, \lambda_2^{(\cdot, \cdot)_n} = 2n, \gamma = \frac{4n}{n-1}$ . Using the rightmost formula in (3.4), one obtains that  $v = \frac{2(3n+p-2)}{n-1} = 4 + \frac{2(n+p)}{n-1}$ , so that  $p = q(n-1) - 1$  and  $v = 2(3+q)$ . Now equation  $v(v-1)/2 - \dim so(V) = 0$  reads  $n(1+p) - 2n^2(1+p) + 2(8+6p+p^2) = 0$  or  $n - 2n^2 + 2(5+p + \frac{3}{1+p}) = 0$ . This implies  $p \in \{1, 2, 5\}$ , and one gets an integer value for  $n$  only for  $p = 1$ . More precisely, in that case  $n = 3$  and  $v = 8$ , so we are in case  $(B_3)_{\omega_3}$ . Type  $C_n$  is easier and treated similarly.

**3.2. Conformal embeddings of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(sp(V))$  with  $\mathfrak{k}$  simple.** In this section we discuss the conformal embeddings of a simple Lie algebra  $\mathfrak{k}$  in  $sp(V)$ . More specifically we consider an irreducible finite dimensional representation  $V$  of  $\mathfrak{k}$  admitting a  $\mathfrak{k}$ -invariant nondegenerate symplectic form  $\langle \cdot, \cdot \rangle$ .

From now on we will denote  $sp(V, \langle \cdot, \cdot \rangle)$  simply as  $sp(V)$ . We let  $\nu : \mathfrak{k} \rightarrow sp(V)$  be the representation map. Let also  $\tau : S^2(V) \rightarrow sp(V)$  be the linear isomorphism such that  $\tau(u)(v) = i(v)(u)$ , where  $i$  is the contraction map, extended to  $S^2(V)$  as an even derivation. More explicitly  $\tau^{-1}(X) = \frac{1}{2} \sum_i X(v_i) v^i$ , where  $\{v_i\}$  is a basis of  $V$  and  $\{v^i\}$  is the corresponding dual basis (i.e.  $\langle v_i, v^j \rangle = \delta_{ij}$ ).

Recall that a Lie superalgebra  $\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_1$  is said to be of Riemannian type if it admits a nondegenerate supersymmetric even invariant form.

**Proposition 3.4.** [29] *Let  $\mathfrak{k}$  be a Lie algebra admitting a nondegenerate ad-invariant symmetric bilinear form  $(\cdot, \cdot)$  and  $V$  an irreducible finite-dimensional representation of  $\mathfrak{k}$  admitting a nondegenerate symplectic  $\mathfrak{k}$ -invariant form  $\langle \cdot, \cdot \rangle$ .*

*Then the space  $\mathfrak{k} \oplus V$  admits a Lie superalgebra structure such that the pair  $(\mathfrak{k} \oplus V, (\cdot, \cdot) \oplus \langle \cdot, \cdot \rangle)$  is of Riemannian type if and only if*

$$(3.7) \quad \sum_i \tau^{-1}(X_i)^2 = 0,$$

where  $\{X_i\}$  is an orthogonal basis of  $\mathfrak{k}$ .

Let us return to our situation: we are assuming that  $\mathfrak{k}$  is a simple Lie algebra and  $V$  is an irreducible finite-dimensional representation of  $\mathfrak{k}$  admitting a nondegenerate invariant symplectic form. Let  $(\cdot, \cdot)$  be the normalized invariant bilinear form on  $\mathfrak{k}$ . Recall that  $(X, Y) = \text{tr}_V(XY)$ . For an invariant symmetric bilinear form on  $\mathfrak{k}$  we choose  $(\cdot, \cdot)|_{\mathfrak{k} \times \mathfrak{k}}$ . Let  $\mathfrak{p}$  be the orthogonal complement of  $\nu(\mathfrak{k})$  in  $\mathfrak{sp}(V)$ . Then  $g_1 = \frac{1}{2}\lambda_1^{\text{tr}V}$ . By Corollary 2.1,  $C_{\mathfrak{k}}$  acts scalarly on  $\mathfrak{p}$  and the eigenvalue is  $\gamma = \lambda_2^{\text{tr}V}$ .

**Lemma 3.5.** *Assume that  $\tilde{V}(k, \mathfrak{k})$  embeds conformally in  $V_k(\mathfrak{sp}(V))$ . Then either  $\mathfrak{k} + V$  admits the structure of a Lie superalgebra of Riemannian type, or  $k = 1$  and*

$$\gamma = \frac{(\dim V + 4) \dim \mathfrak{k}}{\dim \mathfrak{sp}(V)}, \quad g_1 = \frac{1}{2}\gamma - 1.$$

*Proof.* Let  $p_{\mathfrak{k}}, p_{\mathfrak{p}}$  be the projections of  $S^2(V)$  onto  $\mathfrak{k}$  and  $\mathfrak{p}$  respectively corresponding to the direct sum  $S^2(\mathfrak{k}) = \mathfrak{k} \oplus \mathfrak{p}$ .

Write explicitly

$$\begin{aligned} \sum_i (\tau^{-1}(X_i))^2 &= \frac{1}{4} \sum_{i,j,r} X_i(v_j) X_i(v_r) v^j v^r \\ &= \frac{1}{8} \sum_{i,j,r} X_i^2(v_j v_r) v^j v^r - \frac{1}{8} \sum_{i,j,r} X_i^2(v_j) v_r v^j v^r - \frac{1}{8} \sum_{i,j,r} v_j X_i^2(v_r) v^j v^r \\ &= \frac{1}{8} \sum_{i,j,r} X_i^2(v_j v_r) v^j v^r - \frac{\lambda}{4} \sum_{j,r} v_j v_r v^j v^r. \end{aligned}$$

Here  $\lambda$  is the eigenvalue of the action of  $C_{\mathfrak{k}}$  on  $V$ . Noting that  $\sum_i v_i v^i = 0$  we obtain  $\sum_{j,r} v_j v_r v^j v^r = 0$ , hence  $\sum_i (\tau^{-1}(X_i))^2 = \frac{1}{8} \sum_{i,j,r} X_i^2(v_j v_r) v^j v^r$ . Write now

$$\begin{aligned} \sum_{i,j,r} X_i^2(v_j v_r) v^j v^r &= \sum_{i,j,r} X_i^2(p_{\mathfrak{k}}(v_j v_r)) v^j v^r + \sum_{i,j,r} X_i^2(p_{\mathfrak{p}}(v_j v_r)) v^j v^r \\ &= 2g_1 \sum_{j,r} p_{\mathfrak{k}}(v_j v_r) v^j v^r + \gamma \sum_{j,r} p_{\mathfrak{p}}(v_j v_r) v^j v^r \\ &= (2g_1 - \gamma) \sum_{j,r} p_{\mathfrak{k}}(v_j v_r) v^j v^r + \gamma \sum_{j,r} v_j v_r v^j v^r \\ &= (2g_1 - \gamma) \sum_{j,r} p_{\mathfrak{k}}(v_j v_r) v^j v^r. \end{aligned}$$

We now compute  $p_{\mathfrak{k}}(v_j v_r)$  explicitly. We extend  $\langle \cdot, \cdot \rangle$  to  $S^2(V)$  by restricting to symmetric tensors the form  $\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle$  on  $V \otimes V$ .

Set  $u_i = \tau^{-1}(X_i)$  and note that

$$\begin{aligned}\langle u_i, u_j \rangle &= \frac{1}{8} \sum_{r,s} (\langle X_i(v_r), X_j(v_s) \rangle \langle v^r, v^s \rangle + \langle X_i(v_r), v^s \rangle \langle v^r, X_j(v_s) \rangle) \\ &= \frac{1}{8} \sum_{r,s} (-\langle X_j X_i(v_r), v_s \rangle \langle v^r, v^s \rangle + \langle X_i(v_r), v^s \rangle \langle v^r, X_j(v_s) \rangle) \\ &= \frac{1}{8} \sum_r (\langle v^r, X_j X_i(v_r) \rangle + \langle v^r, X_j X_i(v_r) \rangle) = -\frac{1}{4} \text{tr}(X_j X_i).\end{aligned}$$

Recall that  $\text{tr}(X_j X_i) = (X_j, X_i)$ , hence  $\langle u_i, u_j \rangle = -\delta_{ij} \frac{1}{4}$ ; therefore

$$\begin{aligned}p_{\mathfrak{k}}(v_j v_r) &= -4 \sum_t \langle v_j v_r, u_t \rangle u_t = -2 \sum_{t,k} \langle v_j v_r, X_t(v_k) v^k \rangle u_t \\ &= -\sum_{t,k} (\langle v_j, X_t(v_k) \rangle \langle v_r, v^k \rangle + \langle v_r, X_t(v_k) \rangle \langle v_j, v^k \rangle) u_t = -2 \sum_t \langle v_j, X_t(v_r) \rangle u_t.\end{aligned}$$

Upon substituting, we find that

$$\begin{aligned}\sum_i (\tau^{-1}(X_i))^2 &= -\frac{2g_1 - \gamma}{4} \sum_{j,r,t} \langle v_j, X_t(v_r) \rangle u_t v^j v^r \\ &= -\frac{2g_1 - \gamma}{8} \sum_{j,r,t,k} \langle v_j, X_t(v_r) \rangle X_t(v_k) v^k v^j v^r \\ &= -\frac{2g_1 - \gamma}{8} \sum_{r,t,k} X_t(v_k) v^k X_t(v_r) v^r \\ &= -\frac{2g_1 - \gamma}{2} \sum_t (\tau^{-1}(X_t))^2.\end{aligned}$$

If  $\sum_t (\tau^{-1}(X_t))^2 = 0$ , then, by Proposition 3.4,  $\mathfrak{k} + V$  has the structure of a Lie superalgebra of Riemannian type. Otherwise, we must have that

$$(3.8) \quad 2g_1 - \gamma = -2.$$

Recall that, by AP-criterion,  $\frac{\gamma}{2(k+g_1)} = 1$  so  $k = \frac{\gamma - 2g_1}{2} = 1$ . Finally, as central charges must be equal, we obtain

$$\frac{\dim sp(V)}{\dim V/2 + 2} = \frac{\dim \mathfrak{k}}{g_1 + 1},$$

so

$$(3.9) \quad g_1 = \frac{(\dim V/2 + 2) \dim \mathfrak{k}}{\dim sp(V)} - 2, \quad \gamma = \frac{(\dim V + 4) \dim \mathfrak{k}}{\dim sp(V)}.$$

**Proposition 3.6.** *Let  $\mathfrak{k}$  be a simple Lie algebra and  $V$  an irreducible symplectic representation of  $\mathfrak{k}$ . If  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(sp(V))$ , then either  $k = 1$  or  $\mathfrak{k} = sp(V)$ .*

*Proof.* Assume  $k \neq 1$ . By Lemma 3.5,  $\mathfrak{k} + V$  admits the structure of a Lie superalgebra of Riemannian type. We claim that  $\mathfrak{k}' = \mathfrak{k} + V$  is a simple Lie superalgebra. Assume  $\mathfrak{i}$  is a graded ideal of  $\mathfrak{k}'$ ; then  $\mathfrak{i} = (\mathfrak{i} \cap \mathfrak{k}) \oplus (\mathfrak{i} \cap V)$  and since  $\mathfrak{k}$  is simple and  $V$  is irreducible, either  $\mathfrak{k} = \mathfrak{i}$  or  $\mathfrak{i} = V$ . In the former case,  $[\mathfrak{i}, V] \subset \mathfrak{i} = \mathfrak{k}$  and  $[\mathfrak{i}, V] = [\mathfrak{k}, V] \subset V$ . Hence  $[\mathfrak{k}, V] = 0$ , so that  $V$  is the 1-dimensional trivial representation, which is not symplectic. In the latter case

then  $[V, V] \subset \mathfrak{k}$  and  $[V, V] \subset \mathfrak{i} = V$  hence  $[V, V] = 0$ . By invariance of the bilinear form of  $\mathfrak{k}'$ , we have that  $([\mathfrak{k}, V], V) = (\mathfrak{k}, [V, V]) = 0$ ; by non-degeneracy  $[\mathfrak{k}, V] = 0$ , hence again  $V$  is the trivial representation. By Kac classification of finite dimensional simple Lie superalgebras [21], there are three simple Lie superalgebras  $\mathfrak{l}$ , for which  $\mathfrak{l}_0$  is simple:  $B(0, n)$  and the strange ones,  $P(n)$  and  $Q(n)$ . But the strange ones have no even non-zero invariant bilinear form. We are left with  $B(0, n)$ , which corresponds to  $\mathfrak{k}' = sp(V) \oplus V$ .  $\square$

**Remark 3.2.** In the case when  $\mathfrak{k}$  is semisimple but not simple, we can have conformal embeddings in  $V_k(sp(V))$  when  $k = -1/2$ . One example is  $\mathfrak{k} = sl(2) \times so(m)$  and  $V$  is  $2m$ -dimensional irreducible  $\mathfrak{k}$ -module isomorphic to the tensor product  $V_{sl(2)}(\omega_1) \otimes V_{so(m)}(\omega_1)$ . Such conformal embeddings will be studied in Sections 4 and 7.

**3.3. Conformal embeddings of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(sl(V))$  with  $\mathfrak{k}$  simple.** In this section we discuss the conformal embeddings of a simple Lie algebra  $\mathfrak{k}$  in  $sl(V)$ . More specifically we consider an irreducible finite dimensional representation  $V$  of  $\mathfrak{k}$ .

Let  $\tau : V \otimes V^* \rightarrow gl(V)$  be the linear isomorphism such that  $\tau(v \otimes \lambda)(w) = \lambda(w)v$ . More explicitly  $\tau^{-1}(X) = \sum_i X(v_i) \otimes v^i$ , where  $\{v_i\}$  is a basis of  $V$  and  $\{v^i\}$  is the corresponding dual basis in  $V^*$ . Let also  $\tau^*$  be the corresponding map from  $V^* \otimes V$  to  $gl(V^*)$ . Restricting  $\tau^{-1}$  to  $sl(V)$  we obtain a monomorphism of  $\mathfrak{k}$ -modules  $\tau^{-1} : sl(V) \rightarrow V \otimes V^*$ .

