Interstitial energy flux and stress-power for second-gradient elasticity

Angelo Morro
Dipartimento di Informatica, Bioingegneria, Robotica e Ingegneria dei Sistemi, Università Genova, 16145 Genova, Italy

Maurizio Vianello
Dipartimento di Matematica, Politecnico di Milano, 20133 Milan, Italy

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1. Introduction

The dependence of constitutive relations for a solid or a fluid on the first and *second* (or higher) deformation gradient, first proposed in some pioneering papers by Toupin [1, 2], poses a well known conceptual obstacle to the thermodynamical framework of continuum mechanics of so-called non-simple materials, as first shown by Gurtin [3]. The issue has been confronted by means of different approaches, through the introduction either of internal variables [4] or non-standard interaction terms [5–14].

In particular, motivated by the purpose of describing spatial interaction effects of longer range in elastic materials, in a remarkable article, Dunn and Serrin [9] developed a thermodynamic scheme where an additional flux $\mathbf{u}$, the *interstitial energy flux*, was inserted into the balance of energy, beside the heat flow and the standard working of the stress. Such an 'extra flux' was not included in the entropy inequality and this framework was then shown to be sufficient for allowing a dependence of constitutive properties on higher order gradients of deformation. Dunn and Serrin's contribution is all the more interesting in view of the fact that, among other things, a symmetric stress tensor is obtained, thus avoiding the need for micro-polar couples or some other non-standard quantity.

Corresponding author:
Maurizio Vianello, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo 32, 20133 Milan, Italy.
Email: maurizio.vianello@polimi.it

Some authors have taken the alternative approach of postulating an expression for the stress-power (sometimes called the *inner working*) which includes an additional term linear in the second material velocity gradient. This idea is basically at the center of the fundamental contributions by Germain [10–12] and, in some way, is shared by many subsequent developments, due to a variety of authors [15–18].

The purpose of this work is to show that, upon a slight but significant generalization, the approach based on a postulated expression for the stress-power involving a non-standard linear term in the second velocity gradient, as presented by Germain [10–12] and the references cited above, is somewhat equivalent to the scheme of balance equations involving an interstitial energy flux, as proposed by Dunn and Serrin [9]. So far, to our knowledge, the possibility of such an equivalence has gone completely unnoticed in the literature (or, at least, among the roughly 140 papers where Dunn and Serrin [9] is cited, which we have been able to peruse).

To this end, we begin by postulating an expression for the inner working which, formally, is in complete agreement with what can be found in Germain [10] and Fried and Gurtin [16–18], where we introduce an innocuous slight generalization in the expression of the stress-power: we do not take the *hyperstress* $L_{h\alpha\beta}$ (the tensorial coefficient of the material second-gradient of velocity) to be symmetric in the last pair of indexes but allow for a skew-symmetric part which is, of course, powerless, but, interestingly, turns out not to be useless. Next, by use of a key technical detail borrowed from the appendix of Dunn and Serrin [9], we show that this choice makes possible the deduction of constitutive equations for the free energy ψ and the Piola or Cauchy stresses $\hat{\mathbf{T}}$ and $\hat{\mathbf{S}}$ which are *exactly* coincident with what was derived through the introduction of the interstitial energy by Dunn and Serrin [9].

We show that what makes the Cauchy stress tensor symmetric in a second-gradient theory is indeed the powerless part of the inner working of the second velocity gradient. Thus, the generalization we introduce is shown to be well motivated and fruitful, since, moreover, it leads to a set of results which are in agreement with what was found, from a different perspective, by Dunn and Serrin [9]. Indeed, an attractive feature of their work lies in the fact that the stress tensor they obtain is symmetric. For a second-gradient theory this is a significant simplification, in our opinion, since, as mentioned above, it leaves the model free from surface stress couples, which are otherwise needed.

We believe it is of some interest to know that two different approaches (one based on the interstitial energy and the other one on the introduction of a second-gradient inner power) can be (partially, at least) reconciled. We also think that our computations shed further light on the inner working of Dunn and Serrin's approach [9].

We use standard index notation so that our formulae are straightforward and unambiguous (and not so elegant, perhaps), with Greek indexes for the coordinates of points and components of vectors and tensors with respect to a Cartesian coordinate system in the reference configuration $\mathcal{B}$, and small Latin indexes for points and components in $\mathcal{B}_t$, the present configuration at current time.

Thus, we shall consistently write $F_{h\alpha}$ and $F_{h\alpha\beta} = F_{h\beta\alpha}$ for the Cartesian components of the first and second deformation gradient of a motion described by $x_h(p_\alpha, t)$, with Jacobian $J = \det[F_{h\alpha}]$, where p_α is the α component of the reference position vector. As usual, ρ denotes the mass density, with $\rho_0 = \rho J$ the reference density, which we assume to be uniform. The velocity field in the material description is $\dot{x}_h = \partial_t x_h(p_\alpha, t)$ while for the spatial description we write v_i (superposed dots denote material time derivatives, while partial derivatives of a field Φ with respect to spatial and material coordinates are written as $\Phi_{,h}$ or $\Phi_{,\alpha}$). Thus, $\dot{F}_{h\alpha} = \dot{x}_{h,\alpha}$ and $\dot{F}_{h\alpha\beta} = \dot{x}_{h,\alpha\beta}$. Moreover, $v_{i,j}$ and $v_{i,jk} = v_{i,kj}$ are the first and second spatial velocity gradients.

