



Innovative Applications of O.R.

Fluctuation identities with continuous monitoring and their application to the pricing of barrier options

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ABSTRACT

We present a numerical scheme to calculate fluctuation identities for exponential Lévy processes in the continuous monitoring case. This includes the Spitzer identities for touching a single upper or lower barrier, and the more difficult case of the two-barriers exit problem. These identities are given in the Fourier-Laplace domain and require numerical inverse transforms. Thus we cover a gap in the literature that has mainly studied the discrete monitoring case; indeed, there are no existing numerical methods that deal with the continuous case. As a motivating application we price continuously monitored barrier options with the underlying asset modelled by an exponential Lévy process. We perform a detailed error analysis of the method and develop error bounds to show how the performance is limited by the truncation error of the sinc-based fast Hilbert transform used for the Wiener–Hopf factorisation. By comparing the results for our new technique with those for the discretely monitored case (which is in the Fourier-z domain) as the monitoring time step approaches zero, we show that the error convergence with continuous monitoring represents a limit for the discretely monitored scheme.

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1. Introduction

Identities providing the Fourier-z transform of probability distribution functions of the extrema of a random path subject to monitoring at discrete intervals were first published by Spitzer (1956). They were extended to the continuous case by Baxter and Donsker (1957) and to double barriers by Kemperman (1963). The identities for the minimum and maximum of a path, for use with a single upper or lower barrier and for the two-barrier exit problem, are comprehensively described in the discrete monitoring case by Fusai, Germano, and Marazzina (2016), who proposed numerical methods to compute them for exponential Lévy processes. The discretely and continuously monitored identities are in the Fourier-z and Fourier-Laplace domains respectively. This means that, with the application of the inverse z or Laplace transform

as appropriate, they can be used within Fourier transform option pricing methods, which we will use as an example in this paper. The relevance of the Spitzer identity in several fields within operational research is nowadays well recognised. We mention, for example, the application to queuing systems, see the classical contributions by Cohen (1975, 1982) and Prabhu (1974) and more recent work by Bayer and Boxma (1996), Markov chains (Rogers, 1994), insurance (Chi & Lin, 2011), inventory systems (Cohen & Pekelman, 1978; Grassmann & Jain, 1989), and applied probability (Grassman, 1990), as well as in mathematical finance.

Pricing derivatives, especially exotic options, is a challenging problem often covered also in the operations research literature; see e.g. Kou (2008). Fusai et al. (2016) provide extensive references for this, as well as for many non-financial applications of the Hilbert transform and the related topics of Wiener–Hopf factorisation and Spitzer identities in insurance, queuing theory, physics, engineering, applied mathematics, etc. Derivative pricing with Fourier transforms was first investigated by Heston (1993). Carr and Madan (1999) published the first method with both the characteristic function and the payoff in the Fourier domain. Fang and Oosterlee (2008, 2009) devised the COS method based on the

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Table 1
Characteristic exponent of some Lévy processes.

Process	Characteristic exponent $\psi(\xi)$	Rational
Normal	$i\xi\mu - \frac{1}{2}\sigma^2\xi^2$	✓
Kou	$i\xi\mu - \frac{1}{2}\sigma^2\xi^2 + \lambda\left(\frac{(1-\rho)\eta_2}{\eta_2 + i\xi} + \frac{\rho\eta_1}{\eta_1 - i\xi}\right)$	✓
Merton	$i\xi\mu - \frac{1}{2}\sigma^2\xi^2 + \lambda(e^{i\xi\alpha - \frac{1}{2}\beta^2\xi^2} - 1)$	✗
NIG	$\delta\left(\sqrt{\alpha^2 - (\beta + i\xi)^2} - \sqrt{\alpha^2 - \beta^2}\right)$	✗
VG	$-\frac{1}{\nu}\log\left(1 - i\xi\theta\nu + \frac{1}{2}\nu\sigma^2\xi^2\right)$	✗

Fourier-cosine expansion. Innocentis and Levendorskiĭ (2014) coupled piecewise polynomial interpolation with an efficient version of the Fourier transform technique. Kirkby (2017) exploited the frame-projected transition densities, which transform the problem into the Fourier domain and accelerate the convergence of intermediate expectations. The Hilbert transform (King, 2009) has also been successfully employed: by Feng and Linetsky (2008) to price barrier options using backward induction in Fourier space, and by Marazzina, Fusai, and Germano (2012) and Fusai et al. (2016) to compute via the Plemelj–Sokhotsky relations the factorisations required by the Wiener–Hopf method and the Spitzer identities. For a comparison of the two approaches in the discrete monitoring case, see Phelan, Marazzina, Fusai, and Germano (2018). Feng and Linetsky showed that computing the Hilbert transform with the sinc expansion, as studied by Stenger (1993, 2011), gives errors that reduce exponentially as the number of fast Fourier transform (FFT) grid points increases. However, the Feng and Linetsky method cannot be extended to continuously monitored options because its recursive structure makes it an inherently discrete scheme. In contrast (Green, Fusai, & Abrahams, 2010) showed that methods based on the Spitzer identities can be extended to continuous monitoring using the Laplace transform in the time domain rather than the z-transform. Unfortunately, they limited their analysis to the Gaussian case.

In this article we implement a method to numerically calculate the required Wiener–Hopf factors and thence the Spitzer identities in continuous time; we apply this to price continuously monitored options with general exponential Lévy processes. For continuous monitoring, the Wiener–Hopf factorisation can be done analytically if the characteristic exponent is rational (see Eq. (7) and Table 1), i.e. for the Gaussian and Kou double exponential processes, or in some special cases, e.g. when the jumps are only positive or negative. It is also possible to approximate an irrational exponent with a rational one that is easily factored (Kuznetsov, 2010). However, an analytical solution for the continuous monitoring case which is usable for any exponential Lévy process and does not require approximation has not been found yet. Therefore, the importance of our contribution is that it provides a formula to compute the Wiener–Hopf factors with a single barrier or two barriers in the continuous monitoring case. Moreover, we also propose a fast and accurate numerical method to make the computation of the Wiener–Hopf factors operational for any Lévy process, even when the exponent is not rational, like the variance gamma process. In the discrete case an analytical Wiener–Hopf factorisation can be done only for a Gaussian process (Fusai, Abrahams, & Sgarra, 2006), but from a numerical point of view the problem is easier and there are a number of papers dealing with exponential Lévy processes. However, it is well known that the convergence of numerical methods for discrete monitoring to the continuous monitoring limit is very slow; (see e.g. Broadie, Glasserman, & Kou, 1997). Therefore this work contributes to the literature by providing a procedure to determine the finite-time distribution of the extrema and of the

hitting times in the presence of one or two barriers for a process with independent and identically distributed increments, such as a Lévy process, whereas previous numerical methods, like the one by Fusai et al. (2016), dealt only with discrete monitoring. Even if this article is mainly motivated by applications in option pricing, its relevance is very much beyond it. First-passage problems with models based on Markov processes are also ubiquitous in physical, biological, social, actuarial and other sciences. For example, our technique could be used to compute the ruin probability, i.e. the probability that a Lévy process takes value in a set A at a time $T > 0$ given that the process never falls below a barrier B in the interval $[0, T]$, i.e. $P(X(T) \in A, \min_{t \in [0, T]} X(t) > B)$. This is a classical problem in actuarial science and applied probability; see for example Klüppelberg, Kyprianou, and Maller, 2004. For applications in physics and biophysics, see e.g. the review by Bray, Majumdar, and Schehr (2013). Similar problems also arise in statistics, see for example the classical paper by Chernoff (1961), or in studying when a process reaches for the first time an adverse threshold state (a patient dies, or an industrial device breaks down).

This method follows the approach suggested by Green et al. (2010) and is based on the Fusai, Germano and Marazzina (FGM) method (Fusai et al., 2016) with spectral filtering (Phelan et al., 2018). While the latter method is for discrete monitoring and thus in the Fourier-z domain, here we operate in the Fourier-Laplace domain. Besides the discrete Fourier transform (DFT), or actually the fast Fourier transform (FFT), which is a standard technique, we also require a numerical inverse Laplace transform; for the latter we use an algorithm proposed by Abate and Whitt (1992a, 1995), which is based on a Fourier series and is derived in a similar way to their well established numerical inverse z-transform (Abate & Whitt, 1992b). Spectral filters are a powerful technique to improve Fourier-based option pricing, introduced to this field by Ruijter, Versteegh, and Oosterlee (2015). Cui, Kirkby, and Nguyen (2017) and Phelan et al. (2018) showed that multiplying the Fourier input by a spectral filter speeds up the price convergence when the characteristic functions decays slowly. At the end, the error convergence of our procedure is slightly worse than first-order polynomial; we explain this in detail with reference to the truncation error of the sinc-based discrete Hilbert transform. Our results show that the error convergence is consistent with the error bound and the performance of the discretely monitored technique as the monitoring interval goes to zero.

The structure of this paper is as follows. In Section 2 we briefly run through Fourier, Hilbert, Laplace and z transforms and explain how they are used for the calculation of the Spitzer identities. We then present a numerical pricing scheme for continuously monitored options and explain its relationship with the FGM pricing scheme for discrete monitoring. Section 3 provides a discussion of the error convergence of the pricing technique with special reference to the truncation error of the sinc-based Hilbert transform. Section 4 shows the results that were achieved, comparing them with the results for the FGM method for discretely monitored options.

2. Fourier transform methods for option pricing

In this paper we make extensive use of the Fourier transform (see e.g. Kreyszig, 2011; Polyanin & Manzhirov, 1998), an integral transform with many applications. Historically, it has been widely employed in spectroscopy and communications, therefore much of the literature refers to the function in the Fourier domain as its spectrum. According to the usual convention in the financial literature, the forward and inverse Fourier transforms are defined as

$$\hat{f}(\xi) = \mathcal{F}_{x \rightarrow \xi}[f(x)] = \int_{-\infty}^{+\infty} e^{i\xi x} f(x) dx, \quad (1)$$

$$f(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}[\hat{f}(\xi)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\xi x} \hat{f}(\xi) d\xi. \quad (2)$$

Let $S(t)$ be the price of an underlying asset and $x(t) = \log(S(t)/S_0)$ its log-price. To find the price $v(x, t)$ of an option at time $t = 0$ when the initial price of the underlying is $S(0) = S_0$, and thus its log-price is $x(0) = 0$, we need to discount the expected value of the undamped option payoff $\phi(x(T))e^{-\alpha x(T)}$ at maturity $t = T$ with respect to an appropriate risk-neutral probability distribution function (PDF) $p(x, T)$ whose initial condition is $p(x, 0) = \delta(x)$. As shown by Lewis (2001), this can be done using the Plancherel relation,

$$\begin{aligned} v(0, 0) &= e^{-rT} \mathbb{E}[\phi(x(T))e^{-\alpha x(T)} | x(0) = 0] \\ &= e^{-rT} \int_{-\infty}^{+\infty} \phi(x) e^{-\alpha x} p(x, T) dx \\ &= \frac{e^{-rT}}{2\pi} \int_{-\infty}^{+\infty} \hat{\phi}(\xi) \hat{p}^*(\xi + i\alpha, T) d\xi \\ &= e^{-rT} \mathcal{F}_{\xi \rightarrow x}^{-1}[\hat{\phi}(\xi) \hat{p}^*(\xi + i\alpha, T)](0). \end{aligned} \quad (3)$$

Here, $\hat{p}^*(\xi + i\alpha, T)$ is the complex conjugate of the Fourier transform of $e^{-\alpha x} p(x, T)$. To price options using this relation, we need the Fourier transforms of both the damped payoff and the PDF. A double-barrier option has the damped payoff

$$\phi(x) = e^{\alpha x} S_0 (\theta(e^x - e^k))^+ \mathbf{1}_{[l, u]}(x), \quad (4)$$

where $\theta = 1$ for a call, $\theta = -1$ for a put, $\mathbf{1}_A(x)$ is the indicator function of the set A , $k = \log(K/S_0)$ is the log-strike, $u = \log(U/S_0)$ is the upper log-barrier, $l = \log(L/S_0)$ is the lower log-barrier, K is the strike price, U is the upper barrier and L is the lower barrier. The damping factor $e^{\alpha x}$ ensures the integrability of the payoff function; see Feng and Linetsky (2008) for a full discussion of the selection of the damping parameter α , and the online supplementary material for additional results on the choice of α . The Fourier transform of the damped payoff $\phi(x)$ is available analytically,

$$\hat{\phi}(\xi) = S_0 \left(\frac{e^{(1+i\xi+\alpha)a} - e^{(1+i\xi+\alpha)b}}{1+i\xi+\alpha} - \frac{e^{k+(i\xi+\alpha)a} - e^{k+(i\xi+\alpha)b}}{i\xi+\alpha} \right), \quad (5)$$

where for a call option $a = u$ and $b = \max(k, l)$, while for a put option $a = l$ and $b = \min(k, u)$.

