On the use of AR models for SHM: A global sensitivity and uncertainty analysis framework

Alessio Datteo, Giorgio Busca *, Gianluca Quattromani, Alfredo Cigada

Politecnico di Milano, Department of Mechanical Engineering, Via La Masa 1, 20156 Milano, Italy

This paper proposes a complete sensitivity analysis of the use of Autoregressive models (AR) and Mahalanobis Squared Distance in the field of Structural Health Monitoring (SHM). Autoregressive models come from econo- metrics and their use for modelling the response of a physical system has been well established in the last twenty years. However, their aware application in engineering should be supported by knowledge about how they describe phenomena which are well defined by physics. Since autoregressive models are estimated by a least square minimization, statistical tools like Global Sensitivity Analysis and uncertainty propagation are powerful methods to investigate the performance of AR models applied to SHM. These methodologies allow one to understand the role of the uncertainty and uncorrelated noise by a rigorous approach based on statistical motivations. Moreover, it is possible to quantify the link between the mechanical properties of a system and the AR parameters, as well as the Mahalanobis Squared Distance. By fixing a factor prioritization among the variables of a AR model, it is possible to understand which are the parameters playing a main role in damage detection and which type of structural changes is possible to efficiently detect.

Keywords:
Structural Health Monitoring
Autoregressive model
Mahalanobis Squared Distance
Uncertainty propagation
Global Sensitivity Analysis

1. Introduction

The importance of improving the understanding of the performance of structures over their lifetime with information obtained from Structural Health Monitoring (SHM) has been widely documented. The assessment of structural reliability is strictly connected to the quality of information provided by a damage detection process [1–6]. The diagnosis of in-service structures on a continuous real-time basis is of primary importance for aerospace, civil and mechanical engineering. The advantages of Structural Health Monitoring (SHM) are optimal use of a structure, reduced downtime and avoidance of catastrophic failures, moreover it can drastically change the planning of maintenance service with several economic benefits [7]. There are many potentially useful techniques to achieve these aims, and their applicability to a specific situation depends on the size of the acceptable critical damage for a structure.

As mentioned in the given reference, the problem of damage detection has a hierarchical structure. At the lowest level, it is required to recognize damage has occurred or not. At the highest level, damage location and size must be identified for a proper estimation of the residual structure life. One of the most promising approaches to damage identification is based on pattern recognition [8]. Data are measured from a structure and converted, by a process of feature extraction, into a representation where variations due to damage are highlighted.

* Corresponding author.
E-mail address: giorgio.busca@polimi.it (G. Busca).

Vibration based methods have been widely used for identification of various types of damages for several real and laboratory structures. The methods relying on vibration response -also known as Output-Only methods –represent an important category within the vibration based methods for Structural Health Monitoring (SHM). Their use is highly important for in-service structures such as bridges, aircrafts, naval vehicles and others, where the excitation signal is not available.

In the last twenty years, the scientific community focused its attention on exploitation of different types of autoregressive models and features for SHM. In [9] *Sohn* et al. studied *AR* models and features based on analysis of residuals (X-Chart, S-Chart, EWMA). In [10] *Carden* et al. applied the more complex *ARMA* models to SHM application; the approach has been validated with experimental data taken from Z24 bridge. *Sohn* et al. in [11] proposed a linearized version of *ARMA,* the *AR-ARX* model.
The applicability of this approach has been demonstrated with an experimental setup based on an eight degree of freedom mass-spring system. Features based on residuals often assume a Gaussian distribution of sample data sets. This assumption might be misleading, making SHM algorithms less efficient. *Worden* et al. in [12] tried to overcome this problem proposing a more sophisticated data processing called *sequential probability ratio test* (SPRT), which relies on the analysis of extreme value statistics. In [13] *Yao* et al. proposed a comparison between sev-

Published Journal Article available at: https://doi.org/10.1016/j.ress.2017.10.017

Nomenclature

X_t, X_{t-1}, X_0	Signal sample at time $t, t-1, 0$
ϕ_1	Autoregressive parameter of AR(1) model
a_t	Residual between model prediction and signal output at time t
q	Order of the AR(q) model
D_m	Mahalanobis square distance
$[\phi]$	qxp Matrix of Autoregressive reference dataset. p is the number of set acquired
ϕ_i	Generic i-th AR parameter of the model
$\{\phi_\mu\}$	$qx1$ Vector of the means of the Autoregressive reference datase
$[S]$	qxq Covariance Matrix of the Autoregressive reference dataset
$\{\hat{\phi}\}$	$qx1$ Vector of the Autoregressive parameters assumed for the outlier analysis
Y	General output quantity
X_i	General i-th input quantity
$V(Y)$	Total variance of the Y output
$V(E(Y\|X_i))$	Conditional variance of the Y output with respect X_i input
$V(E(Y\|X_{\sim i}))$	Conditional variance of the Y output with respect all inputs except X_i
S_{X_i}	First order global sensitivity index
S_{X_i,X_j}	Second order global sensitivity index
S_{T_i}	Total sensitivity index
$[M]$, $[D]$, $[K]$	Mass matrix, damping matrix, stiffness matrix of a generic mechanical system
m, h, k	Mass, damping ratio and stiffness of 1 d.o.f. mechanicals system
σ_u	Standard deviation of Gaussian white noise excitation
$U(\sim)$	Uniform distribution

eral pattern recognition algorithms using autoregressive models. Moreover, they introduced a new feature extraction technique, called *Cosh spectral distance* (COSH) and validated it with several experimental data.