2. The stress-power

In a significant number of contributions, at least since Germain's work [10], the starting point for a discussion of second-gradient materials is an appropriate expression for the stress-power (or inner working) associated with a part $\mathcal{P}$. Here, as in Germain [10], Fabrizio et al. [15], Fried and Gurtin [18] and Podio-Guidugli and Vianello [19], we write such a quantity as

$$W^{\text{int}}_{\mathcal{P}_t} = \int_{\mathcal{P}_t} [T_{ij} v_{i,j} + G_{ijk} v_{i,jk}]\, \mathrm{d}V, \tag{1}$$

where T_{ij} and G_{ijk} (which is often named *hyperstress*) are basically just seen as coefficients of $v_{i,j}$ and $v_{i,jk}$. In other words, such tensors are assumed to belong to the dual space of the first and second velocity gradient, in the same way as a force can be seen as a co-vector whose pairing with the velocity yields the (zeroth-order) working. In particular, quite naturally, it is usually stated, implicitly or explicitly, that, without loss of generality, one may assume symmetry of G_{ijk} with respect to the second and third index

$$G_{ijk} = G_{ikj}. \tag{2}$$

The motivation is almost obvious: a skew-symmetric part (with respect to the same pair of indexes) *would do no work* for any motion of the body and, thus, would appear to be useless and of no effect.

In the case of Germain's article [10] such an (entirely reasonable) choice can be deduced when it is stated that G_{ijk} belongs to a space of dimension $3 \times 6 = 18$, while in Fried and Gurtin [18] this is explicitly written in equation (26) and motivated on p. 521.

The goal of this paper is to find a connection between the modeling of longer range interactions by means of the interstitial work flux, as in Dunn and Serrin [9], and an alternative approach, as in Degiovanni et al. [5], Germain [10] and Fried and Gurtin [18], which is based on the stress-power expressed with the introduction of the hyperstress G_{ijk}. We show that this second approach leads to an equivalent Cauchy stress tensor and interstitial work flux, provided a more general hyperstress is considered, which is not assumed to be symmetric in the sense of equation (2).

We find it more convenient to develop our presentation following a Lagrangian description and, coherently with this choice, we write the stress-power in the reference configuration in the form

$$W_{\mathcal{P}}^{\text{int}} = \int_{\mathcal{P}} [S_{h\alpha}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta}]\,\mathrm{d}V \tag{3}$$

which, conceptually, is clearly equivalent to equation (1).

A detailed discussion of the relationship between the alternative expressions for the stress-power provided by equations (1) and (3) is contained in Section 3.3 of Podio-Guidugli and Vianello [19]. As shown there, one can derive both

$$\begin{aligned}
\dot{F}_{h\alpha} &= v_{h,k}F_{k\alpha}\\
\dot{F}_{h\alpha\beta} &= v_{h,k}F_{k\alpha\beta} + v_{h,kl}F_{k\alpha}F_{l\beta}
\end{aligned}$$

and the inverse relations

$$v_{h,k} = \dot{F}_{h\alpha}F^{-1}_{\alpha k} \tag{4}$$

$$v_{h,kp} = -\dot{F}_{h\alpha}F^{-1}_{\alpha i}F_{i\gamma\beta}F^{-1}_{\gamma k}F^{-1}_{\beta p} + \dot{F}_{h\gamma\beta}F^{-1}_{\gamma k}F^{-1}_{\beta p}. \tag{5}$$

In view of equations (1) and (3)–(5), as shown in Proposition 3 of Podio-Guidugli and Vianello [19], condition $W_{\mathcal{P}}^{\text{int}} = W_{\mathcal{P}_t}^{\text{int}}$ for all motions and all parts $\mathcal{P}$ is then guaranteed by the relations

$$\begin{aligned}
JT_{hk} &= S_{h\alpha}F_{k\alpha} + L_{h\alpha\beta}F_{h\alpha\beta}\\
JG_{hkp} &= L_{h\alpha\beta}F_{k\alpha}F_{p\beta}
\end{aligned} \tag{6}$$

which can be inverted as

$$\begin{aligned}
S_{h\alpha} &= JT_{hk}F^{-1}_{\alpha k} - L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k}\\
L_{h\alpha\beta} &= JG_{hkl}F^{-1}_{\alpha k}F^{-1}_{\beta l}
\end{aligned} \tag{7}$$

and provide the connection between stress and hyperstress in the Eulerian and Lagrangian description.

As we mentioned before, tensors G_{ijk} and $L_{h\alpha\beta}$, related through equations (6) and (7), are both usually supposed to be symmetric in last their two indexes. This choice is quite understandable, since the skew-symmetric parts with respect to such indexes would give no contribution to the stress-power, or inner working, because the second gradient of the Eulerian or Lagrangian velocity field is, by its nature, symmetric in the last pair of indexes. Thus, as explicitly noted by Gurtin and Fried [18], it seems reasonable that, without loss of generality, one should require G_{hkp} (or $L_{h\alpha\beta}$) to be symmetric in the last two subscripts.