The Fourier transform of the PDF $p(x, t)$ of a stochastic process $X(t)$ is the characteristic function

$$\Psi(\xi, t) = \mathbb{E}[e^{i\xi X(t)}] = \int_{-\infty}^{+\infty} e^{i\xi x} p(x, t) dx = \mathcal{F}_{x \rightarrow \xi}[p(x, t)] = \hat{p}(\xi, t). \quad (6)$$

For a Lévy process the characteristic function can be written as $\Psi(\xi, t) = e^{\psi(\xi)t}$, where the characteristic exponent $\psi(\xi)$ is given by the Lévy–Khincine formula as

$$\psi(\xi) = i\mu\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} (e^{i\xi\eta} - 1 - i\xi\eta \mathbf{1}_{[-1,1]}(\eta)) \nu(d\eta). \quad (7)$$

The Lévy–Khincine triplet (μ, σ, ν) uniquely defines the Lévy process: μ defines the linear drift of the process, σ is the volatility of the diffusion part of the process, and the jump part of the process is specified so that $\nu(\eta)$ is the intensity of a Poisson process with jump size η . Under the risk-neutral measure the parameters of the triplet are linked by the equation

$$\mu = r - q - \frac{1}{2}\sigma^2 - \int_{\mathbb{R}} (e^\eta - 1 - i\eta \mathbf{1}_{[-1,1]}(\eta)) \nu(d\eta), \quad (8)$$

where r is the risk-free interest rate and q is the dividend rate. In general the characteristic function of a Lévy process is available in closed form, for example for the Gaussian (Schoutens, 2003), normal inverse Gaussian (NIG) (Barndorff-Nielsen, 1998), CGMY

(Carr, Geman, Madan, & Yor, 2002), Kou double exponential (Kou, 2002), Merton jump-diffusion (Merton, 1976), Lévy alpha-stable (Nolan, 2018), variance gamma (VG) (Madan & Seneta, 1990) and Meixner (Schoutens, 2003) processes.

Some pricing techniques based on the Fourier transform also use the Hilbert transform, which is an integral transform related to the Fourier transform. Unlike with the Fourier transform, the function under transformation remains in the same domain, rather than moving between the x and ξ domains. The Hilbert transform of a function in the Fourier domain is defined as

$$\begin{aligned} \mathcal{H}[\hat{f}(\xi)] &= \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\hat{f}(\xi')}{\xi - \xi'} d\xi' \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \left(\int_{\xi-1/\epsilon}^{\xi-\epsilon} \frac{\hat{f}(\xi')}{\xi - \xi'} d\xi' + \int_{\xi+\epsilon}^{\xi+1/\epsilon} \frac{\hat{f}(\xi')}{\xi - \xi'} d\xi' \right), \end{aligned} \quad (9)$$

where P.V. denotes the Cauchy principal value. Applying the Hilbert transform in the Fourier domain is equivalent to multiplying the function in the x domain by $-i \operatorname{sgn} x$.

Whilst the Fourier and Hilbert transform operate on the state variable (here the log-price), the Laplace transform is applied to time. The forward and inverse Laplace transforms are

$$\mathcal{L}_{t \rightarrow s}[f(t)] = \tilde{f}(s) := \int_0^{+\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \quad (10)$$

$$\mathcal{L}_{s \rightarrow t}^{-1}[\tilde{f}(s)] = f(t) := \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \tilde{f}(s) ds, \quad (11)$$

where $a \in \mathbb{R}$ is on the right of all singularities of $\tilde{f}(s)$ in the complex plane. The Laplace transform is closely related to the z -transform of a discrete function $f(t_n) = f(n)$, $n \in \mathbb{N}_0$,

$$\mathcal{Z}_{n \rightarrow q}[f(n)] := \sum_{n=0}^{\infty} q^n f(t_n), \quad q \in \mathbb{C}. \quad (12)$$

Given a continuous function $f_c(t)$, we define the discrete function $f_d(t_n)$ consisting of sampled values of the former, where Δt is the sampling interval and $t_n = n\Delta t$ are the sampling times. Then with a z -transform parameter $q = e^{-s\Delta t}$, the Laplace and z -transforms are related in the limit $\Delta t \rightarrow \infty$:

$$\begin{aligned} \mathcal{L}_{t \rightarrow s}[f_c(t)] &= \int_0^{\infty} e^{-st} f_c(t) dt = \lim_{\Delta t \rightarrow 0} \Delta t \sum_{n=0}^{\infty} e^{-sn\Delta t} f_c(n\Delta t) \\ &= \lim_{\Delta t \rightarrow 0} \Delta t \sum_{n=0}^{\infty} (e^{-s\Delta t})^n f_d(t_n) = \lim_{\Delta t \rightarrow 0} \Delta t \sum_{n=0}^{\infty} q^n f_d(t_n) \\ &= \lim_{\Delta t \rightarrow 0} \Delta t \mathcal{Z}[f_d(t_n)]. \end{aligned} \quad (13)$$

2.1. Spitzer identities for continuous monitoring

If we wish to use Eq. (3) to price barrier options, the required characteristic functions are more complicated than the closed-form expressions referred to above. We need the characteristic function of the PDF of the value of a stochastic process $X(t)$ at time $t = T$, conditional on the process remaining inside continuously monitored upper and lower barriers. We use the identities published by Spitzer (1956) which were extended to the continuously monitored case by Baxter and Donsker (1957) and to double-barriers by Kemperman (1963). The Spitzer identities provide the Fourier- z transform of the PDF of a stochastic process $X(t)$ at time $t = T$, conditional on whether $X(t)$ reaches a barrier at discretely monitored times. The Fourier transform is applied to the process values and the z -transform is applied to the discrete monitoring times. Baxter and Donsker (1957) demonstrated that we can obtain the equivalent identities for continuously monitored barriers in the Fourier-Laplace domain. Green et al. (2010) showed that the relationship between the Laplace and z -transforms described in Eq. (13) can

be exploited to price continuously monitored options using the Spitzer identities in the Fourier-Laplace domain.

An important aspect in the calculation of the Spitzer identities is the decomposition of a function $\widehat{f}(\xi)$ into + and - parts, $\widehat{f}_+(\xi) = \mathcal{F}_{x \rightarrow \xi}[f(x)\mathbf{1}_{\mathbb{R}_+}(x)]$ and $\widehat{f}_-(\xi) = \mathcal{F}_{x \rightarrow \xi}[f(x)\mathbf{1}_{\mathbb{R}_-}(x)]$, such that $\widehat{f}(\xi) = \widehat{f}_+(\xi) + \widehat{f}_-(\xi)$. This can be done directly in the Fourier domain using the Plemelj–Sokhotsky relations (Fusai et al., 2016; King, 2009):

$$\widehat{f}_+(\xi) = \frac{1}{2} \{ \widehat{f}(\xi) + i\mathcal{H}[\widehat{f}(\xi)] \}, \quad (14)$$

$$\widehat{f}_-(\xi) = \frac{1}{2} \{ \widehat{f}(\xi) - i\mathcal{H}[\widehat{f}(\xi)] \}. \quad (15)$$

The shift theorem $\mathcal{F}_{x \rightarrow \xi}[f(x+b)] = \widehat{f}(\xi)e^{-ib\xi}$ allows us to obtain the generalised Plemelj–Sokhotsky relations for an arbitrary barrier b :

$$\widehat{f}_{b+}(\xi) = \frac{1}{2} \{ \widehat{f}(\xi) + e^{ib\xi} i\mathcal{H}[e^{-ib\xi} \widehat{f}(\xi)] \}, \quad (16)$$

$$\widehat{f}_{b-}(\xi) = \frac{1}{2} \{ \widehat{f}(\xi) - e^{ib\xi} i\mathcal{H}[e^{-ib\xi} \widehat{f}(\xi)] \}. \quad (17)$$

Calculating the Spitzer identities also requires us to factorise a function, i.e. obtain $\widehat{g}_+(\xi)$ and $\widehat{g}_-(\xi)$ such that $\widehat{g}(\xi) = \widehat{g}_+(\xi)\widehat{g}_-(\xi)$. This is achieved by decomposing the logarithm $h(\xi) = \log \widehat{g}(\xi)$ and then exponentiating the results to obtain $\widehat{g}_+(\xi) = \exp \widehat{h}_+(\xi)$ and $\widehat{g}_-(\xi) = \exp \widehat{h}_-(\xi)$.

Green et al. (2010) dealt with fluctuation identities that can be used for lookback, single-barrier and double-barrier options. Here we concentrate on the identities for single-barrier down-and-out and double-barrier options. The first step is always to factorise $\Phi_c(\xi, s) = s - \psi(\xi) = \Phi_{c+}(\xi, s)\Phi_{c-}(\xi, s)$. For a single-barrier down-and-out option, the Laplace transform of the required characteristic function is

$$\widetilde{p}(\xi, s) = \frac{1 - \Phi_{c-}(\xi, s)P_{cl-}(\xi, s)}{\Phi_c(\xi, s)} = \frac{P_{cl+}(\xi, s)}{\Phi_{c+}(\xi, s)}, \quad (18)$$

where $P_c(\xi, s) = 1/\Phi_{c-}(\xi, s)$ is decomposed with respect to the lower log-barrier l using Eqs. (16) and (17). For a double-barrier option, the Laplace transform of the required characteristic function is

$$\widetilde{p}(\xi, s) = \frac{1 - \Phi_{c-}(\xi, s)J_{cl-}(\xi, s) - \Phi_{c+}(\xi, s)J_{cu+}(\xi, s)}{\Phi_c(\xi, s)}, \quad (19)$$

where $J_{cu+}(\xi, s)$ and $J_{cl-}(\xi, s)$ are the solution to the pair of coupled equations

$$J_{cu+}(\xi, s) = \left[\frac{1 - \Phi_{c-}(\xi, s)J_{cl-}(\xi, s)}{\Phi_{c+}(\xi, s)} \right]_{u+}, \quad (20)$$

$$J_{cl-}(\xi, s) = \left[\frac{1 - \Phi_{c+}(\xi, s)J_{cu+}(\xi, s)}{\Phi_{c-}(\xi, s)} \right]_{l-}. \quad (21)$$

For $u \rightarrow \infty$, $J_{cu+} \rightarrow 0$ and $J_{cl-} \rightarrow P_{cl-}$ we thus recover the Spitzer identity for the single barrier, Eq. (18). The latter can be calculated directly, while so far only an iterative solution has been found (Fusai et al., 2016; Phelan et al., 2018) to the coupled Eqs. (20) and (21).