Assume that  $\tilde{V}(k, \mathfrak{k})$  embeds conformally in  $V_k(sl(V))$ . We choose  $(\cdot, \cdot)$  to be the normalized invariant form on  $sl(V)$ . Recall that  $(X, Y) = tr_V(XY)$ . As an invariant symmetric form on  $\mathfrak{k}$  we choose  $(\cdot, \cdot)_{|\mathfrak{k} \times \mathfrak{k}}$ , so that  $g_1 = \frac{1}{2}\lambda_1^{tr_V}$  and  $\gamma = \lambda_2^{tr_V}$ .

**Lemma 3.7.** *Assume that  $\tilde{V}(k, \mathfrak{k})$  embeds conformally in  $V_k(sl(V))$ . Then either  $\mathfrak{k} = sl(V)$  or  $k = \pm 1$ . Moreover,*

$$(3.10) \quad g_1 = \frac{\dim \mathfrak{k}}{\dim V \mp 1} \mp 1, \quad \gamma = \frac{2 \dim \mathfrak{k}}{\dim V \mp 1},$$

where we take the upper sign for  $k = 1$ , the lower sign for  $k = -1$ .

*Proof.* Let  $p_{\mathfrak{k}}, p_{\mathfrak{p}}, p_{CI_V}$  be the projections of  $V \otimes V^*$  onto  $\mathfrak{k}, \mathfrak{p}$  and  $CI_V$  respectively corresponding to the direct sum  $V \otimes V^* = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathbb{C} \sum_i v_i \otimes v^i$ . Write explicitly

$$\begin{aligned} \sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i) &= \sum_{i,j,r} X_i(v_j) \otimes v^j \otimes X_i(v^r) \otimes v_r \\ &= \sum_{i,j,r} \sigma_{23}(X_i(v_j) \otimes X_i(v^r) \otimes v^j \otimes v_r) \\ &= \frac{1}{2} \sum_{i,j,r} \sigma_{23}(X_i^2(v_j \otimes v^r) \otimes v^j \otimes v_r) - \frac{1}{2} \sum_{i,j,r} X_i^2(v_j) \otimes v^j \otimes v^r \otimes v_r \\ &\quad - \frac{1}{2} \sum_{i,j,r} v_j \otimes v^j \otimes X_i^2(v^r) \otimes v_r \\ &= \frac{1}{2} \sum_{i,j,r} \sigma_{23}(X_i^2(v_j \otimes v^r) \otimes v^j \otimes v_r) - \lambda \sum_{j,r} v_j \otimes v^j \otimes v^r \otimes v_r. \end{aligned}$$



Here  $\lambda$  is the eigenvalue of the action of  $C_{\mathfrak{k}}$  on  $V$ . Write now

$$\begin{aligned} \sum_{i,j,r} \sigma_{23}(X_i^2(v_j \otimes v^r) \otimes v^j \otimes v_r) &= \sum_{i,j,r} \sigma_{23}(X_i^2(p_{\mathfrak{k}}(v_j \otimes v^r)) \otimes v^j \otimes v_r) \\ &+ \sum_{i,j,r} \sigma_{23}X_i^2(p_{\mathfrak{p}}(v_j \otimes v^r)) \otimes v^j \otimes v_r + \sum_{i,j,r} \sigma_{23}X_i^2(p_{\mathbb{C}\sum v_i \otimes v^i}(v_j \otimes v^r)) \otimes v^j \otimes v_r \\ &= 2g_1 \sum_{j,r} \sigma_{23}(p_{\mathfrak{k}}(v_j \otimes v^r) \otimes v^j \otimes v_r) + \gamma \sum_{j,r} \sigma_{23}(p_{\mathfrak{p}}(v_j \otimes v^r) \otimes v^j \otimes v_r). \end{aligned}$$

In the last equality we used the fact that  $C_{\mathfrak{k}}$  acts trivially on  $\sum v_i \otimes v^i$ . Thus

$$\begin{aligned} \sum_{i,j,r} \sigma_{23}(X_i^2(v_j \otimes v^r) \otimes v^j \otimes v_r) &= (2g_1 - \gamma) \sum_{j,r} \sigma_{23}(p_{\mathfrak{k}}(v_j \otimes v^r) \otimes v^j \otimes v_r) \\ &+ \gamma \sum_{j,r} \sigma_{23}(p_{sl(V)}(v_j \otimes v^r) \otimes v^j \otimes v_r) \end{aligned}$$

To compute  $p_{sl(V)}(v_j \otimes v^r)$  we use the trace form on  $V \otimes V^*$ , so that, if  $\langle v \otimes \lambda, w \otimes \mu \rangle = \mu(v)\lambda(w)$ , then

$$p_{sl(V)}(X) = X - \frac{\langle X, \sum v_i \otimes v^i \rangle}{\langle \sum v_i \otimes v^i, \sum v_i \otimes v^i \rangle} \sum_i v_i \otimes v^i.$$

In particular

$$p_{sl(V)}(v_j \otimes v^r) = v_j \otimes v^r - \frac{\delta_{jr}}{\dim V} \sum_i v_i \otimes v^i.$$

We now compute  $p_{\mathfrak{k}}(v_j \otimes v^r)$  explicitly. Set  $u_i = \tau^{-1}(X_i)$  and note that

$$\langle u_i, u_j \rangle = \sum_{r,s} v^s(X_i(v_r))v^r(X_j(v_s)) = \sum_s v^s(X_j X_i(v_s)) = tr(X_j X_i).$$

Recall that  $tr(X_j X_i) = (X_j, X_i)$ , hence  $\langle u_i, u_j \rangle = \delta_{ij}$ ; therefore

$$(3.11) \quad p_{\mathfrak{k}}(v_j \otimes v^r) = \sum_t \langle v_j \otimes v^r, u_t \rangle u_t = \sum_{t,k} \langle v_j \otimes v^r, X_t(v_k) \otimes v^k \rangle u_t = \sum_t v^r(X_t(v_j)) u_t.$$

Substituting we find that

$$\begin{aligned}
& \sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i) \\
&= \frac{1}{2}(2g_1 - \gamma) \sum_{j,r} \sigma_{23}(p_{\mathfrak{k}}(v_j \otimes v^r) \otimes v^j \otimes v_r) + \frac{1}{2}\gamma \sum_{j,r} \sigma_{23}(p_{sl(V)}(v_j \otimes v^r) \otimes v^j \otimes v_r) \\
&\quad - \lambda \sum_{j,r} v_j \otimes v^j \otimes v^r \otimes v_r \\
&= \frac{1}{2}(2g_1 - \gamma) \sum_{j,r,t} \sigma_{23}(v^r(X_t(v_j))) u_t \otimes v^j \otimes v_r + \frac{1}{2}\gamma \sum_{j,r} v_j \otimes v^j \otimes v^r \otimes v_r \\
&\quad - \frac{1}{2}\gamma \sum_{j,i} \frac{1}{\dim V} v_i \otimes v^j \otimes v^i \otimes v_j - \lambda \sum_{j,r} v_j \otimes v^j \otimes v^r \otimes v_r \\
&= -\frac{1}{2}(2g_1 - \gamma) \sum_{r,t,i} X_t(v_i) \otimes X_t(v^r) \otimes v^i \otimes v_r + \frac{1}{2}\gamma \sum_{j,r} v_j \otimes v^j \otimes v^r \otimes v_r \\
&\quad - \frac{1}{2}\gamma \sum_{j,i} \frac{1}{\dim V} v_i \otimes v^j \otimes v^i \otimes v_j - \lambda \sum_{j,r} v_j \otimes v^j \otimes v^r \otimes v_r.
\end{aligned}$$

Now write

$$\begin{aligned}
& \sum_{t,r,i} X_t(v_i) \otimes X_t(v^r) \otimes v^i \otimes v_r = \frac{1}{2} \sum_{t,r,i} X_t^2(v_i \otimes v^r) \otimes v^i \otimes v_r - \lambda \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r \\
&= \frac{1}{2}(2g_1 - \gamma) \sum_{r,i} p_{\mathfrak{k}}(v_i \otimes v^r) \otimes v^i \otimes v_r + \frac{1}{2}\gamma \sum_{r,i} p_{sl(V)}(v_i \otimes v^r) \otimes v^i \otimes v_r \\
&\quad - \lambda \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r \\
&= \frac{1}{2}(2g_1 - \gamma) \sum_{r,i,t} v^r(X_t(v_i)) u_t \otimes v^i \otimes v_r + \frac{1}{2}\gamma \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r \\
&\quad - \frac{1}{2}\gamma \sum_{r,i,s} \frac{\delta_{ri}}{\dim V} v_s \otimes v^s \otimes v^i \otimes v_r - \lambda \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r \\
&= -\frac{1}{2}(2g_1 - \gamma) \sum_{r,t,j} X_t(v_j) \otimes v^j \otimes X_t(v^r) \otimes v_r + \frac{1}{2}\gamma \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r \\
&\quad - \frac{1}{2}\gamma \sum_{r,s} \frac{1}{\dim V} v_s \otimes v^s \otimes v^r \otimes v_r - \lambda \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r.
\end{aligned}$$

Summing up we obtain

$$\begin{aligned}
(3.12) \quad \sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i) &= \frac{(2g_1 - \gamma)^2}{4} \sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i) \\
&\quad \left( -\frac{(2g_1 - \gamma)\gamma}{4} + \frac{(2g_1 - \gamma)\lambda}{2} - \frac{\gamma}{2 \dim V} \right) \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r \\
&\quad + \left( \frac{(2g_1 - \gamma)\gamma}{4 \dim V} + \frac{1}{2}\gamma - \lambda \right) \sum_{r,s} v_s \otimes v^s \otimes v^r \otimes v_r.
\end{aligned}$$

Consider the natural identifications

$$\iota : V \otimes V^* \otimes V^* \otimes V \rightarrow gl(V) \oplus gl(V^*) \rightarrow gl(V) \otimes gl(V)^* \rightarrow End(gl(V))$$

Also recall that  $gl(V) = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathbb{C}I_V$ . We claim that

$$(3.13) \quad \iota\left(\sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i)\right) = -p_{\mathfrak{k}},$$

$$(3.14) \quad \iota\left(\sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r\right) = I_{gl(V)},$$

$$(3.15) \quad \iota\left(\sum_{r,s} v_s \otimes v^s \otimes v^r \otimes v_r\right) = (\dim V)p_{\mathbb{C}I_V}.$$

We check (3.13):

$$\begin{aligned}
\iota\left(\sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i)\right)(v_r \otimes v^s) &= \sum_{i,h,k} (X_i(v^h) \otimes v_h)(v_r \otimes v^s)X_i(v_k) \otimes v^k \\
&= -\sum_{i,h,k} v^h(X_i(v_r))v^s(v_h)X_i(v_k) \otimes v^k \\
&= -\sum_{i,k} v^s(X_i(v_r))X_i(v_k) \otimes v^k,
\end{aligned}$$

which is, by (3.11), the required relation. Relations (3.14), (3.15) are proven with a similar and easier straightforwad computation.

Hence we conclude that the three tensors  $\sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i)$ ,  $\sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r$ ,  $\sum_{r,s} v_s \otimes v^s \otimes v^r \otimes v_r$  are linearly independent unless  $\mathfrak{p} = 0$ . In the latter case  $\mathfrak{k} = sl(V)$ ; in the former we get, in particular,

$$(3.16) \quad \lambda = \frac{(2g_1 - \gamma)\gamma}{4 \dim V} + \frac{1}{2}\gamma.$$

Substituting (3.16) in the coefficient of  $\sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r$  in (3.12) we obtain for this coefficient the expression  $\lambda_2((\lambda_1 - \lambda_2)^2 - 4)/(8 \dim V)$ . Thus (3.12) becomes

$$\begin{aligned}
(3.17) \quad \sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i) &= \frac{(2g_1 - \gamma)^2}{4} \sum_i \tau^{-1}(X_i) \otimes (\tau^*)^{-1}(X_i) \\
&\quad + \frac{\lambda_2((\lambda_1 - \lambda_2)^2 - 4)}{8 \dim V} \sum_{r,i} v_i \otimes v^r \otimes v^i \otimes v_r.
\end{aligned}$$

Hence we obtain

$$(3.18) \quad 2g_1 - \gamma = \pm 2.$$

By AP-criterion,  $\frac{\gamma}{2(k+g_1)} = 1$  so  $k = \frac{\gamma-2g_1}{2} = \pm 1$ .

Finally, as central charges must be equal, we obtain (upper sign for  $k = 1$ , lower sign for  $k = -1$ )

$$\frac{\dim sl(V)}{\dim V \pm 1} = \frac{\dim \mathfrak{k}}{g_1 \pm 1},$$

which gives (3.10).

Recall from [7] the following result.

$(C_n)_{\omega_1}$ : let  $\mathfrak{k}$  be of type  $C_n$ , let  $V = L_{C_n}(\omega_1)$  be the irreducible  $2n$ -dimensional  $C_n$ -module. Then we have a conformal embedding of  $V_{-1}(C_n)$  into  $V_{-1}(A_{2n-1}) = V_{-1}(sl(V))$ .