The crucial detail of this work is that we are *not* making such an assumption. Indeed, within a Lagrangian description, coherent with equation (3), we shall find it convenient to split the hyperstress $L_{h\alpha\beta}$ into a symmetric and a skew-symmetric part (with respect to Greek indexes):

$$L_{h\alpha\beta} = S_{h\alpha\beta} + W_{h\alpha\beta}\,. \tag{8}$$

Of course, while $S_{h\alpha\beta} = S_{h\beta\alpha}$, the skew-symmetric part $W_{h\alpha\beta}$ satisfies the identity

$$W_{h\alpha\beta} = -W_{h\beta\alpha}$$

and, being 'powerless' when inserted into equation (3), seems at first to be useless. Surprisingly, however, it turns out that this is not the case.

Thus, while our contribution is consistent with the theory proposed by Dunn and Serrin [9], it is remarkable that our proposal for a non-zero skew-symmetric part of the hyperstress tensor makes it possible to find a clarifying relationship between the stress-power and interstitial energy approaches.

3. Interstitial energy and stresses

We take as a starting point the (postulated) Lagrangian expression for the stress-power in equation (3) and express such a quantity by means of a volume and a surface integral. We assume that the reference configurations of the body $\mathcal{B}$ and its parts $\mathcal{P}$, over which we perform integrations, applying again and again the Divergence Theorem, are *regular regions*, in the sense of Kellogg [20]. This is a prudent choice which makes us sure that no analytical difficulties might arise in our discussion. For a part $\mathcal{P}$ with outward unit normal m_α on the boundary $\partial\mathcal{P}$, through repeated integrations by parts and applications of the divergence theorem we have

$$\begin{aligned}
W_{\mathcal{P}}^{\text{int}} &= \int_{\mathcal{P}} [S_{h\alpha}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta}]\,\mathrm{d}V \\
&= \int_{\mathcal{P}} [(S_{h\alpha}\dot{x}_h)_{,\alpha} - S_{h\alpha,\alpha}\dot{x}_h + (L_{h\alpha\beta}\dot{F}_{h\alpha})_{,\beta} - L_{h\alpha\beta,\beta}\dot{F}_{h\alpha}]\,\mathrm{d}V \\
&= \int_{\partial\mathcal{P}} [S_{h\alpha}m_\alpha\dot{x}_h + L_{h\alpha\beta}\dot{F}_{h\alpha}m_\beta]\,\mathrm{d}A - \int_{\mathcal{P}} [S_{h\alpha,\alpha}\dot{x}_h + L_{h\alpha\beta,\beta}\dot{F}_{h\alpha}]\,\mathrm{d}V \\
&= \int_{\partial\mathcal{P}} [S_{h\alpha}m_\alpha\dot{x}_h + L_{h\alpha\beta}\dot{F}_{h\alpha}m_\beta]\,\mathrm{d}A - \int_{\mathcal{P}} [S_{h\alpha,\alpha}\dot{x}_h + (L_{h\alpha\beta,\beta}\dot{x}_h)_{,\alpha} - L_{h\alpha\beta,\beta\alpha}\dot{x}_h]\,\mathrm{d}V \\
&= \int_{\partial\mathcal{P}} [(S_{h\alpha} - L_{h\alpha\beta,\beta})m_\alpha\dot{x}_h + L_{h\alpha\beta}\dot{F}_{h\alpha}m_\beta]\,\mathrm{d}A - \int_{\mathcal{P}} [S_{h\alpha,\alpha} - L_{h\alpha\beta,\beta\alpha}]\dot{x}_h\,\mathrm{d}V.
\end{aligned}$$

Now, for

$$\hat{S}_{h\alpha} := S_{h\alpha} - L_{h\alpha\beta,\beta} \tag{9}$$

and

$$w_\beta := L_{h\alpha\beta}\dot{F}_{h\alpha}, \tag{10}$$

the stress-power in equation (3) takes the form

$$W_{\mathcal{P}}^{\text{int}} = \int_{\partial\mathcal{P}} \hat{S}_{h\alpha}m_\alpha\dot{x}_h\,\mathrm{d}A + \int_{\partial\mathcal{P}} w_\beta m_\beta\,\mathrm{d}A - \int_{\mathcal{P}} \hat{S}_{h\alpha,\alpha}\dot{x}_h\,\mathrm{d}V$$

or, in absolute notation,

$$W_{\mathcal{P}}^{\text{int}} = \int_{\partial\mathcal{P}} \hat{\mathbf{S}}\mathbf{m}\cdot\dot{\mathbf{x}}\,\mathrm{d}A + \int_{\partial\mathcal{P}} \mathbf{w}\cdot\mathbf{m}\,\mathrm{d}A - \int_{\mathcal{P}} \mathrm{Div}\,\hat{\mathbf{S}}\cdot\dot{\mathbf{x}}\,\mathrm{d}V.$$

In view of the identity

$$\mathrm{Div}(\hat{\mathbf{S}}^t\dot{\mathbf{x}}) = \hat{\mathbf{S}}\cdot\dot{\mathbf{F}} + \mathrm{Div}\,\hat{\mathbf{S}}\cdot\dot{\mathbf{x}}$$

(a superscript t denotes the transpose) the stress-power can be finally written as

$$W_{\mathcal{P}}^{\text{int}} = \int_{\partial\mathcal{P}} \mathbf{w}\cdot\mathbf{m}\,\mathrm{d}A + \int_{\mathcal{P}} \hat{\mathbf{S}}\cdot\dot{\mathbf{F}}\,\mathrm{d}V = \int_{\mathcal{P}} [\mathrm{Div}\,\mathbf{w} + \hat{\mathbf{S}}\cdot\dot{\mathbf{F}}]\,\mathrm{d}V. \tag{11}$$

It is natural to identify $\hat{\mathbf{S}}$, as defined by equation (9), with the Piola–Kirchhoff stress tensor, in view of its role in equation (11) (for a critical discussion of this point and an alternative variational derivation, the interested reader is referred to Section 5 of Auffray et al. [14]). More interestingly, it seems appropriate to think of $\mathbf{w}$ as the interstitial energy flux vector (per unit area in the reference configuration) first introduced by Dunn and Serrin [9].