2.1.1. Relationship to the Spitzer identities for discrete monitoring

In Section 4 we show numerical results comparing the error convergence obtained using the Spitzer identities for continuous monitoring with the performance of the closely related method using the Spitzer identities for discrete monitoring (Fusai et al., 2016; Green et al., 2010; Phelan et al., 2018).

The relationship between the two methods originates in the connection between the z-transform and the Laplace transform described in Eq. (13). As described in Section 2.1, the first step in pricing continuously monitored barrier options is the calculation of $\Phi_c(\xi, s) = s - \psi(\xi)$ in the Fourier-Laplace domain. The equivalent quantity in the Fourier-z domain for discrete monitoring is $\Phi(\xi, q) = 1 - q\Psi(\xi, \Delta t)$. We can use the relation in Eq. (13) with $q = e^{-s\Delta t}$ to relate the two:

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\Delta t}{\Phi(\xi, q)} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta t}{1 - q\Psi(\xi, \Delta t)} = \lim_{\Delta t \rightarrow 0} \frac{\Delta t}{1 - e^{-s\Delta t} e^{\psi(\xi)\Delta t}} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\Delta t}{1 - e^{(\psi(\xi) - s)\Delta t}} = \frac{1}{s - \psi(\xi)} = \frac{1}{\Phi_c(\xi, s)}. \end{aligned} \quad (22)$$

The same factorisation and decomposition steps described in Section 2.1 (Fusai et al., 2016; Green et al., 2010) are applied to both $\Phi(\xi, q)$ and $\Phi_c(\xi, s)$ to price options with respectively discrete or continuous monitoring.

2.2. Numerical methods

The methods in the previous section are described analytically. However, as they involve some expressions which cannot be solved in closed form, their implementation requires the use of numerical approximation techniques which we discuss in the following sections.

2.2.1. Discrete Fourier and Hilbert transforms and spectral filters

The forward and inverse Fourier transforms shown in Eqs. (1) and (2) are integrals over an infinite domain and in order to implement them numerically we need to approximate each with a discrete Fourier transform (DFT). We implement this in practice using the built-in Matlab FFT function which is based on the FFTW library by Frigo and Johnson (1998).

The calculation of the Hilbert transform of a function $\widehat{f}(\xi)$ can be realised with an inverse/forward Fourier transform pair and multiplication by the sign function in between,

$$i\mathcal{H}[\widehat{f}(\xi)] = \mathcal{F}_{x \rightarrow \xi}[\text{sgn}(x)\mathcal{F}_{\xi \rightarrow x}^{-1}\widehat{f}(\xi)]. \quad (23)$$

However, this results in an error that decreases quadratically with the grid step $\Delta\xi$. In order to obtain exponential error convergence, Feng and Linetsky (2008) and Fusai et al. (2016) have implemented the Hilbert transform using the sinc expansion techniques studied by Stenger (1993, 2011). Stenger showed that, given a function $\widehat{f}(\xi)$ which is analytic in the whole plane, the function and its Hilbert transform can be expressed as

$$\widehat{f}(\xi) = \sum_{k=-\infty}^{+\infty} \widehat{f}(k\Delta\xi) \frac{\sin(\pi(\xi - k\Delta\xi)/\Delta\xi)}{\pi(\xi - k\Delta\xi)/\Delta\xi}, \quad (24)$$

$$\mathcal{H}[\widehat{f}(\xi)] = \sum_{k=-\infty}^{+\infty} \widehat{f}(k\Delta\xi) \frac{1 - \cos(\pi(\xi - k\Delta\xi)/\Delta\xi)}{\pi(\xi - k\Delta\xi)/\Delta\xi}. \quad (25)$$

Stenger (1993) also showed that, when the function $f(\xi)$ is analytic in a strip of the complex plane including the real axis, the expressions in Eqs. (24) and (25) are approximations whose error decays exponentially as $\Delta\xi$ decreases. In addition to discretisation, the infinite sum in Eq. (25) must also be truncated to the grid size M , so that the Hilbert transform approximation becomes

$$\mathcal{H}[\widehat{f}(\xi)] \approx \sum_{k=-M/2}^{+M/2} \widehat{f}(k\Delta\xi) \frac{1 - \cos(\pi(\xi - k\Delta\xi)/\Delta\xi)}{\pi(\xi - k\Delta\xi)/\Delta\xi}. \quad (26)$$

Feng and Linetsky (2008, 2009) showed that if $\widehat{f}(\xi)$ decays at least exponentially as $|\xi| \rightarrow \infty$, i.e. $\widehat{f}(\xi) \leq \kappa \exp(-c|\xi|^\nu)$, then the error in the Hilbert transform and thus in the Plemelj–Sokhotsky relations caused by truncating the series in Eq. (25) is also exponentially bounded. Furthermore, Feng and Linetsky showed that

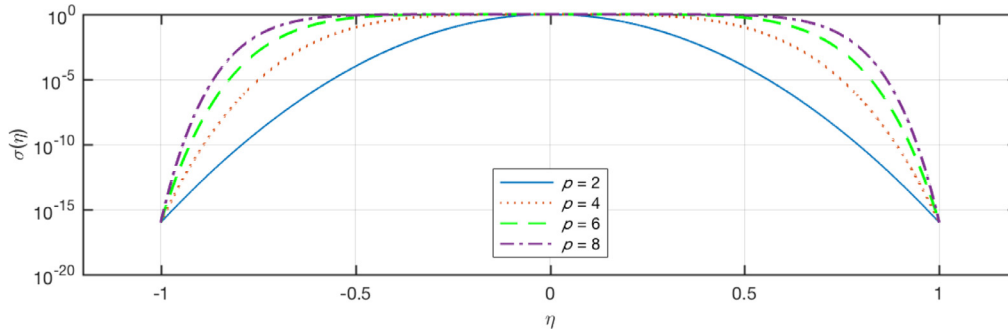


Fig. 1. Shape of the exponential filter plotted with different values of p .

if $\hat{f}(\xi)$ is polynomially bounded then, although the accuracy of the series in Eq. (25) is retained, the error caused by truncating the sum is no longer exponentially bounded. However, it has subsequently been shown that multiplying the input to the Hilbert transform by a filter can improve the error convergence (Phelan et al., 2018).

In the papers by Gottlieb and Shu (1997) and Vandeven (1991), a filter of order p is defined as a function $\sigma(\eta)$ supported on $\eta \in [-1, 1]$ with the properties

- a) $\sigma(0) = 1$, $\sigma^{(l)}(0) = 0$,
- b) $\sigma(\eta) = 0$ for $|\eta| = 1$,
- c) $\sigma(\eta) \in C^{p-1}$.

The scaled variable η is related to ξ in our application as $\eta = \xi/\xi_{\max}$. In this paper we use the exponential filter, which has the form (Gottlieb & Shu, 1997)

$$\sigma(\eta) = e^{-\vartheta|\eta|^p}, \quad (28)$$

where p is even. This does not strictly meet criterion b in Eq. (27) as it does not go exactly to zero when $|\eta| = 1$. However, if we select $\vartheta < \varepsilon \log 10$, where $10^{-\varepsilon}$ is the machine precision, then the filter coefficients are within computational accuracy of the requirements. The exponential filter has the advantages that it has a simple form and that it can be used for any even value of p . Moreover, the order of the filter is a parameter which is directly input to the filter equation. Filter shapes for a range of p values are shown in Fig. 1. Many filters other than the exponential can be used, e.g. the Planck taper (McKechan, Robinson, & Sathyaprakash, 2010; Phelan et al., 2018) and the raised cosine (Ruijter et al., 2015).

2.2.2. Inverse Laplace transform

The Spitzer identities provide the Laplace transform of the characteristic function, so to calculate the option price using Eq. (3) we must apply the inverse Laplace transform. We implement the numerical scheme by Abate and Whitt (1995), which uses the trapezoidal rule to approximate the analytic expression for the inverse Laplace transform shown in Eq. (11) with

$$f(t) \approx \frac{e^{A/2}}{2t} \text{Re} \tilde{f}\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k \text{Re} \tilde{f}\left(\frac{A+2k\pi i}{2t}\right), \quad (29)$$

where $\tilde{f}\left(\frac{A+2k\pi i}{2t}\right)$ is the Laplace transform $\tilde{f}(s)$ with $s = \frac{A+2k\pi i}{2t}$. The value of A controls the accuracy of the approximation; an accuracy of $10^{-\gamma}$ requires $A = \gamma \log(10)$. We then use the Euler transform to accurately approximate this infinite series. First the partial sums

$$b_k = \frac{e^{A/2}}{2t} \text{Re} \tilde{f}\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{j=1}^k (-1)^j \text{Re} \tilde{f}\left(\frac{A+2j\pi i}{2t}\right) \quad (30)$$

are calculated for $k = n_E, \dots, n_E + m_E$. We then take the binomially weighted average (Euler transform) of these terms, resulting in the

approximation

$$f(t) \approx \frac{1}{2^{m_E}} \sum_{k=0}^{m_E} \binom{m_E}{k} b_{n_E+k}. \quad (31)$$

The values of m_E and n_E are selected large enough to give sufficient accuracy, but low enough to avoid unnecessary computational effort. Numerical tests were carried out inverting the Laplace transform of a delayed unit step function $\tilde{f}(s) = e^{-\tau s}/s$ where the delay $\tau = 10$. This is an extreme test case as the step function has a jump discontinuity and Abate and Whitt (1992a) state that the performance bound of $10^{-\gamma} = e^{-A}$ does not apply in the presence of jumps. However it is important to consider the performance of the inverse Laplace transform with discontinuities in the time domain as the value of the contracts that we are pricing will abruptly become zero on expiry. The recovered functions for different values of A , m_E and n_E are shown in Fig. 2 and the errors are shown in Figs. 3 and 4. The empirical results in Fig. 4 show that we can select values for A , m_E and n_E so that, away from the discontinuity, the performance matches the bound of $10^{-\gamma} = e^{-A}$ specified by Abate and Whitt. Furthermore, we show in Sections 3 and 4 that the error bounds and observed results for the pricing procedure are limited by the performance of the sinc-based Hilbert transform. Therefore, we can use the Abate and Whitt inverse Laplace transform method to price mid- to long-dated options.

We base the selection of the parameters for the inverse Laplace transform on empirical results. From Figs. 2 and 3 we can see that the size of the oscillations due to the discontinuity are predominantly affected by m_E and n_E . The error floor is controlled by A ; the values of 18.4, 23 and 27 in Figs. 2–4 correspond to errors of approximately 10^{-8} , 10^{-10} and 10^{-12} . However, Fig. 4 shows that the noise around the error floor is $\approx 10^{-10}$ and therefore there is no advantage in selecting values of A larger than 23. For the pricing calculations we use $A = 23$, $m_E = 61$ and $n_E = 100$, which give a combination of high accuracy and fast computation time. See the online supplementary material for additional results on the choice of the parameters A , m_E and n_E .

2.2.3. Pricing procedure: single-barrier options

We describe the pricing procedure for single-barrier down-and-out options as an example, but the use of the Spitzer identities is equally applicable to other types of barrier options and also to lookback options; the pricing formulae described by Green et al. (2010) include methods for single-barrier up-and-out and knock-in options. The pricing method is adapted from the scheme by Fusai et al. (2016) and Phelan et al. (2018) using the relationship between $\Phi(\xi, q)$ and $\Phi_c(\xi, s)$ described in Section 2.1.1.