**Proposition 3.8.** *Let  $\mathfrak{k}$  be a simple Lie algebra and  $V$  an irreducible representation of  $\mathfrak{k}$ . If  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(sl(V))$ , then either  $k = 1$  or  $\mathfrak{k} = sl(V)$  or we are in case  $(C_n)_{\omega_1}$ .*

*Proof.* We are reduced, by Lemma 3.7, to deal with the case  $k = -1$ . Arguing as in Proposition 3.3, we have that

$$(3.19) \quad sl(V) = \mathfrak{k} \oplus pL(\mu), \text{ where } p \geq 1,$$

with  $\mu$  as in Table 1. Let  $(\cdot, \cdot)_n$  be the normalized form of  $\mathfrak{k}$ ; setting  $v = \dim V$ , one has

$$(3.20) \quad \lambda_2^{(\cdot, \cdot)_n} = \frac{\gamma}{g_1} h^\vee = \frac{2(\dim \mathfrak{k})h^\vee}{1 + v + \dim \mathfrak{k}}.$$

Let us discuss type  $G_2$ . Since  $\mu = \omega_1$ , we know that  $\lambda_2^{(\cdot, \cdot)_n} = 4$ . Hence, by (3.20), we have that  $v = 13$ . But no irreducible representation of  $G_2$  has that dimension. Type  $F_4$  is dealt with similarly.

For type  $C_n$ , we have  $\mu = \omega_2$  and  $\lambda_2^{(\cdot, \cdot)_n} = 2n$ , hence (3.20) gives  $v = 2n$ . This implies  $V = L(\omega_1)$  and we get the conformal embedding  $(C_n)_{\omega_1}$ .

In type  $B_n$ , we have  $\mu = \omega_1$  and  $\lambda_2^{(\cdot, \cdot)_n} = 2n$ , hence (3.20) gives  $v = 2n^2 - n - 2$ . Hence, by dimensional considerations and the results from [8], either  $V$  is the little adjoint representation (i.e., the irreducible module whose highest weight is the highest short root) or it is the spin representation when  $n = 3, 4, 5, 6$ . The former case is excluded since  $2n^2 - n - 2 \neq 2n + 1$  for all  $n$ . The latter case is excluded similarly except when  $n = 6$ . In that case dimensions match, but  $L(\omega_6) \otimes L(\omega_6)$  has a component of type  $L(2\omega_6)$ , which contradicts (3.19).

#### 4. CONFORMAL EMBEDDINGS OF $\tilde{V}(k, \mathfrak{k})$ IN $V_k(\mathfrak{g})$ WITH $\mathfrak{g}$ CLASSICAL AND $\mathfrak{k}$ SEMISIMPLE NON-SIMPLE.

By Dynkin's theory (see Subsection 2.3), the only possibilities for a maximal non-equal rank semisimple non-simple subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$  with  $\mathfrak{g}$  classical are given in the first two columns of Table 2. By equating central charges we find the values for the level  $k$  when conformal embedding might occur. We list these values in the third column of Table 2.

$\mathfrak{g}$	$\mathfrak{k}$	$k$	Ref
$sl(mn)$	$sl(n) \times sl(m)$	1 for all $n, m$ ; $-1$ for $n \neq m$	$(slsl)$
$so(4mn)$	$sp(2n) \times sp(2m)$	$1, -\frac{2(mn-1)}{2mn-m-n}$	$(spsp)$
$so(mn)$	$so(n) \times so(m)$	$1, \frac{4-mn}{m+n+mn}$	$(soso)$
$sp(2mn)$	$sp(2n) \times so(m)$	$-\frac{1}{2}, \frac{mn+2}{2mn+2n-m}$	$(spso)$
$so(2(m+n+1))$	$so(2n+1) \times so(2m+1)$	$1, 1-m-n$	$(BB)$

Table 2

**Theorem 4.1.** (1) The embedding  $sl(n) \times sl(m) \subset sl(nm)$  is conformal for  $k = 1$ . If  $n \neq m$ , it is conformal also for  $k = -1$ . (If  $n = m$ , then the level  $k = -1$  is excluded, since it is critical for the factors  $sl(n)$  of  $sl(n) \times sl(n)$ .)

(2) The embedding  $sp(2n) \times sp(2m) \subset so(4nm)$  is conformal only for  $k = 1$ . (If  $n = m$ , then the level  $k = -1 - \frac{1}{m}$  from  $(spsp)$  is critical for the factors  $sp(2n)$  of  $sp(2n) \times sp(2n)$ .)

(3) The embedding  $so(n) \times so(m) \subset so(nm)$  is conformal only for  $k = 1$ . (If  $n = m$ , then the level  $k = -1 + \frac{2}{m}$  from  $(soso)$  is critical for the factors  $so(n)$  of  $so(n) \times so(n)$ .)

(4) The embedding  $sp(2n) \times so(m) \subset sp(2nm)$  is conformal only for  $k = -1/2$ , except when  $n = 1$  in which case it is conformal for  $k = -1/2, 1$ . (If  $m = 2n + 2$ , then the level  $k = -1/2$  from  $(spso)$  is critical for both factors of  $sp(2n) \times so(2n + 2)$ .)

(5) The embedding  $so(2n + 1) \times so(2m + 1) \subset so(2(n + m + 1))$  is conformal for  $k = 1$  and  $k = 1 - n - m$ , except when  $n = m$  in which case it is conformal only for  $k = 1$ . (If  $m = n$ , then the level  $k = 1 - m - n$  is critical for both factors  $so(2n + 1)$ .)

**Remark 4.1.** The knowledge of decompositions at level 1 is as follows: they are known in a very explicit combinatorial way [36] for  $(slsl)$ ; a formula for cases  $(spsp)$ ,  $(soso)$  involving the combinatorics of Borel stable abelian subspaces is given in [15]; a general formula including also  $(slsl)$  is given in [27].

*Proof of Theorem 4.1.* We apply the AP-criterion. This requires to describe  $\mathfrak{p}$  as a  $\mathfrak{k}$ -module, which can be done by classical invariant theory.

Case  $(slsl)$ . We realize  $sl(nm) = Hom((\mathbb{C}^n \otimes \mathbb{C}^m), (\mathbb{C}^n \otimes \mathbb{C}^m))/\mathbb{C}$  as a  $sl(n) \times sl(m)$ -module. Since  $\mathbb{C}^n \otimes \mathbb{C}^m = L_{sl(n)}(\omega_1) \otimes L_{sl(m)}(\omega_1)$  then

$$End((\mathbb{C}^n \otimes \mathbb{C}^m)) = (L_{sl(n)}(\omega_1) \otimes L_{sl(n)}(\omega_1)) \otimes (L_{sl(m)}(\omega_1) \otimes L_{sl(m)}(\omega_1)).$$

Observe that

$$L_{sl(n)}(\omega_1) \otimes L_{sl(n)}(\omega_1) = L_{sl(n)}(\theta) \oplus L_{sl(n)}(0),$$

hence

$$\begin{aligned} sl(nm) &= (L_{sl(n)}(\theta) \otimes L_{sl(m)}(0)) \oplus (L_{sl(n)}(0) \otimes L_{sl(n)}(\theta)) \oplus (L_{sl(n)}(\theta) \otimes L_{sl(n)}(\theta)) \\ &= (sl(n) \times sl(m)) \oplus (L_{sl(n)}(\theta) \otimes L_{sl(n)}(\theta)). \end{aligned}$$

and

$$\mathfrak{p} = L_{sl(n)}(\theta) \otimes L_{sl(n)}(\theta).$$

We need henceforth to check that

$$\frac{(\theta, \theta + 2\rho)_{sl(n)}}{2m(k + n/m)} + \frac{(\theta, \theta + 2\rho)_{sl(m)}}{2n(k + m/n)} = 1.$$

Here  $(\cdot, \cdot)_{sl(p)}$  is the trace form on  $sl(p)$ . Since  $(\theta, \theta + 2\rho)_{sl(p)} = 2p$  and we have

$$\frac{n}{mk + n} + \frac{m}{nk + m} = 1$$

which has solutions  $k = \pm 1$  except when  $n = m$ , where the only solution is  $k = 1$ .

Case  $(spsp)$ . We realize  $so(4nm) = \bigwedge^2((\mathbb{C}^{2n} \otimes \mathbb{C}^{2m}))$  as a  $sp(2n) \times sp(2m)$ -module. Since

$$(4.1) \quad \bigwedge^2(\mathbb{C}^{2n} \otimes \mathbb{C}^{2m}) = \left( \bigwedge^2(\mathbb{C}^{2n}) \otimes S^2(\mathbb{C}^{2m}) \right) \oplus \left( \bigwedge^2(\mathbb{C}^{2m}) \otimes S^2(\mathbb{C}^{2n}) \right),$$

$\bigwedge^2(\mathbb{C}^{2r}) = L_{sp(2r)}(\omega_2) \oplus L_{sp(2r)}(0)$ , and  $S^2(\mathbb{C}^{2r}) = L_{sp(2r)}(\theta)$ , we obtain that

$$so(4mn) = (L_{sp(2n)}(\theta) \otimes L_{sp(2m)}(\omega_2)) \oplus (L_{sp(2n)}(\omega_2) \otimes L_{sp(2m)}(\theta)) \oplus (sp(2n) \times sp(2m)).$$

and

$$\mathfrak{p} = (L_{sp(2n)}(\theta) \otimes L_{sp(2m)}(\omega_2)) \oplus (L_{sp(2n)}(\omega_2) \otimes L_{sp(2m)}(\theta))$$

We need henceforth to check that

$$\frac{(\theta, \theta + 2\rho)_{sp(n)}}{2m(k + (n+1)/m)} + \frac{(\omega_2, \omega_2 + 2\rho)_{sp(m)}}{2n(k + (m+1)/n)} = 1.$$

and

$$\frac{(\omega_2, \omega_2 + 2\rho)_{sp(n)}}{2m(k + (n+1)/m)} + \frac{(\theta, \theta + 2\rho)_{sp(m)}}{2n(k + (m+1)/n)} = 1.$$

Here  $(\cdot, \cdot)_{sp(r)}$  is the normalized invariant bilinear form of  $sp(r)$ .

It is readily verified that the solutions for the first equation are 1 and  $-(m+1)/m$ , and 1 and  $-(n+1)/n$  for the second equation. If  $m = n$  the values different from 1 coincide with value given in the table; on the other hand this value is critical.

Case (*soso*). We realize  $so(nm) = \bigwedge^2(\mathbb{C}^n \otimes \mathbb{C}^m)$  as a  $so(n) \times so(m)$ -module. Using (4.1)(with  $n, m$  in place of  $2n, 2m$ ), we have that

$$so(mn) = (L_{so(n)}(\theta) \otimes L_{so(m)}(2\omega_1)) \oplus (L_{so(n)}(2\omega_1) \otimes L_{so(m)}(\theta)) \oplus so(n) \times so(m),$$

so that

$$\mathfrak{p} = (L_{so(n)}(\theta) \otimes L_{so(m)}(2\omega_1)) \oplus (L_{so(n)}(2\omega_1) \otimes L_{so(m)}(\theta)),$$

and calculations similar to the previous case lead to statement (3).

Case (*spso*). We realize  $sp(2nm) = S^2((\mathbb{C}^{2n} \otimes \mathbb{C}^m))$  as a  $so(n) \times so(m)$ -module. We have

$$S^2(\mathbb{C}^{2n} \otimes \mathbb{C}^m) = (S^2(\mathbb{C}^{2n}) \otimes S^2(\mathbb{C}^m)) \oplus \left( \bigwedge^2(\mathbb{C}^{2n}) \otimes \bigwedge^2(\mathbb{C}^m) \right),$$

and

$$sp(2nm) = (L_{sp(2n)}(\theta) \otimes L_{so(m)}(2\omega_1)) \oplus (L_{sp(2n)}(\omega_2) \otimes L_{so(m)}(\theta)) \oplus (sp(2n) \times so(m)),$$

so that

$$\mathfrak{p} = (L_{sp(2n)}(\theta) \otimes L_{so(m)}(2\omega_1)) \oplus (L_{sp(2n)}(\omega_2) \otimes L_{so(m)}(\theta)).$$

Again an easy calculation leads to (4).

Case (*BB*). In this case  $\mathfrak{k}$  is the fixed point set of an involutive automorphism of  $\mathfrak{g}$ . From this observation is easy to derive the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus (L_{so(2n+1)}(\omega_1) \otimes L_{so(2m+1)}(\omega_1)),$$

hence we can verify that this embedding is conformal by applying the AP-criterion.

**Remark 4.2.** The branching rules for the conformal embedding in the cases  $m = 2$ ,  $n \geq 3$  of Theorem 4.1(4) were studied in [6]. It was proved that then  $V_{-1/2}(sp(4n))$  is a semisimple  $V_{-1}(sp(2n)) \otimes M(1)$ -module, where  $M(1) = V_1(so(2))$  is a Heisenberg vertex algebra of central charge  $c = 1$ . So it is natural to conjecture that in many cases we shall get semisimple decomposition. But as we shall see below, we have some examples where we will not have semisimplicity.

It is natural to ask about semisimplicity of  $V_{-1}(sl(mn))$  as a  $V^{-m}(sl(n)) \otimes V^{-n}(sl(m))$ -module. Let

$$\Phi : V^{-m}(sl(n)) \otimes V^{-n}(sl(m)) \rightarrow V_{-1}(sl(mn))$$

be the associated homomorphism of vertex algebras. Clearly, if  $m > n$ , then  $V^{-m}(sl(n)) = V_{-m}(sl(n))$  is a simple vertex algebra [23]. But  $\text{Im}(\Phi)$  need not to be simple.



**Example 4.1.** Let  $m = 2n$ . Since  $V^{-2n}(sl(n))$  is a simple vertex algebra, we have

$$\text{Im}(\Phi) = V_{-2n}(sl(n)) \otimes \mathcal{W}_{-n}$$

where  $\mathcal{W}_{-n}$  is a certain quotient of  $V^{-n}(sl(2n))$ . We claim that it is not simple. We noticed in the proof of Theorem 4.1(1) that  $\mathcal{W}_{-n}$  contains a module whose lowest component is isomorphic to  $L_{sl(2n)}(\theta)$  (this module is realized inside of  $V_{-1}(sl(mn))$ ). But the classification of irreducible  $V_{-n}(sl(2n))$ -modules [12] shows that such module can not be  $V_{-n}(sl(2n))$ -module. (It was proved in [12, Section 8] that an irreducible  $V_{-n}(sl(2n))$ -module in the category  $KL_{-n}$  must have lowest component isomorphic to  $L_{sl(2n)}(r\omega_n)$ , for certain  $r \in \mathbb{Z}_{\geq 0}$ .)