In order to better understand such identifications, it is useful to see what would happen had we developed our computations beginning from equation (1). By repeated application of the divergence theorem to the region $\mathcal{P}_t$ with outward unit normal $\mathbf{n}$ on its boundary $\partial\mathcal{P}_t$, we obtain

$$W^{\text{int}}_{\mathcal{P}_t} = \int_{\partial\mathcal{P}_t} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}A + \int_{\mathcal{P}_t} \hat{\mathbf{T}}\cdot\operatorname{grad}\mathbf{v}\,\mathrm{d}V = \int_{\mathcal{P}_t} [\operatorname{div}\mathbf{u} + \hat{\mathbf{T}}\cdot\operatorname{grad}\mathbf{v}]\,\mathrm{d}V,$$

where

$$\hat{T}_{hk} := T_{hk} - G_{hkl,l} \tag{12}$$

and

$$u_h := G_{hkl}v_{h,k}. \tag{13}$$

Notice that, from equation (13), in view of equations (4), (6) and (10) we have

$$\begin{aligned} u_l &= J^{-1}L_{h\alpha\beta}F_{k\alpha}F_{l\beta}v_{h,k} = J^{-1}L_{h\alpha\beta}F_{k\alpha}F_{l\beta}\dot{F}_{h\gamma}F^{-1}_{\gamma k} \\ &= J^{-1}L_{h\gamma\beta}F_{l\beta}\dot{F}_{h\gamma} = J^{-1}F_{l\beta}L_{h\gamma\beta}\dot{F}_{h\gamma} \\ &= J^{-1}F_{l\beta}w_\beta, \end{aligned}$$

which can be written as $J\mathbf{u} = \mathbf{F}\mathbf{w}$. This relation guarantees that the flux of $\mathbf{u}$ through ∂P_t is the same as the flux of $\mathbf{w}$ through ∂P.

It is interesting to notice that $\hat{\mathbf{T}}$, as defined by equation (12), and $\hat{\mathbf{S}}$, as defined by equation (9), are connected through the standard relation $J\hat{\mathbf{T}}\mathbf{F}^{-t} = \hat{\mathbf{S}}$, which follows from some rearrangements:

$$\begin{aligned} J\hat{T}_{hk}F^{-1}_{\alpha k} &= J(T_{hk} - G_{hkl,l})F^{-1}_{\alpha k} = JT_{hk}F^{-1}_{\alpha k} - JG_{hkl,l}F^{-1}_{\alpha k} \\ &= J[J^{-1}(S_{h\gamma}F_{k\gamma} + L_{h\gamma\beta}F_{k\gamma\beta})]F^{-1}_{\alpha k} - J[J^{-1}L_{h\gamma\beta}F_{k\gamma}F_{l\beta}]_{,l}F^{-1}_{\alpha k} \\ &= S_{h\alpha} + L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} - J(J)_{,l}F_{l\beta}L_{h\gamma\beta}F_{k\gamma}F^{-1}_{\alpha k} - JJ^{-1}(L_{h\gamma\beta}F_{k\gamma}F_{l\beta})_{,l}F^{-1}_{\alpha k} \\ &= S_{h\alpha} + L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} + J^{-1}J_{,l}F_{l\beta}L_{h\gamma\beta}F_{k\gamma}F^{-1}_{\alpha k} - L_{h\gamma\beta,l}F_{l\beta}F_{k\gamma}F^{-1}_{\alpha k} - L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} - L_{h\gamma\beta}F_{k\gamma}F_{l\beta,l}F^{-1}_{\alpha k} \\ &= S_{h\alpha} + L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} + F_{l\rho\beta}F^{-1}_{\rho l}L_{h\gamma\beta}F_{k\gamma}F^{-1}_{\alpha k} - L_{h\alpha\beta,\beta} - L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} - L_{h\gamma\beta}F_{l\beta\rho}F^{-1}_{\rho l}F_{k\gamma}F^{-1}_{\alpha k} \\ &= S_{h\alpha} - L_{h\alpha\beta,\beta} = \hat{S}_{h\alpha} \end{aligned}$$

where the identity $J_{,\beta} = JF_{h\alpha\beta}F^{-1}_{\alpha h}$ has been used. We can then regard $\hat{\mathbf{T}}$ as the Cauchy stress tensor provided we prove that $\hat{\mathbf{T}} = \hat{\mathbf{T}}^t$.