Among all the possible strategies for time series modelling, the use of pure AR models is very common. Basically, because the identification of the model is made by a simple least squared minimisation, which requires few computing efforts to be performed, and the uncertainty of the model is usually low. However, pure AR models are only-pole functions and can represent the response of complex systems just by an approximation. The consequences are spurious poles that must be introduced in the model in order to follow the response of the mechanical system, although it depends also on the zeros of the frequency response function of the physical model [14]. The popular application of AR models to SHM relies on their reliable identification of the mechanical properties of a system; however, the main property that should be taken into account, in a SHM context, is their sensitivity to a change of the system they are representing. Since AR models are made by physical and spurious poles, their sensitivity is not a trivial issue. The AR parameters of a model are usually used all together to assess the healthy status of a system, for instance computing a Mahalanobis Square Distance. However, as it will be shown, only few of them are strongly linked to the physical properties of the system; so that, their behaviour is not strictly depending on the physical response they are modelling and to its changes.

Scientific literature lacks examples for the explicit propagation of measurement uncertainty and Global Sensitivity Analysis for damage detection algorithms. *Yao* et al. in [14] proposed a formulation for the sensitivity of Mahalanobis Squared Distance and COSH distance with re- spect to both stiffness reduction and measurement noise level. However, the analysis is based only on an analytical study of the issue in place of a statistical framework. In that work, simulation results and theoretical analysis show some differences due to the approximation adopted for the extrapolation of the sensitivity expression. In [15] *Roy* et al. provided a mathematical formulation to establish the relation between the change in an ARX model coefficients and the normalized stiffness of a structure. The reason behind the choice of ARX model in place of a standard AR model is the coefficients of the ARX model can have a direct correlation with structural stiffness. Such a correlation is however not established for standard AR model.

This work focuses on an accurate analysis of the uncertainty related to vibration-based method. Specifically, it focuses on the use of pure autoregressive models and Mahalanobis Squared Distance, among the most widely adopted approaches in vibration-based methods. Generally, this approach could be extended to any kind of damage feature to quantify its sensitivity to the changes of a system.

The contribution of this paper is the attempt of covering the lack of uncertainty assessment in the SHM literature, performing an Uncertainty Propagation Analysis (UP) and a Global Sensitivity Analysis (GSA) of AR models and Mahalanobis Squared Distance. As it will be proved, a rigorous analysis will demonstrate that pure AR models may hide some weaknesses that could have strong consequences on their feasibility to SHM. This paper will give some guidelines about the variables that strongly affect the performance of AR models for damage detection and about the type of structural changes that might be detected with confidence. The conclusion will be fundamental to those who want to use AR models as tools to get information to predict the safety of aging structures over their service life.

The paper is structured as follows. In Section 2 the background theory of Autoregressive models and Mahalanobis Squared Distance are briefly exposed, in the context of damage detection. The Analysis of Variance is introduced in Section 3. The design of the simulation is discussed in Section 4. Finally, the results of the Analysis of Variance of the AR model and Mahalanobis Squared Distance applied to damage detection are presented and commented in Section 5.

2. Autoregressive models

In the next paragraphs, the background theory of AR models and Mahalanobis Squared Distance is briefly introduced, putting them in the context of dynamic system identification. The reader is asked to refer to a complete background theory provided by the following reference [16].

2.1. Description of a system response function by autoregressive models

Autoregressive models were developed in econometrics as a representation of time-varying processes, in which the output variable depends linearly on its own previous values and on a stochastic term. Nowadays they are used in a wide variety of different fields, for instance Structural Health Monitoring (SHM). Autoregressive models can be implemented to represent the dynamic response of structures. Through these models, it is possible to describe a time series with a lower number of data, which are the parameters of the AR model.

To introduce autoregressive models, let us consider a linear mechanical system. Let us call $u(t)$ the input of a system and $x(t)$ the output. Usually the input could be either a deterministic variable or a stochastic one. In a civil or mechanical structure, which works under operational conditions, the excitation can be described as a random force. Under some strong assumptions [17], operational modal analysis considers the random force as a Gaussian process. In a real case, these assumptions are quite well respected if the data are averaged over a long enough time window. Therefore, under this condition, $u(t) \sim NID(0, \sigma_u^2)$.

Now it is possible to recall the link between a generic dynamic system and an AR model. For sake of simplicity, let us start with the autoregressive model of the first order $AR(1)$, which represents a dynamic response of the first order in the discrete time domain:

$$X_t = \phi_1 X_{t-1} + a_t \tag{1}$$

In Eq. (1) X_t and X_{t-1} are two consecutive samples of the system output, ϕ_1 is the autoregressive parameter and a_t is the residual com- puted between the prediction of the model and the acquired value of the output at the same time; t is the generic discrete time index. The autore- gressive parameter ϕ_1 is usually estimated by a least square approach [16,18]. The a_t term is a stochastic element. Here, the residual term is a Gaussian process with zero mean and variance σ_a^2, i.e. $a_t \sim NID(0, \sigma_a^2)$.

To obtain the complete solution of the first order linear system it is necessary to iterate Eq. (1) and set the initial condition X_0 [16]:

$$\boldsymbol{X}_t = \boldsymbol{\phi}_1^t \boldsymbol{X}_0 + \sum_{j=0}^{t-1} \boldsymbol{\phi}_1^j \boldsymbol{a}_{t-j} \tag{2}$$

where j is the discrete time index from 0 to $t-1$.

The solution of the autoregressive model $AR(1)$ in the form of Eq. (2) is the discrete solution of the dynamic response for a linear system of the first order excited by a random input. In fact, X_t is described by the sum of two terms. The first term on the right side of Eq. (2) is the homogeneous solution made by the product between the initial condition X_0 and ϕ_1^t which is the analogous of the state matrix. Based on the theory of discrete time systems [19], if the eigenvalues of the state matrix are lower than one - i.e. $|\phi_1| < 1$ - the system will be stable and stationary after a certain time t. This term represents the free response of the linear system. The second term on the right-side equation represents the stochastic part, which is linked to the residual term a_{t-j}. This term represents the forced response of the linear system caused by a pure stochastic input $u(t)$, which is defined by its variance σ_u^2.