1. Compute the characteristic exponent $\psi(\xi + i\alpha)$, where α is the damping parameter introduced in Section 2, Eq. (4).

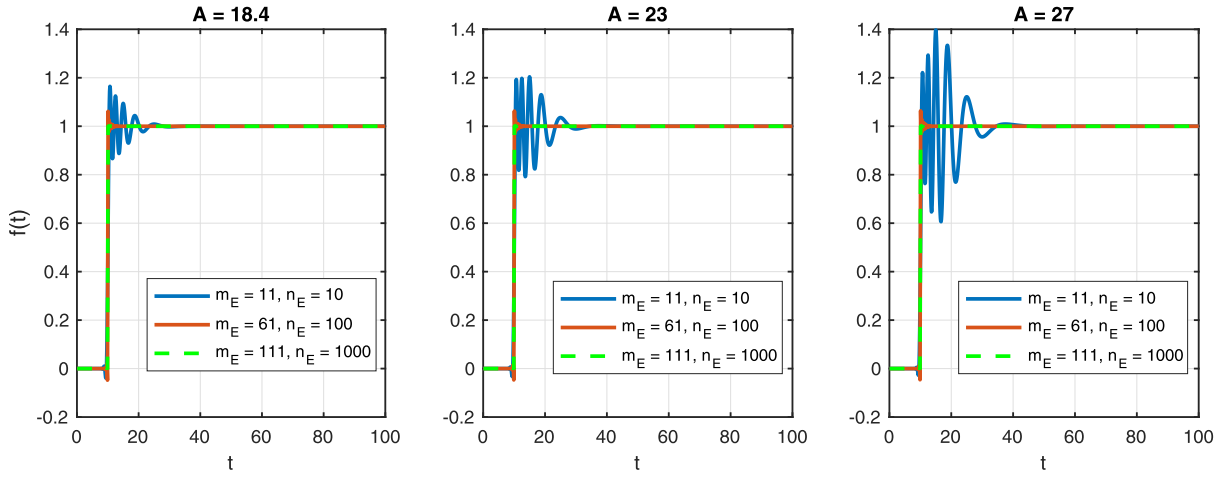


Fig. 2. Output of the inverse Laplace transform of $\tilde{f}(s) = \frac{e^{-10s}}{s}$. Increasing m_E and n_E reduces the size of the oscillations, but it is not improved by increasing A .

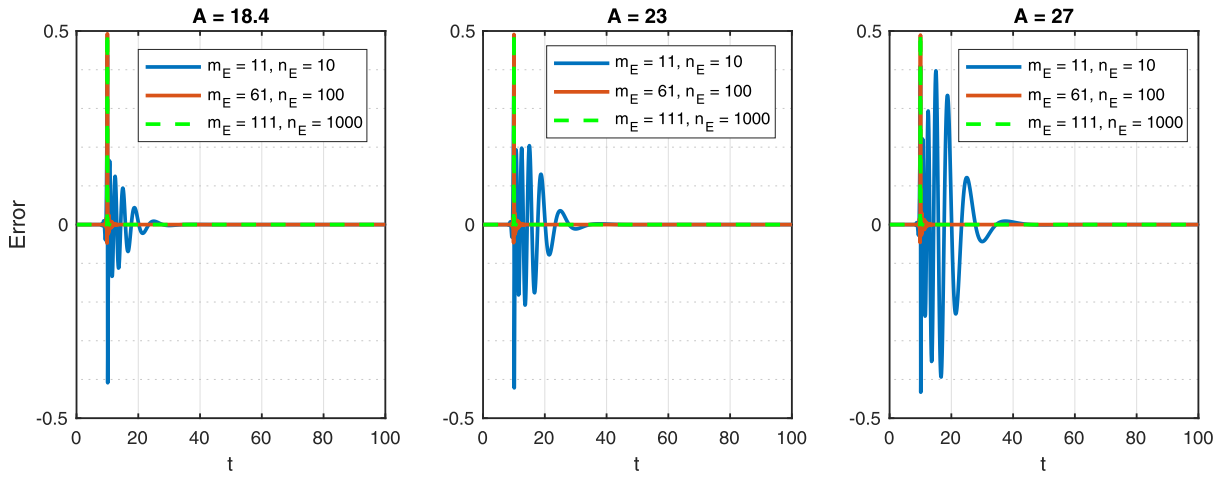


Fig. 3. Error of the inverse Laplace transform of $\tilde{f}(s) = \frac{e^{-10s}}{s}$. Increasing m_E and n_E reduces the size of the errors due to the oscillations, but it is not improved by increasing A .

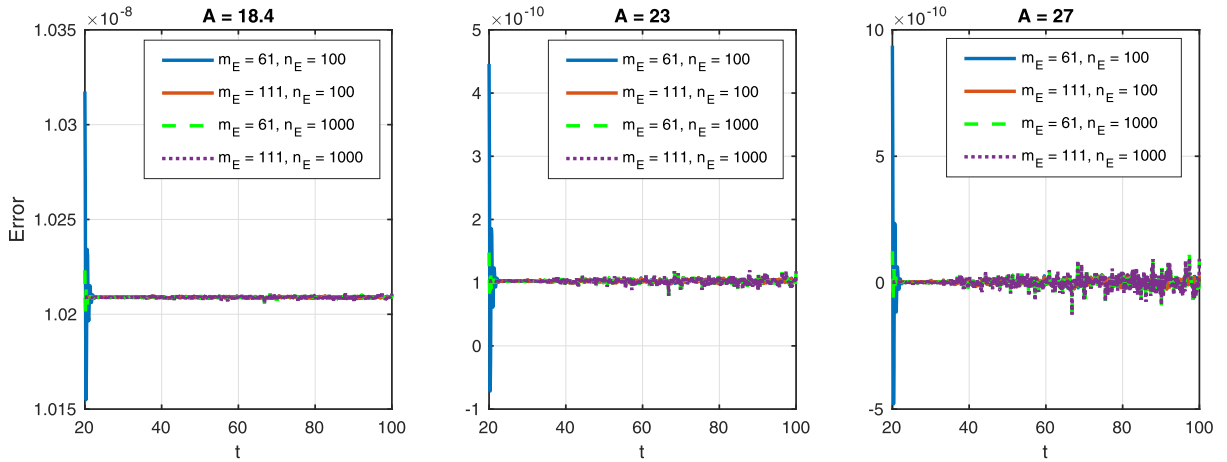


Fig. 4. Error of the inverse Laplace transform of $\tilde{f}(s) = \frac{e^{-10s}}{s}$. Increasing A decreases the error floor, while the latter is unaffected by increasing m_E and n_E . The noise on the error floor is $\approx 10^{-10}$.

2. Use the Plemelj–Sokhotsky relations with the sinc-based Hilbert transform to factorise

$$\Phi_c(\xi, s) := s - \psi(\xi + i\alpha) = \Phi_{c+}(\xi, s)\Phi_{c-}(\xi, s) \quad (32)$$

for all $s = \frac{A+2k\pi i}{2t}$ required for the inverse Laplace transform in Eq. (30).

3. Decompose with respect to l

$$P_c(\xi, s) := \frac{\sigma(\xi/\xi_{\max})}{\Phi_{c-}(\xi, s)} = P_{cl+}(\xi, s) + P_{cl-}(\xi, s), \quad (33)$$

and calculate

$$\tilde{p}(\xi, s) := \frac{P_{cl+}(\xi, s)}{\Phi_{c+}(\xi, s)}, \quad (34)$$

where $\sigma(\xi/\xi_{\max})$ is an exponential filter of order p (see Section 3.2).

4. Calculate the option price as

$$v(0, 0) := \mathcal{F}_{\xi \rightarrow x}^{-1} [\hat{\phi}^*(\xi) \mathcal{L}_s^{-1} \tilde{p}(\xi, s)](0), \quad (35)$$

where $\hat{\phi}^*(\xi)$ is the complex conjugate of the Fourier transform of the damped payoff function given in Eq. (5).

2.2.4. Pricing procedure: double-barrier options

The pricing procedure for double-barrier options is very similar to the method for the single-barrier options described in Section 2.2.3, in that it uses Wiener–Hopf factorisation and decomposition to compute the appropriate Spitzer identity. However, the major difference in this case is that the equations cannot be solved directly and so require the use of a fixed-point algorithm. The steps in the pricing procedure are the same as those for single-barrier down-and-out options described in Section 2.2.3 with the exception of Step 3 which is now replaced by the fixed-point algorithm

3 (a) Set $J_{cu+}(\xi, s) = J_{cl-}(\xi, s) = 0$.

(b) Decompose with respect to l

$$\begin{aligned} P_c(\xi, s) &:= \sigma\left(\frac{\xi}{\xi_{\max}}\right) \frac{1 - \Phi_{c+}(\xi, s) J_{cu+}(\xi, s)}{\Phi_{c-}(\xi, s)} \\ &= P_{cl+}(\xi, s) + P_{cl-}(\xi, s), \end{aligned} \quad (36)$$

and set $J_{cl-}(\xi, s) := P_{cl-}(\xi, s)$.

(c) Decompose with respect to u

$$\begin{aligned} Q_c(\xi, s) &:= \sigma\left(\frac{\xi}{\xi_{\max}}\right) \frac{1 - \Phi_{c-}(\xi, s) J_{cl-}(\xi, s)}{\Phi_{c+}(\xi, s)} \\ &= Q_{cu+}(\xi, s) + Q_{cu-}(\xi, s), \end{aligned} \quad (37)$$

and set $J_{cu+}(\xi, s) := Q_{cu+}(\xi, s)$.

(d) Calculate

$$\begin{aligned} \tilde{p}(\xi, s) &:= \sigma\left(\frac{\xi}{\xi_{\max}}\right) \times \\ &\quad \frac{1 - \Phi_{c-}(\xi, s) J_{cl-}(\xi, s) - \Phi_{c+}(\xi, s) J_{cu+}(\xi, s)}{\Phi_c(\xi, s)}. \end{aligned} \quad (38)$$

(e) If the difference between the new and the old value of $\tilde{p}(\xi, s)$ is less than a predefined tolerance or the number of iterations is greater than a certain threshold then continue, otherwise return to step (b). Numerical tests have shown that an iteration threshold of 5 is sufficient, as higher values do not yield improvements.

3. Error convergence of the pricing procedure

We examine the performance of each stage of the pricing procedure and discuss the respective error bounds. In addition, the effect of each step on the shape of the output function in the Fourier domain is investigated, as this influences the error convergence of later steps. Stenger (1993) showed that the discretisation error in Eq. (25) is exponentially convergent when the function $f(\xi)$ is analytic in a strip of the complex plane including the real axis. Therefore the error calculations here concern the truncation error from the approximation in Eq. (26). The truncation error using the sinc-based Hilbert transform depends on the behaviour of the characteristic exponent as $|\xi| \rightarrow \infty$: Table 1 shows the characteristic exponents of five Lévy processes. The damping parameter α is omitted to make the notation more concise, which is appropriate as its value becomes insignificant for $|\xi| \rightarrow \infty$.