This proves that  $V_{-n}(sl(2n)) \neq \mathcal{W}_{-n}$ , and therefore  $V_{-1}(sl(2n^2))$  is not a semisimple  $V^{-2n}(sl(n)) \otimes V^{-n}(sl(2n))$ -module.

## 5. CONFORMAL EMBEDDINGS OF $\tilde{V}(k, \mathfrak{k})$ IN $V_k(\mathfrak{g})$ WITH $\mathfrak{g}$ OF EXCEPTIONAL TYPE.

We will use the following notation: if  $X$  denotes the type of a simple subalgebra  $\mathfrak{k}$  of a simple Lie algebra  $\mathfrak{g}$ , then  $X^d$  identifies the embedding of Dynkin index  $d$ . We omit the superscript when  $d = 1$ .

Recall that the central charge of the Sugawara Virasoro vector for  $V^k(\mathfrak{g})$ ,  $\mathfrak{g}$  simple, is given by

$$c_k(\mathfrak{g}) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

Also, if  $\mathfrak{k} \subset \mathfrak{g}$  is a reductive subalgebra in  $\mathfrak{g}$ , then we denote by  $c(\mathfrak{k})$  the central charge of the Sugawara Virasoro vector of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(\mathfrak{g})$ .

**Theorem 5.1.** *Let  $\mathfrak{k}$  be a maximal non-equal rank semisimple subalgebra of an exceptional simple Lie algebra  $\mathfrak{g}$ . Then  $\tilde{V}(k, \mathfrak{k})$  is conformally embedded in  $V_k(\mathfrak{g})$  if and only if either  $k = 1$  and  $(\mathfrak{k}, \mathfrak{g})$  belong to the following list*

$$(5.1) \quad \begin{aligned} &(G_2 \times F_4, E_8), \quad (A_2^6 \times A_1^{16}, E_8), \quad (B_2, E_8), \\ &(A_2^{21}, E_7), \quad (G_2 \times C_3, E_7), \quad (G_2 \times A_1^7, E_7), \quad (F_4 \times A_1^3, E_7), \\ &(A_2^9, E_6), \quad (G_2^3, E_6), \quad (C_4, E_6), \quad (G_2 \times A_2^2, E_6), \\ &(G_2 \times A_1^8, F_4), \quad (A_1^{28}, G_2), \end{aligned}$$

or we are in one of the following cases:

$$(5.2) \quad \tilde{V}(-6, G_2 \times F_4) \hookrightarrow V_{-6}(E_8),$$

$$(5.3) \quad \tilde{V}(-4, F_4 \times A_1^3) \hookrightarrow V_{-4}(E_7),$$

$$(5.4) \quad \tilde{V}(-3, G_2 \times A_2^2) \hookrightarrow V_{-3}(E_6),$$

$$(5.5) \quad \tilde{V}(-3, F_4) \hookrightarrow V_{-3}(E_6),$$

$$(5.6) \quad \tilde{V}(-5/2, G_2 \times A_1^8) \hookrightarrow V_{-5/2}(F_4).$$

*Proof.* We start from Dynkin's classification of non-equal rank maximal subalgebras  $\mathfrak{k}$  in exceptional Lie algebras  $\mathfrak{g}$  (see Subsection 2.3). For each pair  $(\mathfrak{k} = \oplus_i \mathfrak{k}_i, \mathfrak{g})$  we determine the values of  $k$  such that  $c_k(\mathfrak{g}) = \sum_i c(\mathfrak{k}_i)$ . We get either  $k = 1$  if  $(\mathfrak{k}, \mathfrak{g})$  appears in (5.1), or the

embeddings displayed in (5.2)–(5.6), or the triples  $(\mathfrak{k}, \mathfrak{g}, k)$  displayed in the following list

$$(5.7) \quad (A_1^{1240}, E_8, 64/75), \quad (A_1^{760}, E_8, 8488/23275), \quad (A_1^{520}, E_8, 5788/15925), \\ (A_2^6 \times A_1^{16}, E_8, -119/474), \quad (A_1^{389}, E_7, 16/39), \quad (A_1^{231}, E_7, 872/2145), \\ (G_2^2 \times A_1^7, E_7, -26/29), \quad (A_1^{24} \times A_1^{15}, E_7, \frac{479 \pm 3\sqrt{46265}}{1524}).$$

The statements for  $k = 1$  are known (cf. [9], [38], [24]). Case (5.5) is treated in [1]. For the other cases we use AP-criterion, verifying condition (2.1) in each of the cases.

Computing explicitly the decomposition via the package SLA [16], it turns out that for the levels  $-1 - h^\vee/6$  we have for (5.6)

$$(5.8) \quad \mathfrak{p} = L_{G_2}(\dot{\theta}_s) \otimes L_{A_1}(4\ddot{\omega}_1).$$

In cases (5.2), (5.3), (5.4), denoting the subalgebra in the leftmost part of each formula by  $\mathfrak{a} \times \ddot{\mathfrak{a}}$ , we have

$$(5.9) \quad \mathfrak{p} = L_{\mathfrak{a}}(\dot{\theta}_s) \otimes L_{\ddot{\mathfrak{a}}}(\ddot{\theta}_s),$$

where  $\dot{\theta}_s$  (resp.  $\ddot{\theta}_s$ ) is the highest short root of  $\mathfrak{a}$  (resp.  $\ddot{\mathfrak{a}}$ ) and the convention that  $\ddot{\theta}_s$  is the highest root if  $\ddot{\mathfrak{a}}$  is simply laced.

Decompositions (5.8), (5.9) allow to check directly equality (2.1). Now we have to exclude the cases appearing in (5.7). We start from the cases when  $\mathfrak{k}$  is of type  $A_1$ : decompose  $\mathfrak{p}$  as a sum of  $A_1$  modules; if  $l$  is the dimension of a summand we should have, by (2.2), that  $\frac{l^2/2+l}{2(dk+1)} = 1$ . But a direct check shows that this is not possible, since the previous equation has no integral solution.

If  $\mathfrak{k}$  is non-simple, then one computes  $\mathfrak{p}$  in a computer assisted way and then checks that (2.1) does not hold.  $\square$

## 6. THE CLASSIFICATION OF CONFORMAL EMBEDDINGS OF MAXIMALLY REDUCTIVE SUBALGEBRAS IN $V_k(\mathfrak{g})$ .

In this section we summarize our results and give the complete classification of conformal embeddings  $\tilde{V}(k, \mathfrak{k}) \subset V_k(\mathfrak{g})$  with  $\mathfrak{g}$  a simple Lie algebra and  $\mathfrak{k}$  a semisimple maximal subalgebra of  $\mathfrak{g}$ .

The conformal embeddings with integrable  $k$  (hence  $k = 1$ ) have been classified long ago: see [9], [20], [24], [38]. We now give the classification of the conformal embeddings at non-integrable level. The following result summarizes the results of the previous sections together with the results of [2].

**Theorem 6.1.** *Assume that  $\mathfrak{k}$  is a maximally reductive subalgebra of a simple Lie algebra  $\mathfrak{g}$ . Then there is a conformal embedding of  $\tilde{V}(k, \mathfrak{k})$  in  $V_k(\mathfrak{g})$  with  $k \neq 1$  only in the following cases:*

(1) for  $\mathfrak{g}$  a simple Lie algebra of classical type and  $\mathfrak{k}$  semisimple

$$\begin{aligned}
\tilde{V}(-1, sl(n) \times sl(m)) &\hookrightarrow V_{-1}(sl(mn)), \quad (m \neq n) \\
\tilde{V}(-\frac{1}{2}, sp(2n) \times so(m)) &\hookrightarrow V_{-\frac{1}{2}}(sp(2nm)), \quad (m \neq 2n+2), \\
\tilde{V}(2 - (n+m)/2, so(n) \times so(m)) &\hookrightarrow V_{2-(n+m)/2}(so(n+m)), \quad (m \neq n) \\
\tilde{V}(-1/2, sp(2n) \times sp(2m)) &\hookrightarrow V_{-1/2}(sp(2(n+m))), \\
\tilde{V}(-1 - (n+m)/2, sp(2n) \times sp(2m)) &\hookrightarrow V_{-1-(n+m)/2}(sp(2(n+m))), \quad (m \neq n) \\
\tilde{V}(-1, sp(2n)) &\hookrightarrow V_{-1}(sl(2n)), \\
\tilde{V}(-2, G_2) &\hookrightarrow V_{-2}(so(7));
\end{aligned}$$

(2) for  $\mathfrak{g}$  an exceptional simple Lie algebra and  $\mathfrak{k}$  semisimple

$$\begin{aligned}
\tilde{V}(-6, A_1 \times E_7), \tilde{V}(-6, A_2 \times E_6), \tilde{V}(-6, G_2 \times F_4) &\text{ in } V_{-6}(E_8), \\
\tilde{V}(-4, A_1 \times D_6), \tilde{V}(-4, A_2 \times A_5), \tilde{V}(-4, F_4 \times A_1^3) &\text{ in } V_{-4}(E_7), \\
\tilde{V}(-3, A_1 \times A_5), \tilde{V}(-3, G_2 \times A_2^2), \tilde{V}(-3, F_4) &\text{ in } V_{-3}(E_6), \\
\tilde{V}(-5/2, A_1 \times C_3), \tilde{V}(-5/2, A_2 \times A_2^2), \tilde{V}(-5/2, B_4), \tilde{V}(-5/2, G_2 \times A_1^8) &\text{ in } V_{-5/2}(F_4), \\
\tilde{V}(-5/3, A_1 \times A_1^3), \tilde{V}(-5/3, A_2) &\text{ in } V_{-5/3}(G_2);
\end{aligned}$$

(3) for  $\mathfrak{k}$  reductive non-semisimple ( $Z$  denotes the one-dimensional center of  $\mathfrak{k}$ )

$$\begin{aligned}
\tilde{V}(-1, A_h \times A_{n-h-1} \times Z) &\hookrightarrow V_{-1}(A_n), \quad (h \geq 1, n-h \geq 2) \\
\tilde{V}(-(n+1)/2, A_h \times A_{n-h-1} \times Z) &\hookrightarrow V_{-(n+1)/2}(A_n), \quad (h \geq 1, h \neq (n-1)/2) \\
\tilde{V}(2-n, D_{n-1} \times Z) &\hookrightarrow V_{2-n}(D_n), \quad \tilde{V}(-2, A_{n-1} \times Z) \hookrightarrow V_{-2}(D_n), \\
\tilde{V}(-1/2, A_{n-1} \times Z) &\hookrightarrow V_{-1/2}(C_n), \\
\tilde{V}(3/2-n, B_{n-1} \times Z) &\hookrightarrow V_{3/2-n}(B_n), \\
\tilde{V}(-3, D_5 \times Z) &\hookrightarrow V_{-3}(E_6), \\
\tilde{V}(-4, E_6 \times Z) &\hookrightarrow V_{-4}(E_7).
\end{aligned}$$

*Proof.* We first discuss the  $\mathfrak{k}$  semisimple case. Combining the results of [2] and of Sections 3–5 above, one obtains the conformal embeddings listed in (1) and (2) except for the embedding  $(B3)_{\omega_3}$ , which is not listed. The reason is that  $(B3)_{\omega_3}$  is a special case of the  $\tilde{V}(2 - (n+m)/2, so(n) \times so(m)) \hookrightarrow V_{2-(n+m)/2}(so(n+m))$  embedding, with  $n = 7$  and  $m = 1$ . To check this claim we have to clarify that we consider two embeddings  $\mathfrak{k} \hookrightarrow \mathfrak{g}$ ,  $\mathfrak{k}' \hookrightarrow \mathfrak{g}$  to be equivalent if there is an automorphism of  $\mathfrak{g}$  mapping  $\mathfrak{k}$  to  $\mathfrak{k}'$ . In such a case, in fact, there is an automorphism of  $V_k(\mathfrak{g})$  that fixes the Virasoro vector and maps  $\tilde{V}(k, \mathfrak{k})$  onto  $\tilde{V}(k, \mathfrak{k}')$ . In the case at hand, let  $\sigma$  be the automorphism of  $so(8)$  induced by the diagram automorphism of order three that maps  $\alpha_1$  to  $\alpha_4$  (with respect to the usual Bourbaki numbering of simple roots). Let  $\pi : so(8) \rightarrow gl(8)$  be the representation defined by  $\pi(X)(v) = \sigma(X)(v)$ . Since the defining representation of  $so(8)$  is  $L_{so(8)}(\omega_1)$ , we see that  $\pi$  is given by the action of  $so(8)$  on  $L_{so(8)}(\omega_4)$ . It is easily checked that the restriction of  $L_{so(8)}(\omega_4)$  to  $so(7)$  is  $L_{so(7)}(\omega_3)$ , which is the spin representation. It follows that the subalgebra of  $so(8)$  corresponding to the action of  $so(7)$  on the spin representation is  $\pi(so(7)) = \sigma(so(7))$ .

By Lemma 2.3, if  $\mathfrak{k}$  is not semisimple, then it is a maximally reductive equal rank subalgebra and conformal embeddings of such subalgebras are determined in [2].  $\square$

## 7. ON CONFORMAL EMBEDDING $sl(2) \times so(m)$ INTO $sp(2m)$ AT $k = -1/2$ .

Recall that the affine vertex algebra  $V_{-1/2}(sp(2m))$  is realized as the even subalgebra of the Weyl vertex algebra  $M_{(m)}$ . By using the conformal embedding of  $sl(2) \times so(m)$  into  $sp(2m)$  at level  $k = -1/2$ , we get an action of  $\widehat{sl(2)} \times \widehat{so(m)}$  on  $M_{(m)}$ . In this section we shall assume that  $m \neq 4$  to exclude the critical level case for  $\widehat{sl(2)}$ .