The first order autoregressive model *AR* (1) cannot represent a dynamic system of higher order. The natural extension of the *AR* (1) model to a higher order system is the Autoregressive Moving Average Model ($ARMA(n, n-1)$) [16,10]. *ARMA* modelling is the most accurate tool to describe the output of linear systems forced with random excitations. The main drawback of using *ARMA* models is the need of a non-linear least square approach to find the MA moving average coefficients [16]. A non-linear least square has not only high computational cost, but it can also have convergence and local minima problems.

To avoid these complications, an autoregressive model $AR(q)$ can approximate the $ARMA(n, n-1)$ model when $q \gg n$, the AR model order and the ARMA model order respectively. The equation of a $AR(q)$ model is the following [9,11]:

$$\boldsymbol{X}_t = \sum_{k=1}^{q} \boldsymbol{\phi}_k \boldsymbol{X}_{t-k} + \boldsymbol{a}_t \tag{3}$$

Where again q is the order of the model, t is the discrete time index when the model is estimated and a_t is the residual. The $AR(q)$ model has only one term for the residual a_t which is calculated at the current time step. The order of the system is generally unknown at priori, as well as the order of the *AR* model suitable to describe the system. Therefore, the process to find out the optimal order of the autoregressive model $AR(q)$ is not trivial. In literature there are some specific techniques to achieve this goal, such as the Akaike's Information Criterion (AIC) or Bayesian information criterion (BIC) [20]. $AR(q)$ models are less accurate in the representation of a dynamic system if compared to the $ARMA(n, n-1)$ models. In general, by modelling the output of a system through *AR* models, it is possible to obtain the eigenvalues of the mechanical system in the discrete time domain; then, mapping these eigenvalues to those in the continuous time domain, the natural frequencies of the system are finally obtained. AR models are only-poles functions. Furthermore, since $AR(q)$ models have a bigger order than the original vibrating system, it has always spurious eigenvalues depending on the chosen order of the model.

Autoregressive parameters are computed by using a least square minimisation and autoregressive model is just an approximation of the response of a system subjected to a random excitation. This means that an analytical solution representing the exact correspondence between the AR parameters and the mechanical system parameters is unknown. An alternative strategy to deal with this situation, it is to study the problem from a statistical point of view, using methods based on the analysis of variance [21].

2.2. Mahalanobis Squared Distance as damage feature

One of the main goals of SHM is detection of early stage damage in a structure. Since it is not possible to directly measure the presence of damage in structures [22], the solution is the definition of some damage sensitive quantities, usually called *damage features*. Vibration-based methods assume that the presence of damage influences the dynamic response of the system [7]. Using an AR model, as a discrete representation of the system output, it is possible to extract some features from the model itself. In the last 15 years, several features have been developed from autoregressive models [9,11–13] and they can be divided in two categories: those based on the residual terms and those based on he autoregressive parameters. Features based on the autoregressive parameters have generally shown to be a more robust damage index [13].

One of the most common features used in SHM applications is the *Mahalanobis Squared Distance* (MSD) of the parameters of the AR model in the context of outlier analysis. The Mahalanobis Squared Distance is the n-dimension generalisation of the Euclidean distance, normalised through the covariance matrix.

The analytical expression of MSD applied to the case of AR parameters is the following:

$$\boldsymbol{D_M} = \left[\left(\{\hat{\boldsymbol{\phi}}\} - \{\boldsymbol{\phi}_\mu\}\right)^T [\boldsymbol{S}]^{-1} \left(\{\hat{\boldsymbol{\phi}}\} - \{\boldsymbol{\phi}_\mu\}\right) \right] \tag{4}$$

Where $\{\hat{\phi}\}$ is the vector made of p AR parameters which represents the potential outlier, $\{\phi_\mu\}$ is the vector of the means of the p AR parameters estimated on a reference dataset $[\phi]$, $[S]$ is the covariance matrix of the reference dataset.

Generally, the literature assumes that multivariate data are normally distributed, so that the MSD can be approximated by a chi-squared distribution in n-dimensional space [23,24]. Under this hypothesis, the threshold to detect the outliers can be estimated as percentile of the reference dataset made by the all the AR parameters collected from the system under the normal condition. When the size of the reference dataset $[\phi]$ is poor or the Gaussian distributions of the variables cannot be assured, the definition of the threshold should follow another method based on Monte Carlo simulation and extreme value statistics [23]. Since in this paper there is an implicit assumption that the reference or training is multivariate Gaussian, the two methods for estimating the threshold should be both valid.

3. Analysis of variance

Finding the relationship between the autoregressive model and the mechanical properties of the structure (mass, stiffness, damping etc.) under stable operational conditions, it is possible to quantify the sensitivity of this model to a change of the system. In this work only the case of stationary operational conditions will be studied. The basic theory of sensitivity and uncertainty propagation will be given in the following [25].

3.1. Uncertainty propagation (UP)

Uncertainty propagation is the process aiming at quantifying the uncertainty in the result of a function, starting from the uncertainty of its inputs. The easiest way to perform this process is the Monte Carlo simulation (MCS) [26]. However, MCS can be computationally heavy and for this reason it could be difficult to be applied. In order to overcome this limit, in the last two decades the use of surrogate models has been increased. Surrogate models aim to reduce simulation time through an approximation in a sample space of the model under investigation.

Among these procedures, the most popular are polynomial chaos expansion (PCE) [27–29], Kriging [30–33], ANOVA decomposition [34–39], neural network [40–42].