The expression for $\mathbf{u}$ given in equation (13) is a special case of the interstitial energy flux derived in the theory proposed in Section 2 of Dunn and Serrin [9]. It is important to point out that, as noted by Dell'Isola and Seppecher [7], the interstitial energy flux $\mathbf{u}$ can be interpreted as the sum of the power of edge contact forces and other types of mechanical interactions. We do not enter into a detailed discussion of this issue, however, and stay within our more limited context.

4. Frame indifference of the stress-power

Now, we impose the requirement of frame invariance on the stress-power per unit volume W^{int}. For $\mathbf{Q} = [Q_{kh}]$ the rotation which connects observers $\mathcal{O}$ and $\mathcal{O}^+$, we easily deduce that

$$F^+_{h\alpha} = Q_{hk}F_{k\alpha} \qquad F^+_{h\alpha\beta} = Q_{hk}F_{k\alpha\beta}$$

from which

$$\begin{aligned} \dot{F}^+_{h\alpha} &= \dot{Q}_{hk}F_{k\alpha} + Q_{hk}\dot{F}_{k\alpha} \\ \dot{F}^+_{h\alpha\beta} &= \dot{Q}_{hk}F_{k\alpha\beta} + Q_{hk}\dot{F}_{k\alpha\beta}. \end{aligned}$$

For

$$W^{\text{int}} = S_{h\alpha}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta}$$
$$W^{\text{int},+} = S^+_{h\alpha}\dot{F}^+_{h\alpha} + L^+_{h\alpha\beta}\dot{F}^+_{h\alpha\beta}$$

it follows that

$$W^{\text{int}} = W^{\text{int},+} \Leftrightarrow \begin{cases} S^+_{k\alpha} = Q_{kh}S_{h\alpha} \\ L^+_{k\alpha\beta} = Q_{kh}L_{h\alpha\beta} \\ S_{h\alpha}F_{k\alpha} + L_{h\alpha\beta}F_{k\alpha\beta} \text{ is symmetric.} \end{cases} \tag{14}$$

The details can be found in Proposition 8 and equation (87) of Podio-Guidugli and Vianello [19].

As we shall see later, the requirement in equation $(14)_3$ is equivalent to frame indifference for the free energy function $\psi(\mathbf{F}, \nabla\mathbf{F}, \theta, \nabla\theta)$. Thus, this is not a condition which we find unnatural.

5. Stress-power and balance of angular momentum

We have not yet made any use of equation (8), which crucially splits $L_{h\alpha\beta}$ into a symmetric and a skew-symmetric part. In this section we show that it is precisely the presence of the powerless skew-symmetric term $W_{h\alpha\beta}$ which makes it possible, in general, to obtain a symmetric stress tensor and, thus, balances the angular momentum.

More precisely, as we prove here below, the (seemingly useless) part $W_{h\alpha\beta}$ of $L_{h\alpha\beta}$ is fully determined by the condition $\hat{\mathbf{T}} = \hat{\mathbf{T}}^t$ as a function of the symmetric part $S_{h\alpha\beta}$. In a sense, we value this connection as the main contribution of our article on this topic of second-gradient materials.

The identification of $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ with the Piola and Cauchy stress are completed by proving that

$$\hat{\mathbf{S}}\mathbf{F}^t = \mathbf{F}\hat{\mathbf{S}}^t \tag{15}$$

which amounts to the symmetry of $\hat{\mathbf{T}} = J^{-1}\hat{\mathbf{S}}\mathbf{F}^t$.

In view of equation (9), the condition in equation (15) takes the form

$$S_{h\alpha}F_{k\alpha} - L_{h\alpha\beta,\beta}F_{k\alpha} = S_{k\alpha}F_{h\alpha} - L_{k\alpha\beta,\beta}F_{h\alpha},$$

which can be rewritten as

$$\begin{aligned} &S_{h\alpha}F_{k\alpha} - (L_{h\alpha\beta}F_{k\alpha})_{,\beta} + L_{h\alpha\beta}F_{k\alpha\beta} \\ &\quad = S_{k\alpha}F_{h\alpha} - (L_{k\alpha\beta}F_{h\alpha})_{,\beta} + L_{k\alpha\beta}F_{h\alpha\beta}. \end{aligned}$$

Assuming that equation $(14)_3$ is satisfied, the symmetry of the Cauchy stress tensor is now guaranteed by

$$L_{h\alpha\beta}F_{k\alpha} = L_{k\alpha\beta}F_{h\alpha}. \tag{16}$$

In view of equation (8), the condition in equation (16) can be easily written as

$$S_{h\alpha\beta}F_{k\alpha} + W_{h\alpha\beta}F_{k\alpha} = S_{k\alpha\beta}F_{h\alpha} + W_{k\alpha\beta}F_{h\alpha},$$

or, after multiplication by $F_{i\beta}$, as

$$S_{h\alpha\beta}F_{k\alpha}F_{i\beta} + W_{h\alpha\beta}F_{k\alpha}F_{i\beta} = S_{k\alpha\beta}F_{h\alpha}F_{i\beta} + W_{k\alpha\beta}F_{h\alpha}F_{i\beta}. \tag{17}$$