3.1. Factorisation

After calculating the characteristic exponent, the next step in the pricing procedure is the numerical factorisation of $\Phi_c(\xi, s) = s - \psi(\xi)$. In order to understand the error convergence we must consider the way that the function behaves for large values of $|\xi|$. The characteristic exponents of the processes listed in Table 1 will take high negative values which will dominate $\Phi_c(\xi, s)$ so that as $|\xi| \rightarrow \infty$ we can approximate $s - \psi(\xi) \sim -\psi(\xi)$. The function to be decomposed in the factorisation stage is therefore $\sim \log[-\psi(\xi)]$. The magnitude of this increases with $|\xi|$, so the bounds for the truncation error of the sinc-based Hilbert transform (Feng and Linetsky, 2008, Theorems 6.4–6.6) cannot be used. Moreover, if we consider the truncation errors from Eq. (26) for positive and negative values of k individually, we obtain two infinite summations that do not converge. However, Table 1 shows that as $|\xi| \rightarrow \infty$ the values of $\psi(\xi)$ and $\psi(-\xi)$ will become increasingly similar. We can exploit this similarity to find a bound by combining the positive and negative truncations: the truncation error of $f(\xi) = \mathcal{H}[\log \Phi_c(\xi, s)]$ is bounded as

$$\begin{aligned} |f_{\Delta\xi}(\xi) - f_{\Delta\xi, M}(\xi)| &= \Delta\xi \left| \sum_{k < -M/2} \log \Phi_c(k\Delta\xi, s) \frac{1 - \cos\left(\pi \frac{\xi - k\Delta\xi}{\Delta\xi}\right)}{\pi(\xi - k\Delta\xi)} \right. \\ &\quad \left. + \sum_{k > M/2} \log \Phi_c(k\Delta\xi, s) \frac{1 - \cos\left(\pi \frac{\xi - k\Delta\xi}{\Delta\xi}\right)}{\pi(\xi - k\Delta\xi)} \right| \\ &< \Delta\xi \sum_{k > M/2} \left| \frac{\log \Phi_c(k\Delta\xi, s)}{\pi(\xi - k\Delta\xi)} \left(1 - \cos\left(\pi \frac{\xi - k\Delta\xi}{\Delta\xi}\right)\right) \right. \\ &\quad \left. + \frac{\log \Phi_c(-k\Delta\xi, s)}{\pi(\xi + k\Delta\xi)} \left(1 - \cos\left(\pi \frac{\xi + k\Delta\xi}{\Delta\xi}\right)\right) \right| \\ &\leq \frac{2\Delta\xi}{\pi} \sum_{k > M/2} \left| \frac{\xi (\log \Phi_c(k\Delta\xi, s) + \log \Phi_c(-k\Delta\xi, s))}{\xi^2 - k^2 \Delta\xi^2} \right. \\ &\quad \left. + \frac{k\Delta\xi (\log \Phi_c(k\Delta\xi, s) - \log \Phi_c(-k\Delta\xi, s))}{\xi^2 - k^2 \Delta\xi^2} \right|, \end{aligned} \quad (39)$$

where $f_{\Delta\xi}(\xi)$ is the value of the infinite summation in Eq. (25) and $f_{\Delta\xi, M}(\xi)$ is the result of the truncated summation in Eq. (26). The two cosines are equal because the difference of their arguments is $2k\pi$; thus $|1 - \cos(\cdot)| \leq 2$ can be factored out.

The next step in bounding the error convergence is to show that the expression in Eq. (39) is dominated by the first sum as $M \rightarrow \infty$. As $\psi(k\Delta\xi) \sim \psi(-k\Delta\xi)$ for $k \rightarrow \infty$, $\log \Phi_c(k\Delta\xi, s) - \log \Phi_c(-k\Delta\xi, s) \rightarrow 0$ as $k \rightarrow \infty$. However, $k\Delta\xi$ is also present in the numerator and increases linearly with k . By determining the rate of decrease of $\log \Phi_c(k\Delta\xi, s) - \log \Phi_c(-k\Delta\xi, s)$, we show that the second term is bounded as $O(1/k^2)$ and therefore the first term dominates Eq. (39). We then calculate a bound for the error based on the first summation term in Eq. (39). These steps are carried out in a slightly different way depending on the form of the characteristic exponents shown in Table 1.

3.1.1. Normal, Merton and Kou processes

For the normal, Merton and Kou processes, when $k \rightarrow \infty$, $\Phi_c(k\Delta\xi)$ becomes dominated by $\sigma(k\Delta\xi)^2 - i\mu k\Delta\xi$ as shown in Table 1. The parameters μ and σ are specific to the distribution. We can therefore approximate the second fraction in the summation by

$$\begin{aligned} &\frac{k\Delta\xi (\log \Phi_c(k\Delta\xi, s) - \log \Phi_c(-k\Delta\xi, s))}{\xi^2 - k^2 \Delta\xi^2} \\ &= \frac{k\Delta\xi}{\xi^2 - k^2 \Delta\xi^2} \log \frac{\Phi_c(k\Delta\xi, s)}{\Phi_c(-k\Delta\xi, s)} \end{aligned}$$

$$\begin{aligned} & \sim \frac{k\Delta\xi}{\xi^2 - k^2\Delta\xi^2} \log \frac{\sigma^2(k\Delta\xi)^2/2 + i\mu(k\Delta\xi)}{\sigma^2(k\Delta\xi)^2/2 - i\mu(k\Delta\xi)} \\ & = \frac{k\Delta\xi}{\xi^2 - k^2\Delta\xi^2} \log \frac{1 + 2i\mu/(\sigma^2 k\Delta\xi)}{1 - 2i\mu/(\sigma^2 k\Delta\xi)}. \end{aligned} \quad (40)$$

The logarithm in Eq. (40) is of the form $\log \frac{1+x}{1-x}$ where $x = \frac{2i\mu}{\sigma^2 k\Delta\xi}$. For $x \rightarrow 0$, $\log \frac{1+x}{1-x} \sim 2x$, thus

$$\begin{aligned} \frac{k\Delta\xi}{\xi^2 - k^2\Delta\xi^2} \log \frac{1 + 2i\mu/(\sigma^2 k\Delta\xi)}{1 - 2i\mu/(\sigma^2 k\Delta\xi)} & \sim \frac{k\Delta\xi}{\xi^2 - k^2\Delta\xi^2} \frac{4i\mu}{\sigma^2 k\Delta\xi} \\ & = \frac{4i\mu}{\sigma^2(\xi^2 - k^2\Delta\xi^2)} \end{aligned} \quad (41)$$

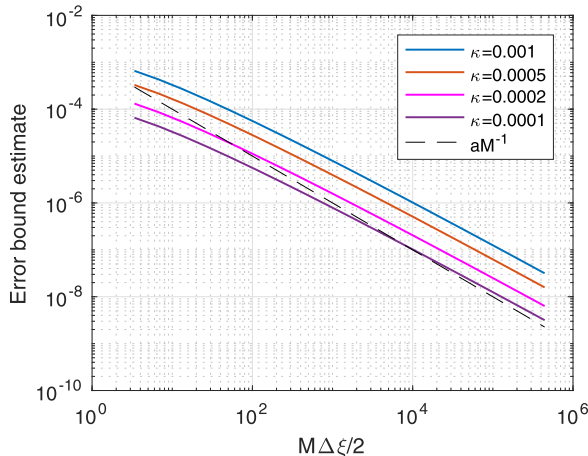
gives an approximation for the second term in Eq. (39). Due to the denominator, this is $O(1/k^2)$. Thus, as $\log \Phi_c(k\Delta\xi, s) + \log \Phi_c(-k\Delta\xi, s)$ increases with k , the error is indeed dominated by the first term in Eq. (39).

For the normal, Kou and Merton processes, $\Phi_c(k\Delta\xi, s)$ and $\Phi_c(-k\Delta\xi, s) \rightarrow 2\log|k\Delta\xi|$ as $k \rightarrow \infty$. Therefore, the error bound is

$$\begin{aligned} |f_{\Delta\xi}(\xi) - f_{\Delta\xi,M}(\xi)| & < \frac{c_1\Delta\xi}{\pi} \sum_{k>M/2} \left| \frac{\log \Phi_c(k\Delta\xi, s)}{\xi^2 - k^2\Delta\xi^2} \right| \\ & < c_2\Delta\xi \sum_{k>M/2} \frac{\log(k^2\Delta\xi^2)}{k^2\Delta\xi^2}, \end{aligned} \quad (42)$$

where c_1 and c_2 are some constants. Here, as Eq. (42) gives the error at fixed values of ξ , i.e. the chosen grid points, the ξ can be absorbed into c_1 . However, as M increases, our range of ξ values increases. Therefore, as there is a linear dependence of the error bound on ξ , we should consider the effect of errors at large values of ξ on the error of the final solution. In doing this we can take account of the shape of the output from the factorisation $\Phi_{c\pm}(\xi, s)$ which decays as $|\xi| \rightarrow \infty$ and the use of a filter on the input to the next step as described in Section 3.2. These effects combine to mean that the error as a proportion of the value of $\Phi_{c\pm}(\xi, s)$ at high $|\xi|$ is less significant to the error of the overall solution than the relationship between the value of M and the error in $\Phi_{c\pm}(\xi, s)$ for low values of $|\xi|$. Approximating the summation by an integral, we obtain

$$|f_{\Delta\xi}(\xi) - f_{\Delta\xi,M}(\xi)| < c_3 \int_{M\Delta\xi/2}^{+\infty} \frac{\log \xi_k}{\xi_k^2} d\xi_k$$



$$\begin{aligned} & = c_3 \left[\frac{\log \xi_k}{\xi_k} + \frac{1}{\xi_k} \right]_{+\infty}^{M\Delta\xi/2} \\ & = c_3 \left(\frac{\log(M\Delta\xi/2)}{M\Delta\xi/2} + \frac{1}{M\Delta\xi/2} \right) \end{aligned} \quad (43)$$

where c_3 is some constant. Having applied the sinc-based discrete Hilbert transform we can calculate the positive and negative functions using the Plemelj–Sokhotsky relations and then exponentiate the results to obtain the Wiener–Hopf factors. Therefore, using the expression in Eq. (43), we can bound the truncation error of the Wiener–Hopf factors, and by extension the total error as the truncation error dominates, as

$$\left| \frac{\Phi_{\Delta\xi,c\pm}(\xi) - \Phi_{\Delta\xi,M,c\pm}(\xi)}{\Phi_{\Delta\xi,c\pm}(\xi)} \right| < \left| 1 - (eM\Delta\xi/2)^{\frac{\kappa}{M\Delta\xi/2}} \right|, \quad (44)$$

where κ is some constant. Here, $\Phi_{\Delta\xi,c\pm}(\xi)$ denotes the (theoretical) Wiener–Hopf factors generated using the series in Eq. (25) and $\Phi_{\Delta\xi,M,c\pm}(\xi)$ denotes the Wiener–Hopf factors calculated using the truncated summation in Eq. (26).

Fig. 5 shows the right-hand side of Eq. (44) plotted against the truncation limit $\xi_{tr} = M\Delta\xi/2$ for different values of κ . This demonstrates that the predicted error bound from the factorisation has a decay that increases in slope as M increases and is slightly shallower than $O(1/M)$ for the values of M which we are using.

3.1.2. Normal inverse Gaussian process

In the case of the NIG process the characteristic exponent is

$$\psi(\xi) = \delta \left(\sqrt{\alpha^2 - (\beta + i\xi)^2} - \sqrt{\alpha^2 - \beta^2} \right). \quad (45)$$

The presence of a square root around the $i\xi$ and ξ^2 terms means that as $|k| \rightarrow \infty$, the equivalent expression to the logarithm in Eq. (40) is $\frac{1}{2} \log \frac{1+2i\beta/(k\Delta\xi)}{1-2i\beta/(k\Delta\xi)}$. Furthermore, $\Phi_c(k\Delta\xi, s)$ and $\Phi_c(-k\Delta\xi, s)$ become dominated by $\log|k\Delta\xi|$ as $|k| \rightarrow \infty$. Therefore the only difference between the truncation error bound for the NIG process and the result in Eq. (43) is the size of the constants.