There are also other popular sensitivity techniques such as Morris' screening method [43–45], stepwise regression based methods [46] and regional sensitivity analysis techniques [47,48]. Since the problem addressed in this paper is linear and the computational cost is affordable, the authors have decided to apply a direct MCS.

How the MC method has to be used is widely described in the supplement to the "Guide to the expression of uncertainty in measurements " (GUM), and the reader may refer to it for further details [49]. The basic way to perform a Monte Carlo simulation follows these steps:

- define the output quantity Y;
- determine the input quantities X_i upon which Y depends;
- set the model relating Y and X_i;
- on the basis of the available knowledge assign Probability Density Functions (PDFs) to X_i;
- propagate the PDFs for the X_i through the model to obtain the PDF for Y, where the propagation is performed sampling the value of X_i from their PDFs several times.

The most common and easy way to sample the input data is to use a random sampling function for several trials, GUM states at least 10ˆ5–10ˆ6 runs. In order to decrease the computational cost, more sophisticated methods should be used, like the Latin Hypercube sampling or quasi-Monte Carlo algorithm, which is the method used in this work [50–52]. Having the PDF of Y it is then possible to apply a GSA and enhance the knowledge about the relationship among Y and the different inputs X_i. The PDF also allows to define the expectation of Y and its uncertainty bandwidth with a specified probability to find Y in that range.

3.2. Global sensitivity analysis (GSA)

Saltelli et al. [25] defines sensitivity analysis (SA) as: *"The study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input "*. In the literature, local sensitivity refers to the sensitivity at a fixed point in the parameter space, while global sensitivity refers to an integrated sensi- tivity over the entire input parameter space.

The methods used to perform GSA can be divided into two kinds of approaches: variance-based and density-based [53–59]. The first type of approaches was developed in nineties and it is very popular because various smart computational methods are available. However, it assumes that variance is a good parameter to represent the distribution of variables, inputs and outputs, in their hyperspace, but this hypothesis is not always true. It has been recognised that variance-based methods are not suitable when the outputs have probability density functions highly skewed or multimodal. In these situations, the second type of approaches can be used. Density-based methods do not rely on a specific moment to describe the outputs but use all the probability density distribution to perform the Global Sensitivity Analysis on the model.

Herein the *model* is the autoregressive model, which is used to fit the vibration signal. Then the inputs will be the variables involved in the generation of the vibration data (the dynamic response of the structure): mass, stiffness, damping and force. The output of the model will be the vector of the autoregressive parameters $\{\phi\}$ which represents the

response of the system. Since the outputs of the model respect the conditions to apply a variance-based approach, the authors have arbitrarily chosen this way to develop GSA. To prove the reliability of the obtained results also a moment-independent method (PAWN) will be used to estimate a GSA on the autoregressive parameters [60,61]. The results of PAWN method will be reported in Appendix B.

Let $\{\phi\} = f([M], [D], [K], \sigma_u)$ be the unknown function linking the outputs – the AR parameters vector $\{\phi\}$ – and the inputs – the mechan- ical properties and random forcing level. The size of vector $\{\phi\}$ will depend on the autoregressive model order chosen to fit the signal. The inputs are the mass matrix [M], the damping matrix [D] and the stiffness matrix [K] of the mechanical system, whose size depends on the complexity of the structure, i.e. the degrees of freedom of the system. The response of the structure also depends upon the type of forcing source. A Gaussian white noise excitation is considered and setting the standard deviation σ_u, which is the amplitude of its average spectrum, the amplitude of the force is defined.

If the probability density function (PDF) of the inputs and the outputs are known, then it is possible to quantify the dependency of the outputs from the inputs. Moreover, it is possible to define the input parameters mostly affecting the outputs. As it is explained in Section 3.1, the populations of the mechanical parameters are generated by Monte Carlo simulations. Once established the populations of the generic inputs X_i – in this case $[M]$, $[D]$, $[K]$ and σ_u – and obtained the population of the generic output Y – in this case the AR parameter vector $\{\phi\}$ - it is possible to derive some evaluations on the conditioned variance and compute the Sobol's indexes.

Sobol [50] defined some indexes based on the variance of the data to quantify global sensitivity.

One is the *first-order global sensitivity index* of the generic input pa- rameter X_i over the output Y [25]

$$S_{X_i} = \frac{V\left(E(Y|X_i)\right)}{V(Y)} \tag{5}$$

where $V(E(Y|X_i))$ is the conditional variance and $V(Y)$ is the total variance of the output. Roughly speaking, each sensitivity index S_{X_i} is obtained by fixing iteratively the variable X_i at a given value x_i^* (with $i = 1$ … K) and letting the other parameters vary randomly according to their probability distribution. S_{X_i} is a number always between 0 and 1. A S_{X_i} value close to 1 indicates a strong link between the considered variables and the outputs.

However, a value of S_{X_i} close to 0 does not mean an irrelevant variable, because the first order index only quantifies the direct effect of each parameters, dropping out the combined effect of more inputs.

The *total effect index* of factor X_i is obtained by conditioning all factors but X_i [25]:

$$S_{T_i} = 1 - \frac{V\left(E(Y|X_{\sim i})\right)}{V(Y)} = \frac{E\left(V(Y|X_{\sim i})\right)}{V(Y)} \tag{6}$$

It is made of all the terms of any order that are including X_i. Note that S_{T_i} and S_{X_i} carry different information. It can be demonstrated that $S_{X_i} = 0$ is a necessary but not sufficient condition to exclude the dependency of Y on X_i. Even if $S_{X_i} = 0$, the input X_i might still be involved in interactions with other inputs so that, although its first-order term is zero, there might be nonzero higher-order terms. On the other hand, if $S_{T_i} = 0$, this is a necessary and sufficient condition for X_i being uninfluential.