For

$$D_{hki} := S_{h\alpha\beta}F_{k\alpha}F_{i\beta} \qquad H_{hki} := W_{h\alpha\beta}F_{k\alpha}F_{i\beta}$$

we have

$$D_{hki} = D_{hik} \qquad H_{hki} = -H_{hik}$$

and the condition in equation (17) takes the final form

$$D_{hki} + H_{hki} = D_{khi} + H_{khi}$$

or, equivalently,

$$D_{hki} - D_{khi} = H_{khi} - H_{hki}. \tag{18}$$

We now borrow from an idea found in Appendix A, p. 122 of Dunn and Serrin [9] and permute indexes to obtain

$$D_{hik} - D_{ihk} = H_{ihk} - H_{hik} \tag{19}$$

and

$$D_{kih} - D_{ikh} = H_{ikh} - H_{kih}. \tag{20}$$

The sum of equations (18), (19) and (20), in view of symmetries and skew-symmetries, gives

$$2D_{hki} - 2D_{ikh} = 2H_{khi}$$

so that

$$W_{k\alpha\beta} F_{h\alpha} F_{i\beta} = S_{h\alpha\beta} F_{k\alpha} F_{i\beta} - S_{i\alpha\beta} F_{k\alpha} F_{h\beta}.$$

It is useful to multiply the above expression by $F^{-1}_{\gamma i}$

$$W_{k\alpha\gamma} F_{h\alpha} = S_{h\alpha\gamma} F_{k\alpha} - F^{-1}_{\gamma i} S_{i\alpha\beta} F_{k\alpha} F_{h\beta} \tag{21}$$

and again by $F^{-1}_{\mu h}$ to obtain

$$W_{k\mu\gamma} = F^{-1}_{\mu h} S_{h\alpha\gamma} F_{k\alpha} - F^{-1}_{\gamma i} S_{i\alpha\mu} F_{k\alpha}. \tag{22}$$

Thus, the symmetry of the Cauchy stress tensor is guaranteed by a unique appropriate choice of the skew-symmetric (and powerless) part $W_{h\alpha\beta}$ of $L_{h\alpha\beta}$, which can be expressed linearly through the symmetric part $S_{h\alpha\beta}$.

Moreover, we anticipate that the skew-symmetric tensor part $W_{h\alpha\beta}$, as determined by equation (22), is what makes the flux **w** frame indifferent.

6. Frame indifference of the interstitial energy flux

Since **w** is a material vector field in the reference configuration, frame indifference is satisfied by condition $w^+_\beta = w_\beta$ for all changes of observer. In view of equations (10) and (14) we write

$$\begin{aligned} w^+_\beta &= L^+_{k\alpha\beta} \dot{F}^+_{k\alpha} = Q_{kh} L_{h\alpha\beta} (Q_{ks} F_{s\alpha})^{\cdot} \\ &= Q_{kh} L_{h\alpha\beta} \dot{Q}_{ks} F_{s\alpha} + Q_{kh} L_{h\alpha\beta} Q_{ks} \dot{F}_{s\alpha} \\ &= Q^t_{hk} \dot{Q}_{ks} L_{h\alpha\beta} F_{s\alpha} + Q^t_{sk} Q_{kh} L_{h\alpha\beta} \dot{F}_{s\alpha} \\ &= W_{hs} L_{h\alpha\beta} F_{s\alpha} + L_{s\alpha\beta} \dot{F}_{s\alpha} \\ &= L_{h\alpha\beta} F_{s\alpha} W_{sh} + w_\beta \end{aligned}$$

where $W_{hs} = Q_{kh} \dot{Q}_{ks}$ is skew-symmetric and arbitrary.

Thus,

$$w^+_\beta = w_\beta \Leftrightarrow L_{h\alpha\beta} F_{s\alpha} = L_{s\alpha\beta} F_{h\alpha}$$

and this is equivalent to the condition in equation (16), discussed before in connection with symmetry of the Cauchy stress $\hat{\mathbf{T}}$. We conclude that the frame invariance of **w** and the symmetry of $\hat{\mathbf{T}}$ are guaranteed by the same property of the stress-power, which is satisfied through the appropriate choice for $W_{h\alpha\beta}$ expressed in equation (22).

7. Balance of energy and entropy inequality

The balance of linear momentum is postulated in the classical form, which locally reduces to

$$\rho_0 \ddot{\mathbf{x}} = \rho_0 \mathbf{b}_0 + \mathrm{Div}\,\hat{\mathbf{S}},$$

while the balance of energy is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \left[\rho_0 e + \tfrac{1}{2}\rho_0 \dot{\mathbf{x}}^2\right] \mathrm{d}V = \int_{\partial\mathcal{P}} [\hat{\mathbf{S}}\mathbf{m}\cdot\dot{\mathbf{x}} + \mathbf{w}\cdot\mathbf{m} - \mathbf{q}_0\cdot\mathbf{m}]\,\mathrm{d}A + \int_{\mathcal{P}} [\rho_0 \mathbf{b}_0\cdot\dot{\mathbf{x}} + \rho_0 r]\,\mathrm{d}V$$

where all terms are classical, except for the interstitial energy flux $\mathbf{w}$, which we add following Dunn and Serrin's [9] approach. The local form is obtained as

$$\rho_0 \dot{e} = \rho_0 r + \underbrace{\hat{\mathbf{S}}\cdot\dot{\mathbf{F}} + \mathrm{Div}\,\mathbf{w}}_{W^{\mathrm{int}}} - \mathrm{Div}\,\mathbf{q}_0,$$

where the role of the stress-power is clear.