Since  $V^{-m/2}(sl(2)) = V_{-m/2}(sl(2))$ , we have that

$$\widetilde{V}(-1/2, sl(2) \times so(m)) \cong V_{-m/2}(sl(2)) \otimes \widetilde{V}_{-2}(so(m))$$

where  $\widetilde{V}_r(\mathfrak{g})$  denotes a quotient (not necessarily simple) of  $V^r(\mathfrak{g})$ . We will use freely this notation in the following sections. Hence we have that  $M_{(m)}$  becomes a  $V_{-m/2}(sl(2)) \otimes \widetilde{V}_{-2}(so(m))$ -module, and we are interested in its decomposition. It is important to notice that this case is exactly an infinite-dimensional analog of the action of dual pair  $sl(2) \times so(m)$  on the polynomial algebra  $\mathbb{C}[z_1, \dots, z_m]$  studied by R. Howe in [22, Section 4]. This classical result has very important applications in the representation theory. R. Howe proved that  $\mathbb{C}[z_1, \dots, z_m]$  is a completely reducible  $sl(2) \times so(m)$ -module. In studying the same problem in the affine setting, we construct a family of singular vectors in  $M_{(m)}$  which exactly correspond to Howe singular vectors in the decomposition of  $\mathbb{C}[z_1, \dots, z_m]$ . We study complete-reducibility problem. As a byproduct, we prove that the affine vertex algebra  $V_{-2}(D_3) = V_{-2}(A_3)$  is realized as a subalgebra of  $M_{(6)}$ . But we show that in general (for  $m \geq 8$ )  $M_{(m)}$  is not completely reducible as  $V_{-m/2}(sl(2)) \otimes \widetilde{V}_{-2}(so(m))$ -module.

We fix the following notation.

- Let  $\widetilde{V}_{-2}(D_n) = \widetilde{V}_{-2}(so(2n))$  be the vertex subalgebra of  $V_{-1/2}(C_{2n})$  generated by the factor  $D_n$  in conformal embedding  $sl(2) \times D_n$  into  $C_{2n}$  at  $k = -1/2$  ( $n \geq 3$ ).
- Let  $\widetilde{V}_{-2}(A_{n-1})$  be the vertex subalgebra of  $V_{-1}(A_{2n-1})$  generated by the factor  $A_{n-1}$  in conformal embedding  $sl(2) \times A_{n-1}$  into  $A_{2n-1}$  at  $k = -1$  ( $n \geq 3$ ).
- Let  $\widetilde{V}_{-2}(B_n) = \widetilde{V}_{-2}(so(2n+1))$  be the vertex subalgebra of  $V_{-1/2}(C_{2n+1})$  generated by the factor  $B_n$  in conformal embedding  $sl(2) \times B_n$  into  $C_{2n+1}$  at  $k = -1/2$  ( $n \geq 2$ ). When  $n = 1$ , the subalgebra generated by the factor  $B_n$  is isomorphic to  $V_{-4}(sl(2))$ .
- Let  $M(1)$  be the Heisenberg vertex algebra of rank (and central charge) 1.

Recall also that  $V_{-1/2}(sp(2m))$  can be realized as a subalgebra of the Weyl vertex algebra  $M_{(m)}$  generated by bosonic fields  $a_i^\pm$  ( $i = 1, \dots, m$ ) with  $\lambda$ -brackets

$$[(a_i^\pm)_\lambda(a_j^\pm)] = 0, \quad [(a_i^+)_\lambda(a_j^-)] = \delta_{i,j}.$$

We shall now consider  $M_{(m)}$  as a module for  $V_{-m/2}(sl(2)) \otimes \widetilde{V}_{-2}(so(m))$ .

### Lemma 7.1.

- (1) Assume that  $m \geq 5$ . Then there exist highest weight modules  $\widetilde{L}_{\widehat{sl(2)}}(-(\frac{m}{2} + k)\Lambda_0 + k\Lambda_1)$ ,  $\widetilde{L}_{\widehat{so(m)}}(-(2+k)\Lambda_0 + k\Lambda_1)$  in the category  $KL$  such that  $M_{(m)}$  contains a  $V_{-m/2}(sl(2)) \otimes$

$\tilde{V}_{-2}(so(m))$ -submodule isomorphic to

$$M^{sub} = \sum_{k=0}^{\infty} \tilde{L}_{\widehat{sl(2)}}(-(\frac{m}{2} + k)\Lambda_0 + k\Lambda_1) \otimes \tilde{L}_{\widehat{so(m)}}(-(2+k)\Lambda_0 + k\Lambda_1).$$

In particular,  $V_{-1/2}(sp(2m))$  contains a  $V_{-m/2}(sl(2)) \otimes \tilde{V}_{-2}(so(m))$ -submodule isomorphic to

$$(M^{sub})^0 = \sum_{k=0}^{\infty} \tilde{L}_{\widehat{sl(2)}}(-(\frac{m}{2} + 2k)\Lambda_0 + 2k\Lambda_1) \otimes \tilde{L}_{\widehat{so(m)}}(-(2+2k)\Lambda_0 + 2k\Lambda_1).$$

(2) Assume that  $m = 3$ . Then  $M_{(3)}$  contains a  $V_{-3/2}(sl(2)) \otimes V_{-4}(sl(2))$ -submodule isomorphic to

$$M^{sub} = \sum_{k=0}^{\infty} L_{\widehat{sl(2)}}(-(\frac{3}{2} + k)\Lambda_0 + k\Lambda_1) \otimes \tilde{L}_{\widehat{sl(2)}}(-(4+2k)\Lambda_0 + 2k\Lambda_1).$$

(Note that we don't claim that sums above are direct!)

*Proof.* (1) The explicit embedding of  $V_{-1/2}(sp(2m))$  into  $M_{(m)}$  is given by

$$\begin{aligned} E_{m+j,m+i} - E_{i,j} &\mapsto: a_i^+ a_j^- :, E_{i,m+j} + E_{j,m+i} \mapsto: a_i^+ a_j^+ :, \\ E_{m+i,j} + E_{m+j,i} &\mapsto: a_i^- a_j^- :. \end{aligned}$$

The embedding of  $sl(2) \times so(m)$  into  $sp(2m)$  is given by

$$\left( \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, B \right) \mapsto \begin{pmatrix} a \cdot Id & b \cdot Id \\ c \cdot Id & -a \cdot Id \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

As a Cartan subalgebra for  $so(m)$  choose the block matrices of type

$$(7.1) \quad \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} \text{ if } m = 2n, \quad \begin{pmatrix} 0 & H & 0 \\ -H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ if } m = 2n + 1,$$

with  $H$  a diagonal  $n \times n$  matrix. Root vectors in  $so(2n)$  are matrices

$$(7.2) \quad \mathcal{E}_\alpha = A - A^t,$$

with  $A = aE_{i,j} + bE_{i,n+j} + cE_{n+i,j} + dE_{n+i,n+j}$  and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} & \text{for } \alpha = \epsilon_i - \epsilon_j, \\ \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix} & \text{for } \alpha = \epsilon_i + \epsilon_j, \\ \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix} & \text{for } \alpha = -\epsilon_i - \epsilon_j. \end{cases}$$

The root vectors for  $so(2n+1)$  are obtained by adding to the root vectors for  $so(2n)$  constructed above the vectors  $\mathcal{E}_{\epsilon_i} = A - A^t$  with  $A = E_{i,2n+1} - \sqrt{-1}E_{n+i,2n+1}$  and  $\mathcal{E}_{-\epsilon_i} = A - A^t$  with  $A = E_{i,2n+1} + \sqrt{-1}E_{n+i,2n+1}$ .

Consider  $\varphi = a_1^+ - \sqrt{-1}a_{n+1}^+$ . Note that it has weight  $\epsilon_1$ . Indeed, let  $h_r$  be the element of the Cartan subalgebra by substituting in (7.1)  $H = \text{diag}(0, \dots, \sqrt{-1}, \dots, 0)$ , with the nonzero

entry in the  $r$ -th position. The action of  $h_r$  is given by  $-\sqrt{-1}(: a_r^+ a_{n+r}^- : - : a_{n+r}^+ a_r^- : )_{(0)}$ , but

$$[\sqrt{-1}(: a_{n+r}^+ a_r^- : - : a_r^+ a_{n+r}^- : )_\lambda \varphi] = -\delta_{r,1} i a_n^+ + \delta_{n+r,n+1} a_1^+ = \delta_{r,1} \varphi.$$

Now we study the action of  $\mathcal{E}_{\epsilon_1-\epsilon_2(0)}, \mathcal{E}_{-\epsilon_1-\epsilon_2(1)}$  on  $\varphi$ . We have that for any root  $\alpha$ , only the  $(0)$ -th product occurs in  $[\mathcal{E}_\alpha \lambda \varphi]$ , hence  $(\mathcal{E}_\alpha)_{(1)} \varphi = 0$ . For  $\mathcal{E}_{\epsilon_1-\epsilon_2(0)}$  we have

$$\begin{aligned} [\mathcal{E}_{\epsilon_1-\epsilon_2} \lambda \varphi] &= [(- : a_1^+ a_2^- : - \sqrt{-1} : a_1^+ a_{n+2}^- : + : a_2^+ a_1^- - \sqrt{-1} : a_2^+ a_{n+1}^- : )_\lambda \varphi] \\ &\quad + [(\sqrt{-1} : a_{n+1}^+ a_2^- : - : a_{n+1}^+ a_{n+2}^- : + \sqrt{-1} : a_{n+2}^+ a_1^- : + : a_{n+2}^+ a_{n+1}^- : )_\lambda \varphi] \\ &= -a_2^+ + a_2^+ - \sqrt{-1} a_{n+2}^+ + \sqrt{-1} a_{n+2}^+ = 0. \end{aligned}$$

In particular

$$[\mathcal{E}_{\epsilon_1-\epsilon_2(0)}, \varphi_{(-1)}] = [\mathcal{E}_{-\epsilon_1-\epsilon_2(1)}, \varphi_{(-1)}] = 0.$$

An obvious induction shows that  $\varphi_{(-1)}^k \mathbf{1}$  is a singular vector for  $\widehat{so(m)}$  of weight  $2\Lambda_0 + k\epsilon_1 = -(2+k)\Lambda_0 + k\Lambda_1$ .

The positive root vector of  $sl(2)$  corresponds to  $\sum_{r=1}^m : a_r^+ a_r^+ :$  and the  $\lambda$ -bracket  $[: a_r^+ a_r^+ :_\lambda \varphi]$  is trivial. Let  $\beta$  be the positive root of  $sl(2)$ , let  $h$  be corresponding coroot; then  $h$  corresponds to  $-\sum_r : a_r^+ a_r^- :$ . Since  $-\sum_r [: a_r^+ a_r^- :_\lambda \varphi] = \varphi$ , we see that the  $sl(2)$ -weight of  $\phi$  is  $\beta/2$ . Thus, arguing as for  $so(m)$ , we see that  $(\phi_{(-1)})^k \mathbf{1}$  is a singular vector for  $\widehat{sl(2)}$  of weight  $-\frac{m}{2}\Lambda_0 + (k/2)\beta = -(\frac{m}{2} + k)\Lambda_0 + k\Lambda_1$ , hence it generates a submodule isomorphic to

$$(7.3) \quad \widetilde{L}_{\widehat{sl(2)}}(-(\frac{m}{2} + k)\Lambda_0 + k\Lambda_1) \bigotimes \widetilde{L}_{\widehat{so(m)}}(-(2+k)\Lambda_0 + k\Lambda_1) \quad (\text{for } m \geq 5).$$

Consider now the case  $m = 3$ . In this case the only positive root is  $\epsilon_1$  whose root vector  $\mathcal{E}_{\epsilon_1}$  maps to

$$- : a_1^+ a_3^- : + \sqrt{-1} a_2^+ a_3^- : + : a_3^+ a_1^- : - \sqrt{-1} : a_3^+ a_2^- : .$$

and

$$[\mathcal{E}_{\epsilon_1} \lambda \varphi] = -a_3^+ + a_3^+ = 0.$$

The  $so(3)$ -weight of  $\phi$  is  $\epsilon_1$ .

We now perform the calculation of the level of the  $\widehat{so(3)}$  action: the trace form of  $sp(6)$  restricted to  $so(3)$  turns out to be  $8(\cdot, \cdot)_{norm}$  where  $(\cdot, \cdot)_{norm}$  is the normalized form of  $so(3)$ . It follows that the  $so(3)$ -level is  $8(-\frac{1}{2}) = -4$ . Hence the  $\widehat{so(3)}$ -weight of  $(\varphi_{(-1)})^k \mathbf{1}$  is  $-4\Lambda_0 + k\epsilon_1 = -(4+2k)\Lambda_0 + 2k\Lambda_1$ . The computation of the  $\widehat{sl(2)}$ -weight of  $(\varphi_{(-1)})^k \mathbf{1}$  is the same as in the previous case. It follows that  $(\varphi_{(-1)})^k \mathbf{1}$  is a singular vector generating

$$L_{\widehat{sl(2)}}(-(\frac{3}{2} + k)\Lambda_0 + k\Lambda_1) \bigotimes \widetilde{L}_{\widehat{sl(2)}}(-(4+2k)\Lambda_0 + 2k\Lambda_1).$$

The irreducibility of  $V_{-3/2}(sl(2)) \cdot (\varphi_{(-1)})^k \mathbf{1} \cong L_{\widehat{sl(2)}}(-(\frac{3}{2} + k)\Lambda_0 + k\Lambda_1)$  follows from the fact that the Weyl module in  $KL_{-3/2}$  with the highest weight  $-(\frac{3}{2} + k)\Lambda_0 + k\Lambda_1$  is irreducible (see Section 9).