In this work, the second-order sensitivity index of two generic input parameters X_i and X_j over the output Y has been used too:

$$S_{X_i,X_j} = \frac{\mathrm{V}\left(E(Y|X_i, X_j)\right)}{V(Y)} - \frac{V\left(E(Y|X_i)\right)}{V(Y)} - \frac{V\left(E(Y|X_j)\right)}{V(Y)} \tag{7}$$

As it will be clear in the next sections this index will provide impor- tant information about the link between the autoregressive parameters (outputs) and the joined effects of the mechanical parameters (inputs). Since the sensitivity analysis of a multioutput system can be heavy, the sensitivity indexes will be evaluated by a quasi-Monte Carlo method and by means of the framework used by Cannavò in [51]. The error associated to the estimation of the sensitivity indexes will be estimated with a confidence level of 68% by the formula reported in [51].

4. Modelling and simulations

The purpose of this paper is to qualify the behaviour of the autoregressive model used to describe the response of a mechanical system in a damage detection context. A one degree of freedom (1dof) system seems to be the best choice to study the relationship between the AR parameters and the mechanical properties of the system. The choice could be considered too simple, but, as will be shown in the next sections, the aspects to take into consideration are several and the use of a multi degree of freedom system (MDOF) could just complicate the description of the outcomes. In the authors' opinion, the interpretation of the results can be extended to a MDOF system, even if the analysis is performed on a 1dof system. Indeed, through modal analysis the response of a linear system can be interpret as the sum of the responses of many 1dof systems. Since the results and the comments refer to a general be- haviour of the AR modelling and its link with the physical poles of the system, there is no reason to expect a different behaviour for a more complex system. Moreover, in Appendix A to this paper, a brief analysis on a MDOF system will be given to prove the generalisation of the main outcomes.

4.1. 1 dof system

The mechanical system considered in this work is a mass-spring-damper system excited by a random input, more precisely by a Gaussian white noise force. At first, it is necessary to establish the normal condition of the system, which means, the condition without any variation due to the occurrences of operational, environmental or damage changes. It is decided to arbitrarily assume a natural frequency of 1 Hz for the system. As for the damping factor, as previously stated, it has been chosen a viscous damping focusing on the case of lightly damped systems only. Lightly damped systems are of interest for SHM applications, because they reflect the behaviour of some typical structures being monitored, for instance steel bridges [62,63]. Therefore, the damping ratio is initially fixed at 0.8%. Assuming the mass of the system equal to 10 kg, the stiffness of the spring is 394.78 N/m for a system with a natural frequency of 1 Hz. For the reference condition, the random input to the mechanical system is a Gaussian white noise with a RMS value of 10 N.

To compute the output response of the dynamic system, different ap- proaches may be implemented. The most common is based on the inte- gration of the equations of motion with a numerical integration method, such as Runge Kutta 45. Although this is a very powerful and effective method, it may be not very efficient when employed inside a Monte Carlo simulation run with up to millions of iterations. In this work, with the purpose to speed up the simulations, a different approach has been implemented, which is based on the *convolution theorem* [64]. It is possible to calculate the response of the system using the convolution integral between the time history of the input and the impulse response of the system or, in the same way, computing the arithmetic product between the frequency response function (FRF) of the system and the spectrum of the input.

It is important to note that the FRF of the system may be obtained in a closed form, once all the mechanical parameters listed before and the spectrum of the Gaussian white noise are imposed. At this point, by making a simple product, the spectrum of the output is computed, but in a more efficient and fast way with respect to an integration approach. Then, through an inverse fast Fourier transform (IFFT) it is possible to get the system output response back to the time domain.

From a numerical point of view, directly computing the spectrum of a numerical generated Gaussian white noise signal may lead to a spectrum which is far distant from the ideal flat one. For this reason, the input spectrum, used in each Monte Carlo simulation to compute the response of the system, has been obtained through an average of 100 Gaussian white noise spectra numerically generated. In this way, a flat power spectrum is obtained for the white noise input, closer to the theoretical one.

The sample frequency has been chosen equal to 20 Hz to have a Nyquist frequency of 10 Hz, which is ten times larger than the natural frequency of the system (1 Hz). Moreover, in order to obtain a spectrum

Table 1
Parameters of 1 dof system related to the nominal condition.

1 dof system	
Nominal conditions	
Mass (m)	10 [kg]
Stiffness (k)	394.78 [N/m]
Damping ratio (h)	0.008 []
Natural Frequency	1 [Hz]
Input	
Nominal condition	
Type	Gaussian white noise
RMS Amplitude (σ_u)	10 [N]
Output	
Time histories	
Type	Acceleration
Sample frequency	20 [Hz]
Duration	100 [s]

frequency resolution of 1/100 Hz, the duration of each time history is chosen as 100 s. All these assumptions for the reference condition of the system are summarised in Table 1.

To determine the FRF of the system, the damping ratio is chosen as input variable in place of the damping nominal value. In this way, once the natural frequency of the system is fixed, it is possible to vary easily the shape of the FRF. However, it was decided to maintain the distinction between mass and stiffness because these two parameters could vary independently in a real case application.

4.2. Generation of databases

The Mahalanobis Squared Distance [24] gives information about the distance of a new unknown condition from a reference one. For that reason, it is needed to have a database - from now on it will be called *DATAREF* - which includes all the time histories related to the reference condition. In this work, it has been decided to allow a limited spread to the mechanical parameters, still considered as a nominal condition. This assumption is related to the fact that in a real case, due to the operational and environmental conditions, these parameters could have small variations, even if the structure is still in the nominal condition. In this paper, it is assumed, that a ±3% variation could be inside the nominal condition and it is due to the intrinsic variability of the physical parameters of the system. The abnormal conditions for all the parameters are fixed in the range of ±30% respect the nominal condition of the system. In other words, to simulate the possibility of an abnormal condition of the system, all the parameters are generated sampling from a uniform distribution whose extremes are fixed at ±30% with respect to their nominal values. Since the damage is unknown, we cannot assume any specific distribution type. We can just give the same distribution to all the parameters (uniform distribution with 30% of variation) to simulate a damage that can equally affect mass, stiffness or damping. The arbitrary maximum damage considered in this work is a change of the dynamic parameters of 30%. This database will be called, since now, *DATAWHAT*.