3.1.3. Variance gamma process

The characteristic function of the VG process is

$$\psi(\xi) = -\frac{1}{\nu} \log \left(1 - i\xi\theta\nu + \frac{1}{2}\nu\sigma^2\xi^2 \right). \quad (46)$$

This is significantly different from the other characteristic exponents that we have considered, being the log of a polynomial. Similarly to the previous methods, we show that as $k \rightarrow \infty$ the decrease

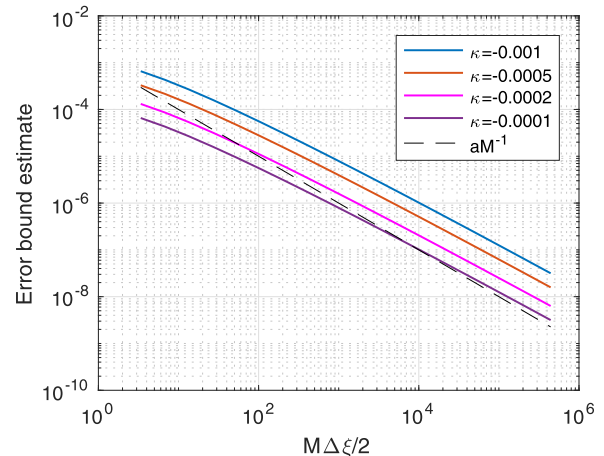


Fig. 5. Right-hand side of Eq. (44) plotted for different values of κ (left: positive values; right: negative values) to show the estimate of the error bound on the sinc-based numerical factorisation of $\Phi_c(\xi, s)$. The predicted error bound from the factorisation has a decay that increases in slope as the truncation limit $\xi_{tr} = M\Delta\xi/2$ increases and is slightly shallower than $O(1/M)$ for the values of M which we are using. Sections 3.1.1–3.1.3 show that this bound applies for the normal, NIG, Kou, Merton and VG processes.

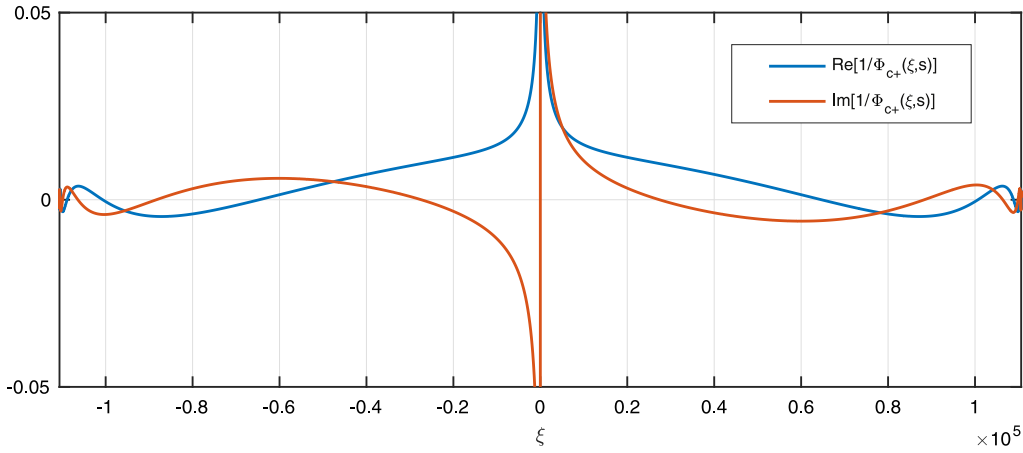


Fig. 6. Example plot of the real and imaginary parts of $\Phi_{c+}(\xi, s)$ plotted against ξ with $s = A/(2t)$, as specified for the Abate and Whitt inverse Laplace transform. Although $|\Phi_{c+}(\xi, s)|$ is bounded by a constant as $|\xi| \rightarrow \infty$, the rate of decay is very slow and we have not been able to determine a decreasing bound.

rate of $\log \frac{\Phi_c(k\Delta\xi, s)}{\Phi_c(-k\Delta\xi, s)}$ is at least $O(1/k)$:

$$\begin{aligned} \log \frac{\Phi_c(k\Delta\xi, s)}{\Phi_c(-k\Delta\xi, s)} &\sim \log \frac{\log(-ik\Delta\xi\theta v + v\sigma^2(k\Delta\xi)^2/2)}{\log(ik\Delta\xi\theta v + v\sigma^2(k\Delta\xi)^2/2)} \\ &= \log \frac{\log\left(1 - \frac{2ik\Delta\xi\theta v}{v\sigma^2(k\Delta\xi)^2}\right) + \log \frac{v\sigma^2(k\Delta\xi)^2}{2}}{\log\left(1 + \frac{2ik\Delta\xi\theta v}{v\sigma^2(k\Delta\xi)^2}\right) + \log \frac{v\sigma^2(k\Delta\xi)^2}{2}} \\ &\sim \log \frac{-\frac{2ik\Delta\xi\theta v}{v\sigma^2(k\Delta\xi)^2} + \log \frac{v\sigma^2(k\Delta\xi)^2}{2}}{\frac{2ik\Delta\xi\theta v}{v\sigma^2(k\Delta\xi)^2} + \log \frac{v\sigma^2(k\Delta\xi)^2}{2}} \\ &= \log \frac{1 - \frac{2i\theta}{\sigma^2 k\Delta\xi \log(v\sigma^2(k\Delta\xi)^2/2)}}{1 + \frac{2i\theta}{\sigma^2 k\Delta\xi \log(v\sigma^2(k\Delta\xi)^2/2)}} \\ &\sim \frac{-4i\theta}{\sigma^2 k\Delta\xi \log(v\sigma^2(k\Delta\xi)^2/2)}. \end{aligned} \quad (47)$$

This decreases quicker than $O(1/k)$ and thus Eq. (39) is dominated by the first term. The expression equivalent to Eq. (42) for the VG process is

$$\begin{aligned} |f_{\Delta\xi} - f_{\Delta\xi, M}| &< \frac{c_1 \Delta\xi}{\pi} \sum_{k > M/2} \left| \frac{\log \Phi_c(k\Delta\xi, s)}{\xi^2 - k^2 \Delta\xi^2} \right| \\ &< c_2 \Delta\xi \sum_{k > M/2} \left| \frac{\log \log(k^2 \Delta\xi^2)}{k^2 \Delta\xi^2} \right|, \end{aligned} \quad (48)$$

where c_1 and c_2 are positive constants. As $|\log \log x|$ is bounded by $|\log x|$ as $x \rightarrow \infty$, the factorisation error of the method with the VG process is also bounded as in Eq. (44).

3.2. Decomposition error

The output of the factorisation is shown in Fig. 6. The next step in the calculation is to find the positive part with respect to l of $P_c(\xi, s) = \frac{1}{\Phi_{c-}(\xi, s)}$. We can attempt to bound the truncation error of this calculation by combining the errors from the positive and negative truncations as before:

$$\begin{aligned} |f_{\Delta\xi}(\xi) - f_{\Delta\xi, M}(\xi)| &< \frac{\Delta\xi}{\pi} \sum_{k > M/2} \left| \frac{P_c(k\Delta\xi)}{\xi - k\Delta\xi} + \frac{P_c(-k\Delta\xi)}{\xi + k\Delta\xi} \right| < \frac{\Delta\xi}{\pi} \times \\ &\sum_{k > M/2} \left| \frac{\xi[P_c(k\Delta\xi) + P_c(-k\Delta\xi)] + k\Delta\xi[P_c(k\Delta\xi) - P_c(-k\Delta\xi)]}{\xi^2 - (k\Delta\xi)^2} \right|. \end{aligned} \quad (49)$$

Fig. 6 shows that for high $|\xi|$, $|P_c(\xi) - P_c(-\xi)|$ is bounded from above by a constant; however, we do not have a decreasing bound for $|P_c(\xi) - P_c(-\xi)|$. Therefore we can only bound a part of Eq. (49) as

$$\begin{aligned} \sum_{k > M/2} \left| \frac{k\Delta\xi[P_c(k\Delta\xi) - P_c(-k\Delta\xi)]}{\xi^2 - (k\Delta\xi)^2} \right| &< c_1 \sum_{k > M/2} \left| \frac{k\Delta\xi}{\xi^2 - (k\Delta\xi)^2} \right| \\ &< c_2 \sum_{k > M/2} \frac{1}{k\Delta\xi}, \end{aligned} \quad (50)$$

where c_1 and c_2 are some positive constants, but a harmonic series does not converge. We can compare this with the discretely monitored version from Fusai et al. (2016) where the first date is taken out of the scheme, meaning that the function undergoing decomposition is multiplied with the characteristic function. For processes other than VG, this restores the exponential slope of the function for high values of ξ , which again means that the truncation error of the sinc-based discrete Hilbert transform is exponentially bounded. To improve the error of the decomposition in the continuously monitored case we can improve the slope of the function on the input to the Hilbert transform by using a spectral filter. We use an exponential filter which has previously achieved good results in option pricing applications (Phelan et al., 2018; Ruijter et al., 2015). The filter is described by Eq. (28) and its shape is shown in Fig. 1. Numerical tests have shown that the use of this filter improves the error of the decomposition step so that it no longer limits the error convergence of the pricing scheme. However, the overall error of the pricing procedure will be continue to be limited by the error from the initial factorisation step as described in Eq. (44) and shown in Fig. 5.

4. Results

We present results for the Spitzer–Laplace pricing procedure for continuously monitored single and double-barrier options for the NIG, Kou and VG processes. We also show that the error convergence represents a limiting case of the performance of the FGM method for discretely monitored options as $N \rightarrow \infty$ and $\Delta t \rightarrow 0$, where N is the number of monitoring dates and Δt is the time interval between them.

4.1. Results for the Spitzer–Laplace method for continuously monitored options

The error convergence for single-barrier down-and-out options is shown in Fig. 7. Fig. 8 shows the results for double-barrier options. The computed prices are given in Tables 2 and 3 for single

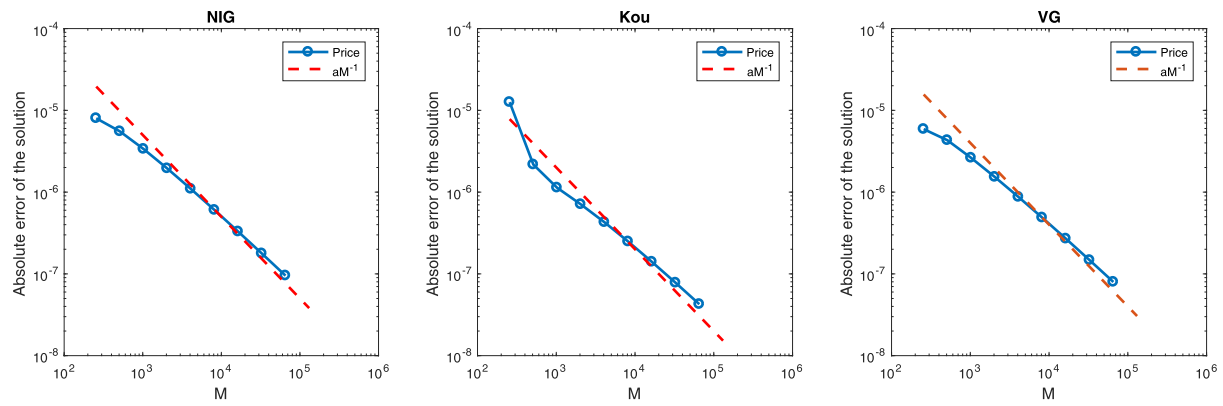


Fig. 7. Error convergence for a continuously monitored single-barrier option. The error convergence conforms to the calculated error bound (only the first point of the Kou process deviates slightly from the overall behaviour) and shows the typical sub-polynomial error convergence for higher values of M .