**Remark 7.1.** It is interesting to investigate whether  $M_{(m)} = M^{sub}$ ,  $V_{-1/2}(sp(2m)) = (M^{sub})^0$ . In Section 9, we prove these equalities hold in the case  $m = 3$ . But in Section 10 we show that in the case  $m = 8$  there is a subsingular vector  $P_{high}^-$  outside of  $M^{sub}$ . A possible reason for a difference between vertex-algebraic and classical setting is that subsingular vectors belong to the kernel of Zhu's functor (cf. Remark 10.1).



We have inclusions  $A_{n-1} \hookrightarrow D_n \hookrightarrow sp(4n)$ . We let  $\overline{V}_{-2}(A_{n-1})$  be the vertex subalgebra of  $V_{-1/2}(sp(4n))$  generated by the elements in  $A_{n-1}$ .

**Theorem 7.2.**

- (1)  $\overline{V}_{-2}(A_{n-1}) \otimes M(1)$  is conformally embedded into  $\tilde{V}_{-2}(D_n)$  for  $n \geq 3$ .
- (2)  $\tilde{V}_{-2}(D_n)$  is simple if and only if  $n = 3$ .
- (3)  $\tilde{V}_{-2}(B_n)$  is not simple for  $n \geq 3$ .

*Proof.* The proof of assertion (1) follows from the following observations:

- (a)  $\tilde{V}_{-1}(A_{2n-1}) \otimes M(1)$  is conformally embedded into  $V_{-1/2}(C_{2n})$  and  $\tilde{V}_{-1}(A_{2n-1}) \otimes M(1) = V_{-1}(A_{2n-1}) \otimes M(1)$  [2, Theorem 5.1, (4)];
- (b) by Theorem 4.1,  $V_{-n}(sl(2)) \otimes \overline{V}_{-2}(A_{n-1})$  is conformally embedded into  $V_{-1}(A_{2n-1})$ ;
- (c)  $V_{-n}(sl(2)) \otimes \tilde{V}_{-2}(D_n)$  is conformally embedded into  $V_{-1/2}(C_{2n})$ : see Theorem 4.1 (4) in case  $n = 1$ .

Start with the following inclusion of vertex algebras:

$$(7.4) \quad \overline{V}_{-2}(A_{n-1}) \otimes M(1) \hookrightarrow \tilde{V}_{-2}(D_n).$$

Tensoring both members of (7.4) with  $V_{-n}(sl(2))$ , we obtain a chain:

$$(7.5) \quad V_{-n}(sl(2)) \otimes \overline{V}_{-2}(A_{n-1}) \otimes M(1) \hookrightarrow V_{-n}(sl(2)) \otimes \tilde{V}_{-2}(D_n) \rightarrow V_{-1/2}(C_{2n}),$$

where, by (c), the rightmost map is a conformal embedding. So, to prove (1), it suffices to prove that the embedding  $V_{-n}(sl(2)) \otimes \overline{V}_{-2}(A_{n-1}) \otimes M(1) \rightarrow V_{-1/2}(C_{2n})$  is conformal. This follows from the chain

$$V_{-n}(sl(2)) \otimes \overline{V}_{-2}(A_{n-1}) \otimes M(1) \rightarrow V_{-1}(A_{2n-1}) \otimes M(1) \rightarrow V_{-1/2}(C_{2n}),$$

where the leftmost map is conformal by (b) and the rightmost is conformal by (a).

We now prove statements (2) and (3). Recall that vertex algebras  $V_{-2}(D_n)$  (for  $n \geq 4$ ) and  $V_{-2}(B_n)$  (for  $n \geq 3$ ) have only finitely many irreducible modules in the category  $\mathcal{O}$  [11], [13]. On the other hand, Lemma 7.1 shows that  $\tilde{V}_{-2}(D_n)$  and  $\tilde{V}_{-2}(B_n)$  have infinitely many irreducible modules, and therefore we conclude that  $\tilde{V}_{-2}(D_n)$  (for  $n \geq 4$ ) and  $\tilde{V}_{-2}(B_n)$  (for  $n \geq 3$ ) cannot be simple.

The simplicity of  $\tilde{V}_{-2}(D_3)$  follows from the following facts:

- The maximal ideal in  $V^{-2}(D_3)$  is generated by a unique singular vector  $v_{sing}$  of conformal weight 2 (the vector  $\sigma(v_0)$  from [12, Theorem 8.2], in the case  $A_3 = D_3$ ).
- By [23, [Theorem 0.2.1],  $\overline{V}_{-2}(A_2) = V_{-2}(A_2)$ . Since  $V_{-2}(A_2) \otimes M(1)$  is conformally embedded into  $\tilde{V}_{-2}(D_3)$ , then  $v_{sing} = 0$  in  $\tilde{V}_{-2}(D_3)$ . Indeed, it was proved in [1] that in the case of conformal embeddings, a singular vector at conformal weight 2 must vanish. Here we have only one singular vector of such conformal weight.

So  $\tilde{V}_{-2}(D_3) = V_{-2}(D_3)$ .

**Remark 7.2.** In our forthcoming paper [5] we prove that the simple vertex algebra  $V_{-2}(B_2)$  embeds into  $M_{(5)}$ . We expect that in the cases  $m = 5, 6$   $M_{(m)}$  is completely reducible. But for  $m \geq 7$  the vertex algebras  $\tilde{V}_{-2}(so(m))$  are non-simple, having a simple maximal ideal (cf. [5]).

**Remark 7.3.** Since  $V_{-1/2}(C_n)$  can be realized using Weyl vertex algebra  $M_{(n)}$ , the previous theorem gives an explicit bosonic realization of the simple vertex algebra  $V_{-2}(A_3) = V_{-2}(D_3)$ . Our result gives a chiralization of [12, Theorem 8.13], where it was proved that there is an embedding of Zhu's algebra  $A(V_{-2}(A_3))$  into the Weyl algebra.

The conformal embedding in Theorem 7.2 (1) gives a proper framework for studying conformal embedding  $A_{n-1} \times Z$  in  $D_n$  at  $k = -2$  for which decomposition is still unknown in the cases  $n = 3, 4$ .

We shall now describe branching rules for conformal embedding  $A_{n-1} \times Z$  in  $D_n$  realized inside  $V_{-1/2}(C_{2n})$ . For a proof we need an important observation which gives a refinement of [2, Theorem 2.4].

**Proposition 7.3.** *Assume that an affine vertex algebra  $\overline{V}_k(\mathfrak{g}_0)$  is conformally embedded into  $\tilde{V}_k(\mathfrak{g})$  and  $\tilde{V}_k(\mathfrak{g})$  is a vertex subalgebra of a simple vertex algebra  $\mathcal{U}$ . In the hypothesis of [2, Theorem 2.4], we have*

$$\tilde{V}_k(\mathfrak{g}) = \bigoplus_{q \in \mathbb{Z}} \tilde{V}_k(\mathfrak{g})^{(q)}, \quad 0 \neq \tilde{V}_k(\mathfrak{g})^{(q)} \cdot \tilde{V}_k(\mathfrak{g})^{(r)} \subset \tilde{V}_k(\mathfrak{g})^{(q+r)} \quad (q, r \in \mathbb{Z}),$$

and each  $\tilde{V}_k(\mathfrak{g})^{(q)}$  is a cyclic  $\overline{V}_k(\mathfrak{g}_0)$ -module (i.e., highest weight  $\widehat{\mathfrak{g}}_0$ -module).

**Corollary 7.4.** *Assume that  $m \geq 5$ . Then*

$$\tilde{V}_{-2}(D_m) = \sum_{\ell \in \mathbb{Z}} \tilde{V}_{-2}(D_m)^{(\ell)}$$

and each  $\tilde{V}_{-2}(D_m)^{(\ell)}$  is a highest weight  $\widehat{gl(m)}$ -module at level  $-2$ .

*Proof.* In the case  $m \geq 5$ , we proved in [2, Theorem 5.1] that

$$V_{-2}(D_m) = \sum_{\ell \in \mathbb{Z}} V_{-2}(D_m)^{(\ell)},$$

and each  $V_{-2}(D_m)^{(\ell)}$  is an irreducible  $V_{-2}(gl(m))$ -module. Our proof used a fusion rules method. We shall now extend this result to a non-simple vertex algebra  $\tilde{V}_{-2}(D_m)$ . Note that  $\tilde{V}_{-2}(D_m)$  is realized as a subalgebra of the simple vertex algebra  $M_{(2m)}$ . As in the proof of [2, Theorem 5.1 (3)] we conclude that the conditions of [2, Theorem 2.4] hold for  $m \geq 5$ . Now Proposition 7.3 implies that

$$(7.6) \quad \tilde{V}_{-2}(D_m) = \sum_{\ell \in \mathbb{Z}} \tilde{V}_{-2}(D_m)^{(\ell)},$$

and each  $\tilde{V}_{-2}(D_m)^{(\ell)}$  is a highest weight  $\overline{V}_{-2}(A_{m-1}) \otimes M(1)$ -module. Moreover,  $V_{-2}(D_m)^{(\ell)}$  is a simple quotient of  $\tilde{V}_{-2}(D_m)^{(\ell)}$ .

## 8. HOWE DUAL PAIRS AND THE LINSHAW-SCHWARZ-SONG'S METHOD

In this section we shall combine results from previous Section and the methods from the paper [31]. As a consequence we will get a new realization of Howe dual pairs of affine vertex algebras at negative levels.

We now recall the setting of [31]. Given a vector space  $V$ , denote by  $\mathcal{S}(V)$  the vertex algebra with even generators  $\beta^x, \gamma^{x'}, x \in V, x' \in V^*$  with  $\lambda$ -brackets

$$[\beta^x_\lambda \gamma^{x'}] = x'(x), [\beta^x_\lambda \beta^y] = 0, [\gamma^{x'}_\lambda \gamma^{y'}] = 0.$$

If  $V$  is a module for a reductive Lie algebra  $\mathfrak{g} = \sum_i \mathfrak{g}_i$ , then one can define a vertex algebra map  $\hat{\tau} : \otimes_i V^{-k_i}(\mathfrak{g}_i) \rightarrow \mathcal{S}(V)$  by setting

$$\hat{\tau}(X) = - \sum_i : \gamma^{x'_i} \beta^{X \cdot x_i} :,$$

where  $\{x_i\}$  is a basis of  $V$  and  $\{x'_i\}$  is its dual basis, and  $k_i$  is the ratio between the trace form induced by  $V$  and the normalized invariant form of  $\mathfrak{g}_i$ . We specialize to  $V = \mathbb{C}^2 \otimes \mathbb{C}^m$  and  $\mathfrak{g} = \mathfrak{gl}(2) \times \mathfrak{gl}(m)$ .

Given a pair  $(U, \langle \cdot, \cdot \rangle)$ , where  $U$  is a vector space and  $\langle \cdot, \cdot \rangle$  is a symplectic form, denote by  $M(U)$  the universal vertex algebra with generators  $u \in U$  and  $\lambda$ -bracket defined by  $[u_\lambda v] = \langle u, v \rangle$ . Note that choosing  $U = \mathbb{C}^{2m}$  with the standard symplectic form given by  $\langle e_i, e_j \rangle = \delta_{j, i+m}$  for  $i = 1, \dots, m; j = 1, \dots, 2m$ , one has that the map  $e_i \mapsto a_i^+, e_{i+m} \mapsto a_i^-, i = 1, \dots, m$ , gives an isomorphism between  $M(\mathbb{C}^{2m})$  and  $M_{(m)}$ . On the other hand, choosing  $U = V \oplus V^*$  with the symplectic form such that  $V$  and  $V^*$  are isotropic and  $\langle x, x' \rangle = x'(x)$  for  $x \in V$  and  $x' \in V^*$ , one obtains that  $\mathcal{S}(V) = M(V \oplus V^*)$ .

Given pairs  $(U, \langle \cdot, \cdot \rangle), (U', \langle \cdot, \cdot \rangle')$  then any linear isomorphism  $\phi : U \rightarrow U'$  preserving the symplectic structures induces an isomorphism of vertex algebras between  $M(U)$  and  $M(U')$ . In particular we define  $\phi : (\mathbb{C}^2 \otimes \mathbb{C}^m) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^m)^* \rightarrow \mathbb{C}^{4m}$  by setting

$$\begin{aligned} e_1 \otimes e_j &\mapsto \frac{1}{\sqrt{2}}(e_j + \sqrt{-1}e_{m+j}), \quad e_2 \otimes e_j \mapsto \frac{1}{\sqrt{2}}(e_{2m+j} + \sqrt{-1}e_{3m+j}), \\ e^1 \otimes e^j &\mapsto \frac{1}{\sqrt{2}}(e_{2m+j} - \sqrt{-1}e_{3m+j}), \quad e^2 \otimes e^j \mapsto -\frac{1}{\sqrt{2}}(e_j - \sqrt{-1}e_{m+j}), \end{aligned}$$

The symplectic isomorphism  $\phi$  induces an isomorphism

$$\Phi : \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^m) \rightarrow M_{(2m)}.$$

We now compute  $\Phi(\hat{\tau}(V^{-m}(sl(2)) \otimes V^{-2}(sl(m))))$ . We claim that

$$\Phi(\hat{\tau}(V^{-m}(sl(2)) \otimes V^{-2}(sl(m)))) = V_{-m}(sl(2)) \otimes \bar{V}_{-2}(A_{m-1}).$$

It is enough to compute the image under  $\Phi \circ \hat{\tau}$  on generators of  $sl(2) \times sl(m)$ :

$$\begin{aligned} \Phi(\hat{\tau}(e)) &= \Phi(-\sum_{i=1}^m : \gamma^{e^2 \otimes e^i} \beta^{e_1 \otimes e_i} :) = -\frac{1}{2} \sum_{i=1}^{2m} : a_i^+ a_i^+ :. \\ \Phi(\hat{\tau}(h)) &= \Phi(-\sum_{i=1}^m (: \gamma^{e^1 \otimes e^i} \beta^{e_1 \otimes e_i} : - : \gamma^{e^2 \otimes e^i} \beta^{e_2 \otimes e_i} :)) = -\sum_{i=1}^{2m} : a_i^+ a_i^- :. \\ \Phi(\hat{\tau}(f)) &= \Phi(-\sum_{i=1}^m : \gamma^{e^1 \otimes e^i} \beta^{e_2 \otimes e_i} :) = -\frac{1}{2} \sum_{i=1}^{2m} : a_i^- a_i^- :. \end{aligned}$$

If  $i \neq j$ , with notation as in (7.2)

$$\begin{aligned} \Phi(\hat{\tau}(E_{ij})) &= \Phi(-\sum_{r=1}^2 : \gamma^{e^r \otimes e^j} \beta^{e_r \otimes e_i} :) = : a_i^+ a_j^- : + \sqrt{-1} : a_{m+i}^+ a_j^- : \\ &\quad - \sqrt{-1} : a_i^+ a_{m+j}^- : + : a_{m+i}^+ a_{m+j}^- : - : a_j^+ a_i^- : - \sqrt{-1} : a_j^+ a_{m+i}^- : \\ &\quad + \sqrt{-1} : a_{m+j}^+ a_i^- : - : a_{m+j}^+ a_{m+i}^- : = (\mathcal{E}_{\epsilon_j - \epsilon_i})_{(-1)} \mathbf{1}. \end{aligned}$$

Finally, if  $i = j$ ,

$$\Phi(\hat{\tau}(E_{ii})) = \Phi(-\sum_{r=1}^2 : \gamma^{e^r \otimes e^i} \beta^{e_r \otimes e_i} :) = \sqrt{-1}(: a_i^+ a_{m+i}^- : - : a_{m+i}^+ a_i^- :) = -(h_i)_{(-1)} \mathbf{1},$$

hence  $\Phi(\hat{\tau}(E_{ii} - E_{jj})) \in \bar{V}_{-2}(A_{m-1})$ .