For both the databases - *DATAWHAT* and *DATAREF* - the amplitude of the white noise input varies around ±50%. This choice is made to emphasise the fact that, even if the standard deviation of the Gaussian white noise may vary a lot due to external factors (for example think about environmental excitation of civil structures), this will not change the sensitivity of the Mahalanobis Squared Distance used as SHM feature. This is true as long as the input maintains the peculiarity of the Gaussian white noise.

Concerning the database, *DATAREF* is the reference database used to calculate the covariance matrix and the average vector of the AR parameters, which are needed to perform the Mahalanobis Squared Distance (cfr. Eq. (6)). *DATAREF* is made by 10^4 MC simulations, which randomly extract the input parameters (mass, stiffness, damping ratio

Table 2
Input variation for DATAREF.

DATAREF **Reference conditions**	
Database dimension	10^4
Mass (m)	U (10 [kg] ± 3%)
Stiffness (k)	U (394.78 [N/m] ± 3%)
Damping ratio (h)	U (0.008 [] ± 3%)
RMS Force Amplitude (σ_u)	U (10 [N] ± 50%)

Table 3
Input variation for DATAWHAT.

DATAWHAT **Unknown conditions**	
Database dimension	10^6
Mass (m)	U (10 [kg] ± 30%)
Stiffness (k)	U (394.78 [N/m] ± 30%)
Damping ratio (h)	U (0.008 [] ± 30%)
Fore Amplitude (σ_u)	U (10 [N] ±50%)

and force amplitude) from the uniform probability distributions in the ranges described by Table 2.

DATAWHAT is made up by 10^6 time histories generated using the Monte Carlo approach from the uniform input distributions reported in Table 3.

From each combination of the input variables it is possible to extract the dynamic response function as describe in 4.1. For all the considered time histories, it is finally computed the AR model by means of a least square approach. The order of the model is estimated by the BIC method (Section 2.2) and it is fixed at 10. Fig. 1 sums up the generation of the two datasets and their use for the GSA analysis.

5. Results and discussion

Since the Mahalanobis Squared Distance is a function of AR parameters (cfr. Eq. (4)), it is reasonable to perform the GSA and UP analyses on these parameters at first and then on this damage feature in order to complete the description of the statistical behaviour of the process. Generally, this approach could be applied to any damage feature in order to quantify its sensitivity to several changes of a system. Section 5.1 reports the results obtained by the AR parameters samples, generated by a Monte Carlo simulation, applied to the 1 dof system de- scribed in Section 4.1. The results from the samples of the Mahalanobis Squared Distance –estimated by means of the AR parameters samples –are shown in Section 5.2. Intuitively, it is reasonable to expect that the two sets of analyses will give the same interpretation of the statistical behaviour of the mechanical system, especially in terms of links between the variation of the mechanical parameters and the distribu- tion of the autoregressive quantities (AR parameters and Mahalanobis Squared Distance).

5.1. Autoregressive parameters

The following two sections (Sections 5.1.1 and 5.1.2) will sum up the main results obtained by performing the GSA analysis and the UP analysis on the AR parameters, calculated from the samples of the data set *DATAWHAT*. GSA is necessary to understand which mechanical parameters – mass, stiffness, damping ratio and force magnitude – have more influence on the variability of the AR parameter populations but, above all, how much is their contribution to define this variability. When this first screening on the data is performed, the uncertainty propagation analysis is used to quantify the uncertainty of the outputs – in this case the AR parameters of the autoregressive model - which are directly connected to the reliability of the Mahalanobis Squared Distance used as damage feature.

5.1.1. Global Sensitivity Analysis results

The numerical values of Table 4 show the results obtained by GSA for the first order sensitivity index, the second order sensitivity index and the total sensitivity index, estimated as defined in Section 3.2. The error estimation of these indexes has given values lower than 5% with a confidence level of 68%. Recalling the system of Section 4.1, the sensitivity indexes are estimated on the structural response to a white noise input force without adding uncorrelated noise to the output signal. The results of the first order index show that the AR parameters ϕ_i from 1 to 9 (over 10) are highly affected by the variation of mass and stiffness. The damping ratio has a negligible effect on the variability of the AR parameters probably because it was decided to focus attention only on lightly damped systems. It is reasonable to expect a larger contribution of this parameter for highly damped systems. Finally, the magnitude of the force used to excite the system has no effect on the variability of the AR parameters, as expected. Indeed, the autoregressive model is time discrete approximation of the response of the system and its representation projected in the Z domain is an all poles function [13]. This confirmation can be used as a clue of the goodness of the analysis.

Although the BIC method (Section 4.2) suggested to use 10 AR parameters to model the signal, the last two parameters ϕ_9 and ϕ_{10} seem to be overfitting factors since their sensitivity to changes of the mechan- ical properties of the structure is low. Even if the order of the model is probably too high and the proper one should be 8, the 10th order model is maintained because overfitting problems are common in real applications and it could be useful how to deal with them and their contribution to the damage feature definition.