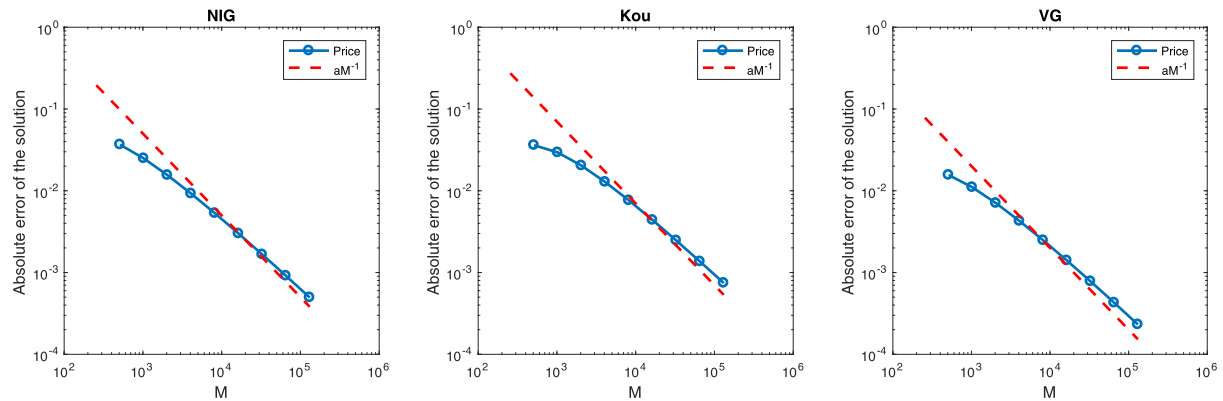


Fig. 8. Error convergence for the continuously monitored double-barrier option. The absolute error is worse than that for the single-barrier option, but the error convergence conforms to the calculated error bound.

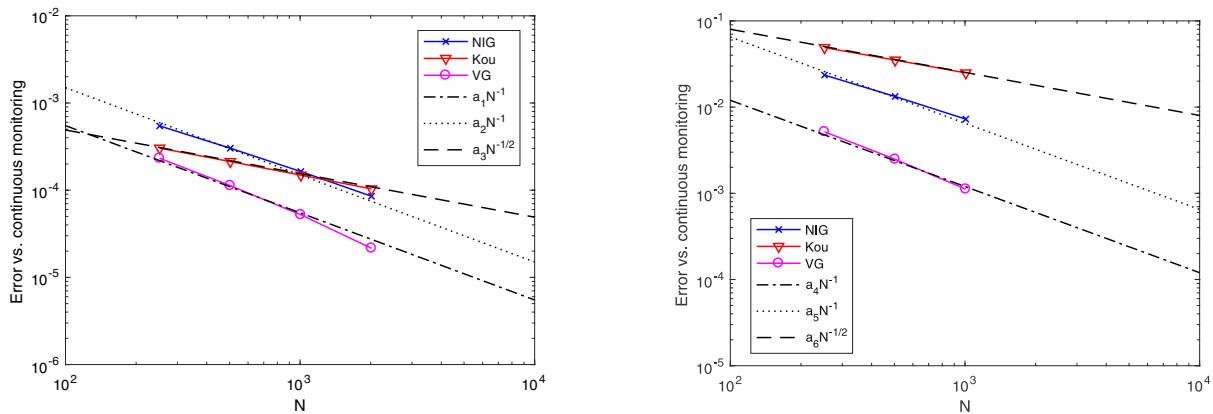


Fig. 9. Error for discretely monitored barrier options (left: single barrier; right: double barrier) used as an approximation of the price for the continuously monitored case, plotted as a function of the number of monitoring dates N . The error is calculated as the difference between the prices for discrete and continuous monitoring at the maximum grid size $M = 2^{17}$. The error for the Kou process converges as $O(1/\sqrt{N})$, whereas the error for the NIG and VG processes converges approximately as $O(1/N)$.

Table 2

Prices calculated for single-barrier options with the contract details and process parameters described in Table 6 in Appendix A and $M = 2^{17}$.

Process	Price
Normal inverse Gaussian	4.77403523401E–2
Kou	4.32042632202E–2
Variance gamma	4.70627023105E–2

Table 3

Prices calculated for double-barrier options with the contract details and process parameters described in Table 6 in Appendix A and $M = 2^{17}$.

Process	Price
Normal inverse Gaussian	2.78787488E–2
Kou	3.30368034E–2
Variance gamma	2.82666693E–2

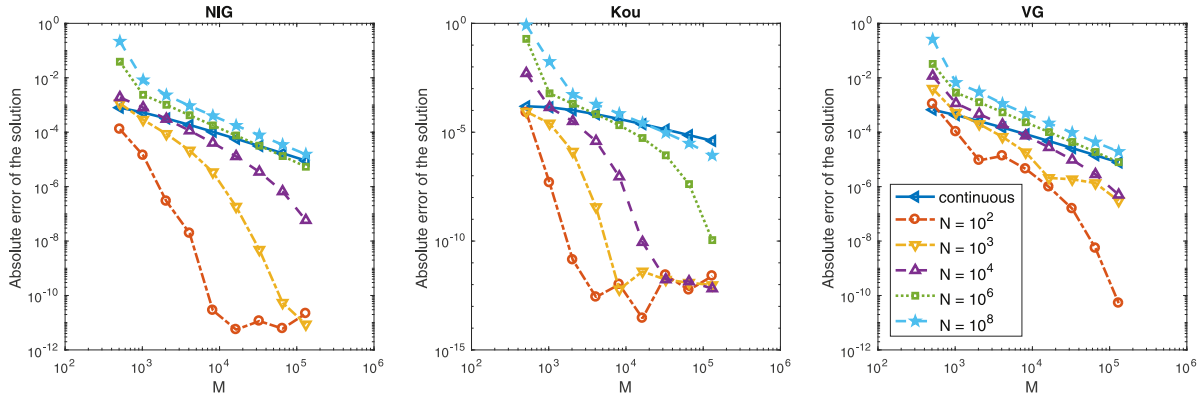


Fig. 10. Error as a function of the grid size M for continuously monitored single-barrier options compared to discretely monitored options as the number of monitoring dates N increases. The error for each number of dates (including continuous monitoring) is calculated against the price for the same number of dates with 2^{18} grid points. For all processes, as $\Delta t \rightarrow 0$ the slope of the error convergence of the discretely monitored scheme approaches that of the continuously monitored scheme.

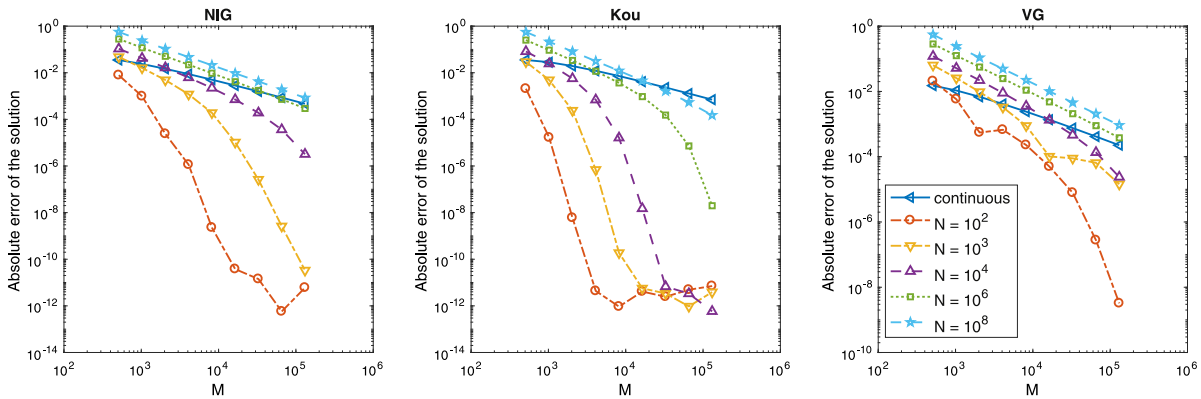


Fig. 11. Error as a function of the grid size M for continuously monitored double-barrier options compared to discretely monitored options as the number of monitoring dates N increases. The error for each number of dates (including continuous monitoring) is calculated against the price for the same number of dates with 2^{18} grid points. For all processes, as $\Delta t \rightarrow 0$ the slope of the error convergence of the discretely monitored scheme approaches that of the continuously monitored scheme.

and double-barrier options respectively. Although the absolute error is worse for double-barrier options, the speed of error convergence is very similar for all cases and is slightly worse than $O(1/M)$, which concurs with the simulated results for the factorisation error shown in Fig. 5. The details of the contract and underlying processes are shown in Table 6 in Appendix A. See the online supplementary material for additional results on the effect of a spectral filter on the error convergence.

4.2. Comparison with the error convergence of the Spitzer-z pricing method for discretely monitored options

In Section 2.1.1 we described the relationship between the Fourier-Laplace based method for continuously monitored options and the FGM method, based in the Fourier-z domain, for discretely monitored options. The latter method, as measured for a single barrier in Fusai et al. (2016) and double barriers in Phelan et al. (2018), with the number of monitoring dates up to $N \approx 10^3$, is exponentially convergent with the number of grid points for the NIG and Kou processes and better than second order polynomially convergent for the VG process. Therefore we investigate the performance of the discretely monitored method with a very high number of dates (i.e. as $\Delta t \rightarrow 0$), to better understand the difference in performance between the two pricing schemes.

In Green et al. (2010) the error between the discretely and continuously monitored prices was shown to be bounded as $O(1/\sqrt{N})$,

where N is the number of monitoring dates. We therefore also consider whether lower errors might be achieved by approximating the price for a continuously monitored option with the price for a discretely monitored option with a very high number of monitoring dates.