As  $gl(m) = \mathbb{C}I \oplus sl(m)$ , we denote by  $V^{-2}(gl(m))$  the vertex algebra  $V^{-2}(sl(m)) \otimes M(1)$ . Note that the above computations show that  $\Phi(\hat{\tau}(V^{-2}(gl(m)))) \subset \tilde{V}_{-2}(D_m)$ .

**Proposition 8.1.** *For  $m \geq 3$  we have*

$$(8.1) \quad Com(V_{-m}(sl(2)), M_{(2m)}) = \tilde{V}_{-2}(so(2m)).$$

*Proof.* It is clear that  $Com(V_{-m}(sl(2)), M_{(2m)}) \supset \tilde{V}_{-2}(so(2m))$ . By [31, Theorem 4.3],  $\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^m)^{sl_2[t]}$  is generated by  $\hat{\tau}(V^{-2}(gl(m)))$  together with generators

$$\begin{aligned} D_{i,j} &=: \beta^{e_1 \otimes e_i} \beta^{e_2 \otimes e_j} : - : \beta^{e_2 \otimes e_i} \beta^{e_1 \otimes e_j} :, & \{i, j\} &\subset \{1, \dots, m\}, \\ D'_{i,j} &=: \gamma^{e^1 \otimes e^i} \gamma^{e^2 \otimes e^j} : - : \gamma^{e^2 \otimes e^i} \gamma^{e^1 \otimes e^j} :, & \{i, j\} &\subset \{1, \dots, m\}. \end{aligned}$$

We already saw that  $\Phi(\hat{\tau}(V^{-2}(gl(m)))) \subset \tilde{V}_{-2}(so(2m))$ . As for  $\Phi(D_{i,j}), \Phi(D'_{i,j})$  we have

$$\begin{aligned} 2\Phi(D_{i,j}) &=: a_i^+ a_j^- : + \sqrt{-1} : a_i^+ a_{m+j}^- : + \sqrt{-1} : a_{m+i}^+ a_j^- : - : a_{m+i}^+ a_{m+j}^- : \\ &- : a_j^+ a_i^- : - \sqrt{-1} : a_j^+ a_{m+i}^- : - \sqrt{-1} : a_{m+j}^+ a_i^- : + : a_{m+j}^+ a_{m+i}^- := (\mathcal{E}_{-\epsilon_j - \epsilon_i})_{(-1)} \mathbf{1}. \end{aligned}$$

Likewise

$$\begin{aligned} -2\Phi(D'_{i,j}) &=: a_j^+ a_i^- : - \sqrt{-1} : a_{m+j}^+ a_i^- : - \sqrt{-1} : a_j^+ a_{m+i}^- : - : a_{m+j}^+ a_{m+i}^- : \\ &- : a_i^+ a_j^- : + \sqrt{-1} : a_i^+ a_{m+j}^- : + \sqrt{-1} : a_{m+i}^+ a_j^- : + : a_{m+i}^+ a_{m+j}^- := (\mathcal{E}_{\epsilon_i + \epsilon_j})_{(-1)} \mathbf{1}. \end{aligned}$$

**Corollary 8.2.** *The following vertex subalgebras form Howe dual pairs:*

- (1)  $V_{-m}(sl(2))$  and  $\tilde{V}_{-2}(D_m)$  inside  $V_{-1/2}(C_{2m})$  for  $m \geq 3$ .
- (2)  $V_{-m}(sl(2))$  and  $\overline{V}_{-2}(A_{m-1})$  inside  $V_{-1}(A_{2m-1})$  for  $m \geq 5$ .

*Proof.* The proof of assertion (1) follows from Proposition 8.1.

By using Corollary 7.4 we get

$$(8.2) \quad \tilde{V}_{-2}(D_m) = \sum_{\ell \in \mathbb{Z}} \tilde{V}_{-2}(D_m)^{(\ell)},$$

and each  $\tilde{V}_{-2}(D_m)^{(\ell)}$  is a highest weight  $\overline{V}_{-2}(A_{m-1}) \otimes M(1)$ -module. So  $V_{-m}(sl(2))$  and  $\overline{V}_{-2}(A_{m-1}) \otimes M(1)$  form a Howe pair inside the charge zero component of  $M_{(2m)}$ , which, by [6], is isomorphic to  $V_{-1}(A_{2m-1}) \otimes M(1)$ . This proves assertion (2).  $\square$

**Remark 8.1.** Note that Corollary 8.2 in the case  $m = 4$  partially proves the Conjecture (5.3) of D. Gaiotto from his recent paper [19]. The conjecture is written as

$$\frac{Sb(\mathbb{C}^{16})}{\widehat{SU}(2)_{-4}} \cong \widehat{SO}(8)_{-2}.$$

Since, in Gaiotto's notation,  $Sb(\mathbb{C}^{16})$  is precisely our  $M_{(8)}$ , the coset is indeed the affine vertex algebra  $\tilde{V}_{-2}(D_4)$ , but it is not simple (as is probably expected in [19]).

### 9. THE DECOMPOSITION IN THE CASE $m = 3$ AND AN $q$ -SERIES IDENTITY

In this section we prove the complete reducibility of  $M_{(3)}$  as  $V_{-3/2}(sl(2)) \otimes V_{-4}(sl(2))$ -module and find the explicit decomposition. It is interesting that this decomposition gives a vertex-algebraic interpretation of the  $q$ -series identity from [26, Example 5.2] (see also [35, Theorem 4]). This suggests that other decompositions from the previous section are related to certain  $q$ -series identities.

Let

$$\phi(q) = \prod_{n \geq 1} (1 - q^n).$$

We shall first identify characters of  $V_k(sl(2))$ -modules for  $k = -3/2$  and  $k = -4$ .

**Lemma 9.1.** *Let  $\ell \in \mathbb{Z}_{\geq 0}$ . We have:*

$$\begin{aligned} ch_q L_{\widehat{sl(2)}}(-(\frac{3}{2} + \ell)\Lambda_0 + \ell\Lambda_1) &= q^{3/8} \phi(q)^{-3} (\ell + 1) q^{\frac{\ell(\ell+2)}{2}}, \\ ch_q L_{\widehat{sl(2)}}(-(4 + 2\ell)\Lambda_0 + 2\ell\Lambda_1) &= q^{-1/4} \phi(q)^{-3} \sum_{i=0}^{\ell} (-1)^{\ell-i} (2i + 1) q^{-\frac{i(i+1)}{2}}. \end{aligned}$$

*Proof.* For  $r \in \mathbb{Z}_{\geq 0}$  and  $k \in \mathbb{C}$  denote by  $V^k(r\omega_1)$  the generalized Verma module

$$V^k(r\omega_1) = U(\widehat{sl(2)}) \otimes_{U(\mathfrak{p})} L_{sl(2)}(r\omega_1),$$

where  $\mathfrak{p} = sl(2) \otimes \mathbb{C}[t] + \mathbb{C}K$  and  $L_{sl(2)}(r\omega_1)$  denotes the irreducible  $(r + 1)$ -dimensional  $sl(2)$ -module, regarded as  $\mathfrak{p}$ -module on which  $K$  acts by  $k\text{Id}$  and  $sl(2) \otimes t\mathbb{C}[t]$  acts trivially.

The proof of the lemma is a consequence of the following facts from the structure theory of Verma modules for  $\widehat{sl(2)}$  (we omit details):

- (1)  $V^{-3/2}(\ell\omega_1)$  is irreducible for every  $\ell \geq 0$ , hence

$$V^{-3/2}(\ell\omega_1) = L_{\widehat{sl(2)}}(-(\frac{3}{2} + \ell)\Lambda_0 + \ell\Lambda_1).$$

One can prove the irreducibility by using the fact that the Hamiltonian reduction maps the Weyl module  $V^{-3/2}(\ell\omega_1)$  to an irreducible module for the Virasoro algebra at central charge  $c = -2$ . We omit details, since they are similar to the proof of [4, Theorem 5.3].

- (2) The vector space of all singular vector in  $V^{-4}(2\ell\omega_1)$  is spanned by the set  $\{v_i \mid i = 0, \dots, \ell\}$  where  $v_i$  is the unique (up to a constant) singular vector of  $\mathfrak{g}$ -weight  $2i\omega_1$ .
- (3) If  $\ell \geq 1$ , the maximal submodule of  $V^{-4}(2\ell\omega_1)$  is irreducible, and it is generated by the singular vector  $v_{\ell-1}$ .

**Theorem 9.2.**  *$M_{(3)}$  is a completely reducible  $V_{-3/2}(sl(2)) \otimes V_{-4}(sl(2))$ -module and the following decomposition holds*

$$(9.1) \quad M_{(3)} = \bigoplus_{\ell=0}^{\infty} \left( L_{\widehat{sl(2)}}(-(\frac{3}{2} + \ell)\Lambda_0 + \ell\Lambda_1) \otimes L_{\widehat{sl(2)}}(-(4 + 2\ell)\Lambda_0 + 2\ell\Lambda_1) \right).$$

*Proof.* Note that the  $q$  character of  $M_{(3)}$  is

$$\begin{aligned}
\text{ch}_q M_{(3)} &= q^{-c/24} \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{-6} \\
&= q^{1/8} \prod_{n=1}^{\infty} (1 - q^{n-1/2})^{-6} \quad (\text{since } c = -3) \\
&= q^{1/8} \left( \frac{\phi(q)}{\phi(q^{1/2})} \right)^6 = q^{1/8} \frac{1}{\phi(q)^6} \left( \frac{\phi(q)^2}{\phi(q^{1/2})} \right)^6 \\
&= q^{1/8} \frac{1}{\phi(q)^6} \Delta(q^{1/2})^6
\end{aligned}$$

where

$$\Delta(q) = \sum_{n \in \mathbb{Z}_{\geq 0}} q^{n(n+1)/2} = \frac{\phi(q^2)^2}{\phi(q)}.$$

Using Lemma 9.1 we get that the  $q$ -character of the right side of (9.1) is given by

$$\frac{q^{1/8}}{\phi(q)^6} \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} (-1)^{\ell-i} (\ell+1)(2i+1) q^{\frac{\ell(\ell+2)-i(i+1)}{2}}.$$

Therefore the proof of the theorem is now reduced to the following identity:

$$(9.2) \quad \frac{\phi(q)^{12}}{\phi(q^{1/2})^6} = \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} (-1)^{\ell-i} (\ell+1)(2i+1) q^{\frac{\ell(\ell+2)-i(i+1)}{2}}.$$

Now we shall see that (9.2) follows from the Kac-Wakimoto identity [26, Example 5.2]:

$$\Delta(q)^6 = -\frac{1}{8} \sum_{(j,k) \in S} (-1)^{\frac{1}{4}(j-1)(k+1)} (j^2 - k^2) q^{\frac{1}{4}(jk-3)},$$

where

$$S = \{(j, k) \mid j, k, \frac{1}{2}(j-k) \in 2\mathbb{Z}_{\geq 0} + 1, j > k \geq 1\}.$$

For every  $(j, k) \in S$ , one can see that there are unique  $\ell, i \in \mathbb{Z}_{\geq 0}$ ,  $i \leq \ell$  such that

$$j = 2\ell + 2i + 3, \quad k = 2\ell - 2i + 1.$$

Hence we have:

$$\begin{aligned}
\Delta(q)^6 &= -\frac{1}{8} \sum_{(j,k) \in S} (-1)^{\frac{1}{4}(j-1)(k+1)} (j^2 - k^2) q^{\frac{1}{4}(jk-3)} \\
&= \left(-\frac{1}{8}\right) \cdot 2 \cdot 4 (-1) \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} (-1)^{\ell-i} (\ell+1)(2i+1) q^{\ell(\ell+2)-i(i+1)} \\
&= \sum_{\ell=0}^{\infty} \sum_{i=0}^{\ell} (-1)^{\ell-i} (\ell+1)(2i+1) q^{\ell(\ell+2)-i(i+1)},
\end{aligned}$$

proving (9.2).