The second order index between mass and stiffness could give more information about the relationship among these two mechanical parameters and the autoregressive model. However, its value is almost negligible with respect to the first order index. This means that the variation of he joint effect of mass and stiffness is less significant than the direct effect of the variation of mass or stiffness separately. This final observation opens to the inference about the relation between these two mechanical parameters (m and k) and the autoregressive parameters Φ_i. Since the first order indexes are the most significant ones, the conclusion is that the relationship between m and k on one side and the output Φ_i is mainly additive [25], i.e. $\phi_i \sim (m + k)$.
Regarding the total sensitivity index, the results show that every AR parameter ϕ_i owes its dispersion to the direct effect of the single mechanical parameter variation (m and k), which is represented by the first order index, whereas the higher order effects are almost negligible, as seen for the second order index. Indeed, the total sensitivity index quantifies the effect of all the terms of any order which link the mechanical parameters (m and k) to the AR parameter ϕ_i. In this case, the first- order index and the total sensitivity index have almost the same value. It must be noticed that the total indexes of ϕ_9 and ϕ_{10} are not reported in Table 4 since the analysis gives a consistent level of uncertainty for the estimation of these values. This is probably due to the weak relation these two AR parameters have with the mechanical system because they are basically overfitting parameters.

Looking their total sensitivity indexes, it is possible to affirm the damping ratio and the force amplitude have a negligible effect on the autoregressive parameters.

Since the variability of the AR parameters depending on the variation of the mechanical quantities is almost entirely described by the first-order index, it is possible to focus on these indexes and add some analyses about the effect of uncorrelated noise. The two plots in Fig. 2 show the effect of uncorrelated noise on the link between the mechanical parameters (m and k) and the AR parameters ϕ_i. Adding noise to the system response signals reduces the amount of variability of the AR parameters ϕ_i which can be directly correlated to the variation of the mechanical parameters. This because the AR model is fitting both the mechanical response and the added noise. As can be seen by the plots, the effect of noise is the same for the first sensitivity index of mass and stiffness. Not all the AR parameters ϕ_i are affected by noise in the same

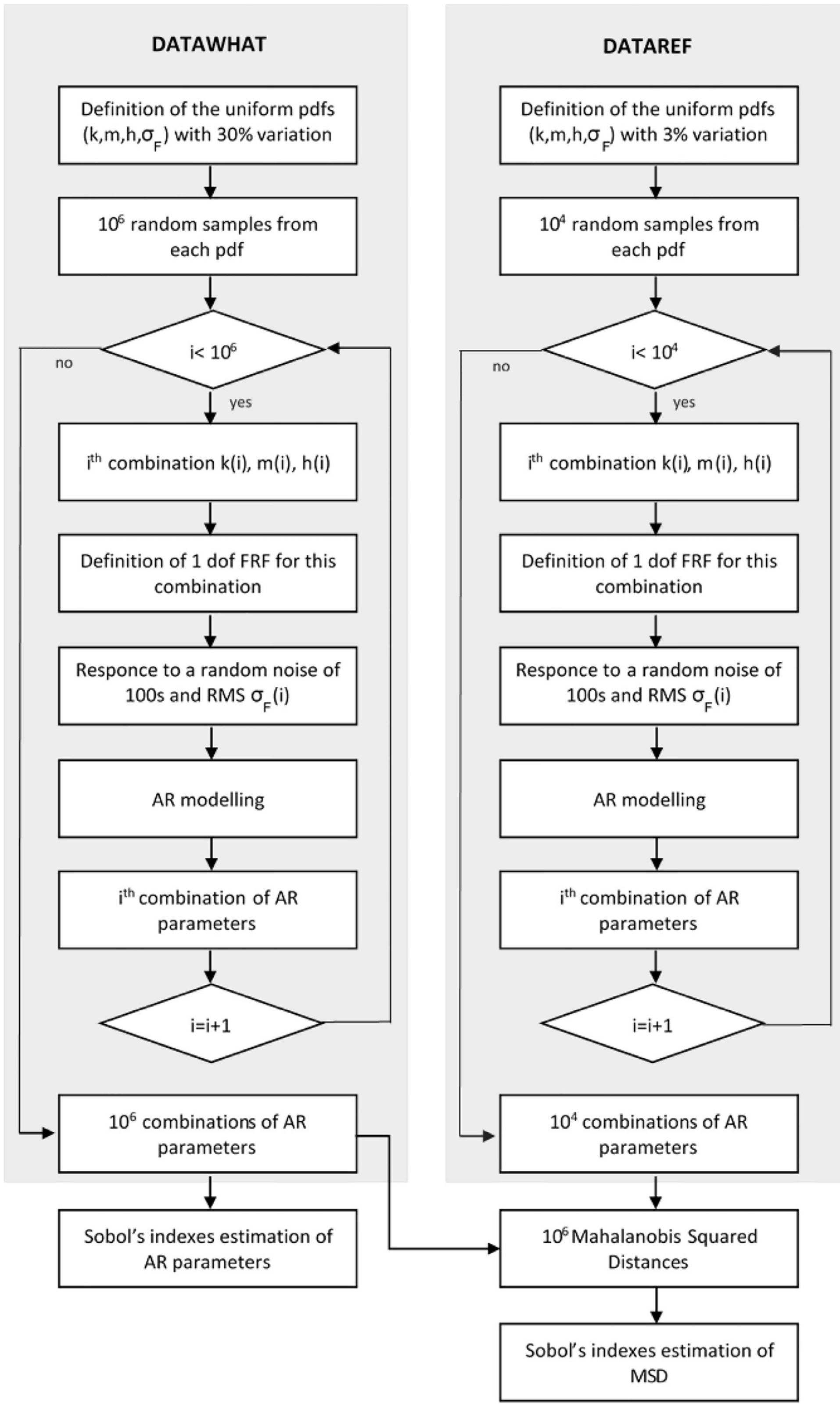

Fig. 1. Data generation and analysis flow chart.

way. The parameters from ϕ_3 to ϕ_6 seem to be more robust to the negative presence of uncorrelated noise. In the worst case – SNR of 20 dB – the first order sensitivity index decreases from 0.5 to 0.4. This probably occurs because these are more correlated to the mechanical parameters of the system. Indeed, the AR parameters ϕ_9 and ϕ_{10} – which are overfitting parameters not really required to properly reconstruct the signal – are strongly affected by the presence of noise and their first-order sensitivity index goes to zero.