We use the same implementation as the one described in Fusai et al. (2016) for single-barrier options and Phelan et al. (2018) for double-barrier options, although the maximum number of monitoring dates is far higher than would ever be used for discretely monitored options in practice. Due to the $O(1/\sqrt{N})$ error bound between the prices for continuously and discretely monitored options, we must select a very large number of monitoring dates in order for this effect to be less significant than the error from the continuously monitored pricing method. The error convergence of the discrete pricing method as $N \rightarrow \infty$ (or $\Delta t \rightarrow 0$) is shown in Figs. 10 and 11. The results show that as $\Delta t \rightarrow 0$, the error convergence for discrete monitoring degrades until it approaches that of continuously monitored options. Moreover, it demonstrates that, rather than being an anomaly, the error convergence of the continuously monitored method is consistent with that of the discretely monitored method as $\Delta t \rightarrow 0$. This can be understood by considering how $\Psi(\xi, \Delta t)$ changes with Δt for the discrete example. As $\Delta t \rightarrow 0$, $\Psi(\xi, \Delta t) = e^{\psi(\xi)\Delta t}$ decays to 0 more and more slowly as $|\xi| \rightarrow \infty$. Therefore the error convergence of the pricing technique for continuously monitored barrier options is a limit of the error convergence for the discrete case as $\Delta t \rightarrow 0$. The relationship

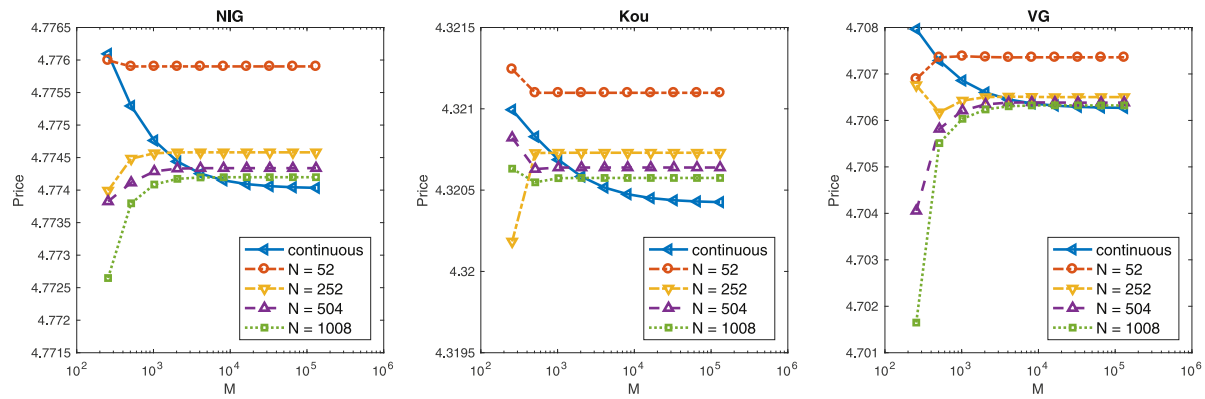


Fig. 12. Price plotted against the grid size M for continuously monitored single-barrier options compared to discretely monitored options as the number of monitoring dates N increases. Note that the larger the value of N , the closer the price is to the continuously monitored option price.

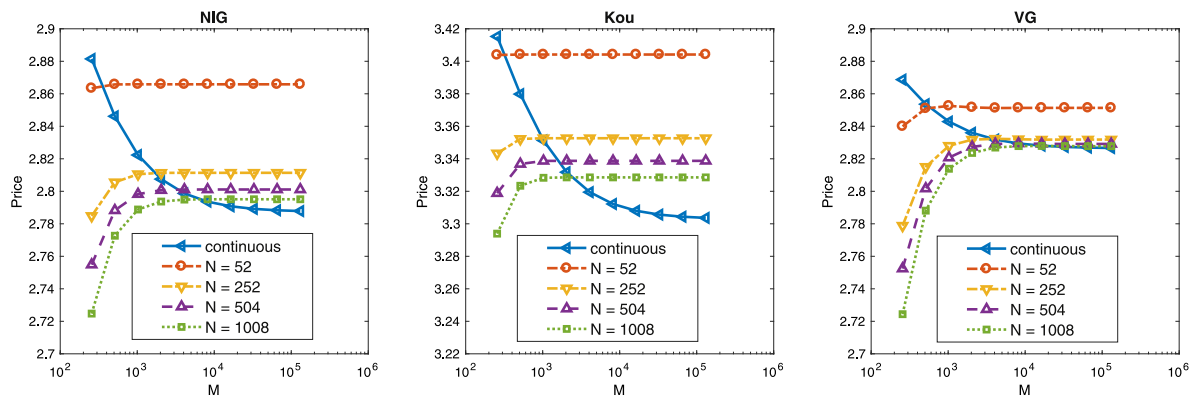


Fig. 13. Price plotted against grid size M for continuously monitored double-barrier options compared to discretely monitored options as the number of monitoring dates N increases. Note that the larger the value of N , the closer the price is to the continuously monitored option price.

between the error convergence of the methods with discrete and continuous monitoring is examined in more detail in the online supplementary material.

Computed prices for continuously and discretely monitored options are plotted against M in Figs. 12 and 13. In addition, computation times of the pricing methods for the discrete and continuously monitored methods are shown in Tables 4 and 5. Figs. 12 and 13 show that, as expected, the larger the number of monitoring dates the closer the price is to the continuously monitored price. However, they also show that the direction of convergence depends on the type of option and the process being used. Therefore, in order to obtain the CPU times in Tables 4 and 5 we take the lowest time where the convergence error for the discretely monitored method is significantly (i.e., ten times) lower than the error compared to the price for the continuously monitored case with maximum M . This shows that for relatively high errors, $\approx 10^{-4}$ for a single barrier and $\approx 10^{-2}$ for double barriers, the discretely monitored method is slightly quicker. However, the discretely monitored method is unable to achieve the lower errors, $\approx 10^{-6}$ for a single barrier and $\approx 10^{-4}$ for double barriers, which are attained by the continuously monitored method and therefore is not a sufficiently accurate approximation.

Can we can achieve a better approximation of the continuous method by increasing the number of monitoring dates further? Previous literature, e.g. Green et al. (2010), has shown that the convergence of the discrete method to the continuous method with increasing monitoring dates is $O(1/\sqrt{N})$. From Fig. 9 we can observe that, although the discrete method with the Kou process

Table 4

CPU times and errors for the continuously monitored method and the discretely monitored method as an approximation to continuous monitoring for the single-barrier case. The CPU times for the discretely monitored price are chosen for the grid size M which gives the lowest CPU time where the convergence error is significantly (about ten times) lower than the error compared to the continuously monitored price.

Process	N	Error	CPU time/s	M
NIG	continuous	3.21E-04	0.07	1024
	1008	1.60E-04	0.07	4096
	continuous	1.86E-04	0.13	2048
	continuous	5.74E-05	0.50	8192
	continuous	1.69E-05	2.03	32768
Kou	252	3.02E-04	0.01	512
	continuous	1.57E-04	0.02	256
	504	2.03E-04	0.01	512
	continuous	1.57E-04	0.02	256
	1008	1.47E-04	0.02	1024
	continuous	1.57E-04	0.02	256
	continuous	6.69E-05	0.12	2048
	continuous	2.35E-05	0.49	8192
VG	continuous	7.36E-06	2.07	32768
	252	2.31E-04	0.04	2048
	continuous	2.57E-04	0.06	1024
	504	1.13E-04	0.11	4096
	continuous	1.49E-04	0.17	2048
	1008	5.29E-05	0.14	8192
	continuous	4.74E-05	0.49	8192
	continuous	1.43E-05	2.01	32768
	continuous	4.19E-06	11.52	131072

Table 5

CPU times and errors for the continuously monitored method and the discretely monitored method as an approximation to continuous monitoring for the double-barrier case. The CPU times for the discretely monitored price are chosen for the grid size M which gives the lowest CPU time where the convergence error is significantly (about ten times) lower than the error compared to the continuously monitored price.

Process	N	Error	CPU time/s	M
NIG	252	2.35E–02	0.16	4096
	continuous	2.38E–02	0.11	512
	504	1.32E–02	0.14	4096
	continuous	1.48E–02	0.24	1024
	1008	7.24E–03	0.29	8192
	continuous	8.78E–03	0.54	2048
	continuous	1.58E–03	5.03	16384
	continuous	4.69E–04	20.94	65536
	252	4.90E–02	0.03	1024
	continuous	3.54E–02	0.07	256
Kou	504	3.51E–02	0.07	2048
	continuous	3.54E–02	0.07	256
	1008	2.47E–02	0.04	1024
	continuous	2.84E–02	0.14	512
	continuous	7.23E–03	1.19	4096
	continuous	2.33E–03	5.30	16384
	continuous	7.03E–04	21.05	65536
	252	5.10E–03	0.08	2048
	continuous	6.86E–03	0.24	1024
	504	2.51E–03	0.13	4096
VG	continuous	2.40E–03	1.15	4096
	1008	1.15E–03	0.29	8192
	continuous	1.36E–03	2.44	8192
	continuous	7.56E–04	5.16	16384
	continuous	2.25E–04	21.21	65536

does indeed have this rate of convergence, it achieves approximately $O(1/N)$ with the NIG and VG processes. Therefore, if we wished to decrease the error of the discrete approximation so that it is significantly (i.e. ten times) less than the continuous case then we would have to increase the maximum number of monitoring dates in Tables 4 and 5 by 100 times for the NIG and VG processes and by 200^2 times for the Kou process. We can see from Figs. 10 and 11 that at $M = 2^{17}$ the discrete methods with these numbers of dates have an error which is worse than the required accuracy of ten times better than the continuous method. Thus the only possibility would be to also increase M , and by extension the CPU time of the discrete monitoring method, causing its computational cost to be greater than that for continuous monitoring.

5. Conclusions

We showed that the numerical method for calculating the discretely monitored Spitzer identities described by Fusai et al. (2016) can be modified for continuous monitoring by using the Fourier–Laplace domain instead of the Fourier- z domain. We implemented this with the inverse Laplace transform by Abate and Whitt (1992a, 1995) which achieves an accuracy of approximately 10^{-11} , sufficient for our chosen application of pricing barrier options. We presented results showing that the conversion from discrete to continuous monitoring means that exponential convergence is no longer achieved, but instead the error convergence becomes sub-polynomial due to the performance of the Wiener–Hopf factorisation. By examining the effect of truncating the sinc-based discrete Hilbert transform, we were able to provide an error bound which is well matched to the observed accuracy of the pricing procedure for continuously monitored options.

It is notable that previous papers have shown that the discretely monitored case achieves exponential convergence (Fusai et al., 2016; Phelan et al., 2018), but the continuous case described here does not. However, our numerical results show that, as the number of monitoring dates increases and $\Delta t \rightarrow 0$, the error convergence for the discretely monitored case degrades and approaches that of the continuously monitored case. Thus, the performance of the technique for continuously monitored barriers is consistent with previous results, being a limit of the performance of the discretely monitored case.

Furthermore we have compared the error vs. computational time of the continuously monitored scheme with that of an approximate solution generated by the discretely monitored scheme with a high number of monitoring dates. We show that, for higher errors, the discrete scheme may produce a rapidly calculated approximation to the continuously monitored scheme, but when lower errors are required the continuously monitored scheme must be used.

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Appendix A.

Table 6 contains all the parameters of the numerical experiments presented in Section 4.

Table 6

Parameters for the numerical tests and processes used; $\Psi(\xi, t)$ is the characteristic function of the process that models the log return of the underlying asset.

	Description	Symbol	Value
Option parameters	Maturity	T	1
	Initial spot price	S_0	1
	Strike	K	1.1
	Upper barrier (double barrier)	U	1.40
	Lower barrier (double barrier)	L	0.60
	Upper barrier (down-and-out)	U	$+\infty$
	Lower barrier (down-and-out)	L	0.80
	Risk-free interest rate	r	0.05
	Dividend rate	q	0.02
Model	$\Psi(\xi, t)$	Symbol	Value
NIG	$e^{-t(\sqrt{\alpha^2 - (\beta + i\xi)^2} + \sqrt{\alpha^2 - \beta^2})}$	α	15
		β	–5
		δ	0.5
Kou	$e^{-t\left(\frac{\sigma^2 \xi^2}{2} - \lambda\left(\frac{(1-p)\eta_2}{\eta_2 + i\xi} + \frac{p\eta_1}{\eta_1 - i\xi} - 1\right)\right)}$	p	0.3
		λ	3
		σ	0.1
		η_1	40
		η_2	12
VG	$(1 - iv\xi\theta + v\sigma^2\xi^2/2)^{-t/v}$	θ	1/9
		σ	$\sqrt{3}/9$
		v	0.25

Supplementary material

Supplementary material associated with this article can be found, in the online version, at [10.1016/j.ejor.2018.04.016](https://doi.org/10.1016/j.ejor.2018.04.016).

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