We follow Dunn and Serrin [9] and write the entropy inequality in the reduced Lagrangian local form as

$$\rho_0(\dot{\psi} + \eta\dot{\theta}) - \hat{\mathbf{S}}\cdot\dot{\mathbf{F}} - \mathrm{Div}\,\mathbf{w} + \frac{\mathbf{q}_0\cdot\nabla\theta}{\theta} \le 0. \tag{23}$$

Of course, the fact that the interstitial energy flux is *not* subjected to the same treatment as the heat flux $\mathbf{q}_0$, is a very touchy point, which is discussed at great length by Dunn and Serrin [9]. Our goal here is much more limited, and we do not discuss such a delicate issue but only remark that $\mathrm{Div}\,\mathbf{w}$ is a mechanical power.

8. A second-gradient free energy

We make a very simple assumption for the free-energy ψ, and take it as a function of the first and second deformation gradients, together with the temperature and its gradient:

$$\psi(\mathbf{F}, \nabla\mathbf{F}, \theta, \nabla\theta), \qquad \psi(F_{h\alpha}, F_{h\alpha\beta}, \theta, \theta_{,\gamma}).$$

The frame indifference of ψ is expressed by the condition that

$$\psi(\mathbf{QF}, \mathbf{Q}\nabla\mathbf{F}, \theta, \nabla\theta) = \psi(\mathbf{F}, \nabla\mathbf{F}, \theta, \nabla\theta)$$

for all rotations $\mathbf{Q}$. This requirement is discussed in detail on p. 115 of Dunn and Serrin [9] and, here, can be shown to be equivalent to

$$\psi_{F_{h\alpha}} F_{k\alpha} + \psi_{F_{h\alpha\beta}} F_{k\alpha\beta} = \psi_{F_{k\alpha}} F_{h\alpha} + \psi_{F_{k\alpha\beta}} F_{h\alpha\beta}, \tag{24}$$

which shall be investigated in more detail in a moment.

It is now useful to compute

$$\dot{\psi} = \psi_\theta \dot{\theta} + \psi_{F_{h\alpha}} \dot{F}_{h\alpha} + \psi_{\theta_{,\gamma}} \dot{\theta}_{,\gamma} + \psi_{F_{h\alpha\beta}} \dot{F}_{h\alpha\beta}$$

and

$$\begin{aligned}
\mathrm{Div}\,\mathbf{w} = w_{\beta,\beta} = \left(L_{h\alpha\beta}\dot{F}_{h\alpha}\right)_{,\beta} &= L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta} \\
&= L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} + S_{h\alpha\beta}\dot{F}_{h\alpha\beta} + \underbrace{W_{h\alpha\beta}\dot{F}_{h\alpha\beta}}_{\equiv 0} \\
&= L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} + S_{h\alpha\beta}\dot{F}_{h\alpha\beta}.
\end{aligned}$$

Thus, the entropy inequality in equation (23) takes the form

$$\rho_0\left(\psi_\theta\dot{\theta} + \psi_{F_{h\alpha}}\dot{F}_{h\alpha} + \psi_{\theta_{,\gamma}}\dot{\theta}_{,\gamma} + \psi_{F_{h\alpha\beta}}\dot{F}_{h\alpha\beta} + \eta\dot{\theta}\right) - \hat{S}_{h\alpha}\dot{F}_{h\alpha} - L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} - S_{h\alpha\beta}\dot{F}_{h\alpha\beta} + \theta_{,\gamma} q^0_\gamma/\theta \le 0$$

and from this, with the usual line of argument, we deduce

$$\begin{gathered}\psi_\theta + \eta = 0, \quad \psi_{\theta_{,\gamma}} = 0, \quad \rho_0 \psi_{F_{h\alpha}} = \hat{S}_{h\alpha} + L_{h\alpha\beta,\beta}, \\ \rho_0 \psi_{F_{h\alpha\beta}} = S_{h\alpha\beta},\end{gathered} \tag{25}$$

and the classical condition on the heat flux

$$\theta_{,\gamma} q^0_\gamma \leq 0, \quad \nabla\theta \cdot \mathbf{q}_0 \leq 0.$$

After multiplication by ρ_0 and in view of equation $(25)_{3,4}$, the condition in equation (24) is transformed into

$$(\hat{S}_{h\alpha} + L_{h\alpha\beta,\beta})F_{k\alpha} + S_{h\alpha\beta}F_{k\alpha\beta} = (\hat{S}_{k\alpha} + L_{k\alpha\beta,\beta})F_{h\alpha} + S_{k\alpha\beta}F_{h\alpha\beta}.$$

Finally, equation (9) shows that equation (24) can be expressed as

$$S_{h\alpha}F_{k\alpha} + S_{h\alpha\beta}F_{k\alpha\beta} = S_{k\alpha}F_{h\alpha} + S_{k\alpha\beta}F_{h\alpha\beta} \tag{26}$$

which coincides with equation $(14)_3$. Thus, the frame indifference of the free energy $\psi(\mathbf{F}, \nabla\mathbf{F}, \theta, \nabla\theta)$ implies the invariance of the stress-power under a change of observer.