### 10. THE CASE $m = 8$ AND AN APPLICATION TO CONFORMAL EMBEDDINGS

As anticipated in Remark 7.1, we now show that in the case  $m = 8$  there is a subsingular vector  $P_{high}^-$  outside of  $M^{sub}$ . As a consequence, the decomposition in this case is completely different from the one in the classical (i.e., non-affine) case. We believe that a similar pattern will occur when  $m \geq 8$ .

We have the following facts.

- The vertex algebra  $M_{(8)}$  is a module for  $\tilde{V}_{-2}(D_4) \otimes V_{-4}(sl(2))$ .
- $\tilde{V}_{-2}(D_4)$  is non-simple, and

$$\text{Com}(V_{-4}(sl(2)), M_{(8)}) \supset \tilde{V}_{-2}(D_4).$$

- $(\varphi_{(-1)})^2 \mathbf{1}$  is a singular vector which generates a  $\tilde{V}_{-2}(D_4) \otimes V_{-4}(sl(2))$ -submodule  $\tilde{L}_{sl(2)}(-6\Lambda_0 + 2\Lambda_1) \otimes \tilde{L}_{so(8)}(-4\Lambda_0 + 2\Lambda_1)$ . So

$$(\varphi_{(-1)})^2 \mathbf{1} = \tilde{w}_1 \otimes \tilde{w}_2,$$

where  $\tilde{w}_1$  (resp.  $\tilde{w}_2$ ) is a highest weight vector in  $\tilde{L}_{sl(2)}(-6\Lambda_0 + 2\Lambda_1)$  (resp.  $\tilde{L}_{so(8)}(-4\Lambda_0 + 2\Lambda_1)$ ).

- $\tilde{L}_{sl(2)}(-6\Lambda_0 + 2\Lambda_1)$  is a certain quotient of the generalized Verma  $\widehat{sl(2)}$ -module  $V^{-4}(2\omega_1)$  which has a singular vector

$$v_0 = \left( f(-1) - \frac{1}{2}h(-1)f(0) - \frac{1}{2}e(-1)f(0)^2 \right) v_{2\omega_1}$$

such that

$$V_{-4}(sl(2)) = U(\widehat{sl(2)}) \cdot v_0 \subset V^{-4}(2\omega_1).$$

So the vertex algebra  $V_{-4}(sl(2))$  is embedded into  $V^{-4}(2\omega_1)$ .

Set  $n = m/2 = 4$  and define

$$\begin{aligned} b_i^+ &= \frac{a_i^+ - \sqrt{-1}a_{n+i}^+}{\sqrt{2}}, & b_i^- &= \frac{a_i^- + \sqrt{-1}a_{n+i}^-}{\sqrt{2}}, \\ b_{n+i}^+ &= \frac{a_i^- - \sqrt{-1}a_{n+i}^-}{\sqrt{-2}}, & b_{n+i}^- &= \frac{a_i^+ + \sqrt{-1}a_{n+i}^+}{\sqrt{-2}}. \end{aligned}$$

Then we have

$$[(b_i^\pm)_\lambda b_j^\pm] = 0, \quad [(b_i^+)_\lambda (b_j^-)] = \delta_{i,j}.$$

Define the following vectors:

$$\begin{aligned} P^+ &= (f(-1) - \frac{1}{2}h(-1)f(0) - \frac{1}{2}e(-1)f(0)^2)(b_1^+)^2, \\ P_{high}^- &= ((b_1^+)^{-2}b_{n+1}^+ - (b_{n+1}^+)^{-2}b_1^+), \\ P_{low}^- &= ((b_1^-)^{-2}b_{n+1}^- - (b_{n+1}^-)^{-2}b_1^-). \end{aligned}$$

By direct calculation we have:

**Lemma 10.1.**

- (1)  $P_{high}^-$  is a highest weight vector for  $sl(2) \times so(8)$  of weight  $(0, 2\omega_1)$ .
- (2)  $P_{low}^- \in U(so(8)) \cdot P_{high}^-$ .

**Proposition 10.2.** *We have:*

- (1)  $P^+ \neq 0$ .
- (2)  $P^+ \in \text{Com}(V_{-4}(sl(2)), M_{(8)});$  i.e.,  $(sl(2) \otimes \mathbb{C}[t]).P^+ = 0$ .
- (3)  $P^+ \in \widetilde{V}_{-2}(D_4)$  and  $P^+$  is a singular vector for  $\widehat{so(8)}$  of weight  $-4\Lambda_0 + 2\Lambda_1$ .
- (4)  $P_{high}^-, P_{low}^- \notin M_{(8)}^{(sub)}$ .

**Corollary 10.3.**  $P_{high}^-$  is a subsingular vector in  $M_{(8)}^{(sub)}$  and  $P_{high}^- + M_{(8)}^{(sub)} \in M_{(8)}/M_{(8)}^{(sub)}$  generates  $V_{-4}(sl(2)) \otimes \widetilde{V}_{-2}(D_4)$ -module isomorphic to  $V_{-4}(sl(2)) \otimes \widetilde{L}_{so(8)}(-4\Lambda_0 + 2\Lambda_1)$ , where  $\widetilde{L}_{so(8)}(-4\Lambda_0 + 2\Lambda_1)$  is a highest weight  $\widetilde{V}_{-2}(D_4)$ -module.

*Proof.* We recall the following formulas for generators for  $\widetilde{V}_{-2}(D_4)$ :

$$\begin{aligned}
(\mathcal{E}_{\epsilon_j - \epsilon_i})_{(-1)} \mathbf{1} &= : a_i^+ a_j^- : + \sqrt{-1} : a_{n+i}^+ a_j^- : - \sqrt{-1} : a_i^+ a_{n+j}^- : + : a_{n+i}^+ a_{n+j}^- : \\
&\quad - : a_j^+ a_i^- : - \sqrt{-1} : a_j^+ a_{n+i}^- : + \sqrt{-1} : a_{n+j}^+ a_i^- : - : a_{n+j}^+ a_{n+i}^- : \\
&= (a_i^+ + \sqrt{-1} a_{n+i}^+)(a_j^- - \sqrt{-1} a_{n+j}^-) - (a_j^+ - \sqrt{-1} a_{n+j}^+)(a_i^- + \sqrt{-1} a_{n+i}^-) \\
&= -2(b_{n+i}^- b_{n+j}^+ + b_j^+ b_i^-), \\
(\mathcal{E}_{\epsilon_i + \epsilon_j})_{(-1)} \mathbf{1} &= : a_j^+ a_i^- : - \sqrt{-1} : a_{n+j}^+ a_i^- : - \sqrt{-1} : a_j^+ a_{n+i}^- : - : a_{n+j}^+ a_{n+i}^- : \\
&\quad - : a_i^+ a_j^- : + \sqrt{-1} : a_i^+ a_{n+j}^- : + \sqrt{-1} : a_{n+i}^+ a_j^- : + : a_{n+i}^+ a_{n+j}^- : \\
&= (a_j^+ - \sqrt{-1} a_{n+j}^+)(a_i^- - \sqrt{-1} a_{n+i}^-) - (a_i^+ - \sqrt{-1} a_{n+i}^+)(a_j^- + \sqrt{-1} a_{n+j}^-) \\
&= -2\sqrt{-1}(b_j^+ b_{n+i}^+ - b_i^+ b_{n+j}^+).
\end{aligned}$$

The generators of  $V_{-4}(sl(2))$  can be expressed as follows:

$$\begin{aligned}
e &= \sqrt{-1} \sum_{i=1}^n b_i^+ b_{n+i}^-, \\
f &= -\sqrt{-1} \sum_{i=1}^n b_i^- b_{n+i}^+, \\
h &= -\sum_{i=1}^n (b_i^+ b_i^- - b_{n+i}^+ b_{n+i}^-).
\end{aligned}$$

By direct calculation we get

$$\begin{aligned}
P^+ &= f(-1)((b_1^+)_{(-1)})^2 \mathbf{1} + e(-1)((b_5^+)_{(-1)})^2 \mathbf{1} - \sqrt{-1} h(-1)((b_1^+)_{(-1)}) b_5^+ \\
&= -3\sqrt{-1}((b_1^+)_{-2} b_5^+ - (b_5^+)_{-2} b_1^+) - \sqrt{-1} \sum_{i=2}^4 (b_1^+ b_i^- + b_5^+ b_{4+i}^-)(b_1^+ b_{4+i}^+ - b_5^+ b_i^+) \\
(10.1) &= -\frac{1}{4} \sum_{i=2}^4 (\mathcal{E}_{\epsilon_1 - \epsilon_i})_{(-1)} (\mathcal{E}_{\epsilon_1 + \epsilon_i})_{(-1)} \mathbf{1} \in \widetilde{V}_{-2}(D_4).
\end{aligned}$$

Note that (10.1) gives a non-trivial projection of the singular vector in  $V^{-2}(D_4)$  from [37, Theorem 3.1] in the case  $n = 1, \ell = 4$ .

Assume that  $P_{low}^- \in M_{(m)}^{(sub)}$ . Since it has conformal weight 2 we should have

$$P_{low}^- = P^-(0) + P^-(1) + P^-(2),$$

where

$$\begin{aligned} P^-(0) &\in V_{-4}(sl(2)) \otimes \tilde{V}_{-4}(D_4) \\ P^-(1) &\in \tilde{L}_{\widehat{sl(2)}}(-6\Lambda_0 + 2\Lambda_1) \otimes \tilde{L}_{\widehat{so(8)}}(-4\Lambda_0 + 2\Lambda_1), \\ P^-(2) &\in \tilde{L}_{\widehat{sl(2)}}(-8\Lambda_0 + 4\Lambda_1) \otimes \tilde{L}_{\widehat{so(8)}}(-6\Lambda_0 + 4\Lambda_1). \end{aligned}$$

By using fusion rules and the fact that  $P^+$  is singular vector in  $\tilde{V}_{-4}(D_4)$  one easily sees that  $P_{(3)}^+ P^-(i) = 0$  for  $i = 0, 1, 2$ . But

$$P_{(3)}^+ P_{low}^- = \nu \mathbf{1} \quad \nu \neq 0,$$

a contradiction. This proves that  $P_{low}^- \notin M_{(m)}^{(sub)}$ .

Note that  $P_{high}^-$  is subsingular for  $\widehat{sl(2)}$ :

$$\begin{aligned} e(0)P_{high}^- &= 0, \\ e(1)P_{high}^- &= (b_1^+)^2, \\ e(2)P_{high}^- &= 0, \\ f(0)P_{high}^- &= 0, \\ f(1)P_{high}^- &= -(b_5^+)^2, \\ f(2)P_{high}^- &= 0. \end{aligned}$$

So  $P_{high}^-, P_{low}^- \notin \text{Com}(V_{-4}(sl(2)), M_{(8)})$ .

Finally we conclude this section with one observation which gives an argument why our subsingular vectors do not appear in the classical Howe setting.

**Remark 10.1.** By taking suitable conformal vector, one can realize the Zhu's algebra of the Weyl vertex algebra  $M_{(m)}$  as the classical Weyl algebra  $A_m$  with generators  $x_i, \frac{\partial}{\partial x_i}$  and commutation relation  $[\frac{\partial}{\partial x_i}, x_j] = \delta_{i,j}$ . It is easy to see that the Zhu's functor maps subsingular vector  $P_{high}^-$  to zero. In our opinion this explain why subsingular vectors which do appear in our analysis, do not appear in classical settings.

**10.1. An application to conformal embeddings.** Here we want to study conformal embedding  $A_3 \times Z \hookrightarrow D_4$ ,  $Z$  1-dimensional abelian, at level  $k$ , and prove that the subalgebra of  $V_{-2}(D_4)$  generated by  $A_3$  is simple.

Note first that in previous sections appeared two vertex algebras associated to  $A_3$ :  $\overline{V}_{-2}(A_3)$  and  $\tilde{V}_{-2}(D_3) = \tilde{V}_{-2}(A_3)$ . Recall that  $\overline{V}_{-2}(A_3)$  is the vertex subalgebra of  $V_{-1}(sl(8)) \subset V_{-1/2}(C_8)$  generated by the elements of  $sl(4)$  in the conformal embedding  $sl(2) \times sl(4) \hookrightarrow sl(8)$  at  $k = -1/2$ . On the other hand  $\tilde{V}_{-2}(D_3)$  is a subalgebra of  $V_{-1/2}(C_6)$  generated by the elements of  $so(6)$  in the conformal embedding  $sl(2) \times so(6) \hookrightarrow sp(12)$  at  $k = -1/2$ .

Since  $\tilde{V}_{-2}(D_3)$  is simple by Theorem 7.2 and  $\overline{V}_{-2}(A_3)$  is not simple by Example 4.1 we conclude that  $\overline{V}_{-2}(A_3) \neq V_{-2}(A_3)$ .

**Lemma 10.4.** *The vertex subalgebra of  $V_{-2}(D_4)$  generated by the elements in  $A_3$  is simple.*

*Proof.* By direct calculation one sees that

$$(10.2) \quad w = (\mathcal{E}_{-\epsilon_1 - \epsilon_4})_{(0)} P^+ \in \overline{V}_{-2}(A_3)$$

is a non-trivial projection of the singular vector in  $V^{-2}(A_3)$  from [12, Theorem 8.2]. Since vector  $w$  generates the maximal ideal of  $\overline{V}_{-2}(A_3)$ , now relation (10.2) shows that  $w$  belongs to the maximal ideal in  $\tilde{V}_{-2}(D_4)$ , and hence we get a non-vanishing homomorphism  $V_{-2}(A_3) \rightarrow V_{-2}(D_4)$ . The claim follows.  $\square$

Thus we conclude:

**Corollary 10.5.** *We have:*

- (1)  $\overline{V}_{-2}(A_3) \otimes M(1)$  is conformally embedded into  $\tilde{V}_{-2}(D_4)$ , but it is not embedded into  $V_{-2}(D_4)$ .
- (2)  $V_{-2}(A_3) \otimes M(1)$  is conformally embedded into  $V_{-2}(D_4)$ .

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