We now turn to equation $(25)_3$ which we use as a starting point for deriving an expression for $\hat{S}_{h\alpha}$. In view of equation (8), with an integration by parts we have

$$\begin{aligned}\hat{S}_{h\alpha} &= \rho_0\psi_{F_{h\alpha}} - L_{h\alpha\beta,\beta} \\ &= \rho_0\psi_{F_{h\alpha}} - S_{h\alpha\beta,\beta} - W_{h\alpha\beta,\beta} \\ &= \rho_0\psi_{F_{h\alpha}} - \rho_0(\psi_{F_{h\alpha\beta}})_{,\beta} - W_{h\alpha\beta,\beta} \\ &= \rho_0\big(\psi_{F_{h\alpha}} - (\psi_{F_{h\alpha\beta}})_{,\beta}\big) - W_{h\alpha\beta,\beta}.\end{aligned} \tag{27}$$

From equation (22) it follows that

$$W_{h\alpha\beta,\beta} = \rho_0\big(F^{-1}_{\alpha k}\psi_{F_{k\gamma\beta}}F_{h\gamma} - F^{-1}_{\beta k}\psi_{F_{k\gamma\alpha}}F_{h\gamma}\big)_{,\beta}$$

and, by substitution in equation (27), we have

$$\hat{S}_{h\alpha} = \rho_0\big[\psi_{F_{h\alpha}} - (\psi_{F_{h\alpha\beta}})_{,\beta} + \big(F^{-1}_{\beta k}\psi_{F_{k\gamma\alpha}}F_{h\gamma} - F^{-1}_{\alpha k}\psi_{F_{k\gamma\beta}}F_{h\gamma}\big)_{,\beta}\big]$$

which is equal to equation (3.4) on p. 114 of Dunn and Serrin [9].

Finally, if we wish to compute the Cauchy stress tensor $\hat{\mathbf{T}}$ we write

$$\hat{T}_{hk} = \frac{1}{J}\hat{S}_{h\alpha}F_{k\alpha} = \frac{\rho}{\rho_0}\hat{S}_{h\alpha}F_{k\alpha} = \rho\big(\psi_{F_{h\alpha}}F_{k\alpha} - (\psi_{F_{h\alpha\beta}})_{,\beta}F_{k\alpha}\big) - \frac{\rho}{\rho_0}W_{h\alpha\beta,\beta}F_{k\alpha} \tag{28}$$

and, from equation (21), we deduce that

$$W_{h\alpha\beta}F_{k\alpha} = \rho_0\big(\psi_{F_{k\alpha\beta}}F_{h\alpha} - F^{-1}_{\beta i}\psi_{F_{i\alpha\gamma}}F_{h\alpha}F_{k\gamma}\big).$$

Moreover, since

$$W_{h\alpha\beta,\beta}F_{k\alpha} = \big(W_{h\alpha\beta}F_{k\alpha}\big)_{,\beta} - \underbrace{W_{h\alpha\beta}F_{k\alpha\beta}}_{\equiv 0} = \big(W_{h\alpha\beta}F_{k\alpha}\big)_{,\beta}$$

from equation (28) we deduce

$$\hat{T}_{hk} = \rho\big(\psi_{F_{h\alpha}}F_{k\alpha} - (\psi_{F_{h\alpha\beta}})_{,\beta}F_{k\alpha}\big) - \rho\big(\psi_{F_{k\alpha\beta}}F_{h\alpha} - F^{-1}_{\beta i}\psi_{F_{i\alpha\gamma}}F_{h\alpha}F_{k\gamma}\big)_{,\beta},$$

an expression which we manipulate into

$$\hat{T}_{hk} = \rho\big[\big(\psi_{F_{h\alpha}}F_{k\alpha} + \psi_{F_{h\alpha\beta}}F_{k\alpha\beta}\big) + \big(F^{-1}_{\beta i}\psi_{F_{i\alpha\gamma}}F_{h\alpha}F_{k\gamma}\big)_{,\beta} - \big(\psi_{F_{h\alpha\beta}}F_{k\alpha} + \psi_{F_{k\alpha\beta}}F_{h\alpha}\big)_{,\beta}\big]. \tag{29}$$

Again, this is exactly the same expression for $\mathbf{T}$ ($\hat{\mathbf{T}}$, here) found in equation $(3.2)_1$ on p. 114 of Dunn and Serrin [9]. A glance at equation (29) confirms that equation (26) makes $\hat{T}_{hk}$ symmetric.

9. Conclusions

This article is framed within the context of non-simple materials and, in particular, elastic materials for which the strain energy ψ is a function of the first and second deformation gradients $F_{h\alpha}$ and $F_{h\alpha\beta}$. Following a now standard approach, we assume as the inner power per unit volume the expression

$$S_{h\alpha}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta}$$

but, as a novelty, we investigate the consequences of letting $L_{h\alpha\beta}$ be non-symmetric in the last two indexes. Indeed, we show that the skew-symmetric part of $L_{h\alpha\beta}$, despite being powerless, is uniquely determined by the condition that the Cauchy stress tensor $\mathbf{T}$ be symmetric, thus avoiding the introduction of micro-couples. Moreover, the expression for $\mathbf{T}$ we derive from the strain energy $\psi(F_{h\alpha}, F_{h\alpha\beta})$ is fully consistent with the result found by Dunn and Serrin [9] in their seminal article with a different approach: rather than a hyperstress term in the inner power, as in Dunn and Serrin [9], an additional interstitial energy flux is included in the first law of thermodynamics.

Conflict of interest

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