To complete the analysis, Table 5 shows the GSA performed on the same system with a nominal damping ratio of 5%, higher than the pre-

Table 4
GSA results of the autoregressive parameters modelling the response of the system without noise. S_m is the first order sensitivity index for the mass, S_k for the stiffness, S_h for damping ratio, and S_u for the amplitude of the force; S_mk is the second order sensitivity index for the joint effect of mass and stiffness; St_m is the total sensitivity index related to the mass, St_k to the stiffness, St_h to the damping ratio, and St_u to the amplitude of the force.

	I Order					II Order		Total Sens. Index			
	S_m	S_k	S_h	S_u		S_mk		St_m	St_k	St_h	St_u
phi1	0.492	0.460	0.0054	0.000	**phi1**	0.023	**phi1**	0.532	0.499	0.026	0.017
phi2	0.502	0.472	0.000	0.000	**phi2**	0.020	**phi2**	0.527	0.497	0.007	0.005
phi3	0.503	0.473	0.000	0.000	**phi3**	0.019	**phi3**	0.526	0.497	0.005	0.004
phi4	0.503	0.474	0.000	0.000	**phi4**	0.018	**phi4**	0.526	0.496	0.004	0.004
phi5	0.502	0.474	0.000	0.000	**phi5**	0.018	**phi5**	0.526	0.498	0.006	0.006
phi6	0.502	0.474	0.000	0.000	**phi6**	0.018	**phi6**	0.526	0.498	0.007	0.007
phi7	0.496	0.471	0.000	0.000	**phi7**	0.017	**phi7**	0.529	0.504	0.016	0.015
phi8	0.485	0.461	0.000	0.003	**phi8**	0.007	**phi8**	0.539	0.517	0.045	0.043
phi9	0.406	0.400	0.004	0.001	**phi9**	0.014	**phi9**	n.a	n.a	n.a	n.a
phi10	0.117	0.129	0.087	0.001	**phi10**	0.008	**phi10**	n.a	n.a	n.a	n.a

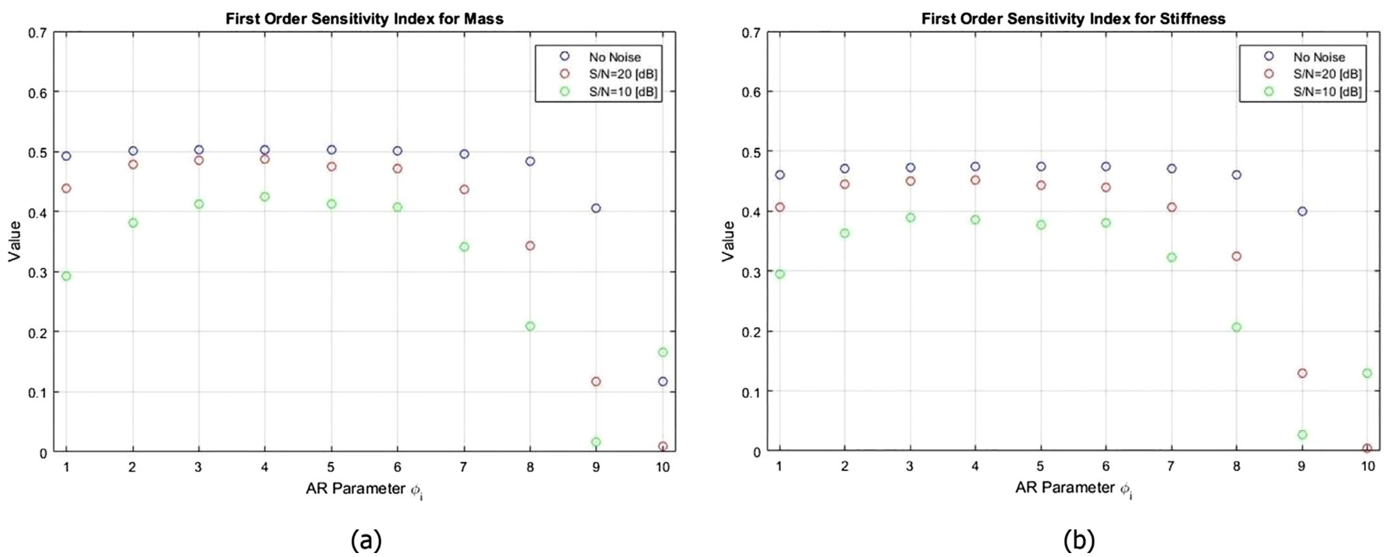

Fig. 2. First order sensitivity indexes of the autoregressive parameters modelling the response of the system adding noise: blue points are data without noise, red points are data with 20 dB SNR and green points are data with 10 dB SNR: (a) first order sensitivity index for mass; (b) first order sensitivity index for stiffness. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 5
GSA results of the autoregressive parameters modelling the response of the system without noise and 5% damping ratio.

	I Order					Total Sens.Index			
	S_m	S_k	S_h	S_u		St_m	St_k	St_h	St_u
phi1	0.377	0.331	0.235	0.003	**phi1**	0.430	0.385	0.256	0.018
phi2	0.482	0.437	0.046	0.003	**phi2**	0.515	0.471	0.055	0.006
phi3	0.498	0.456	0.016	0.002	**phi3**	0.525	0.483	0.023	0.004
phi4	0.503	0.464	0.006	0.002	**phi4**	0.528	0.488	0.013	0.004
phi5	0.505	0.467	0.004	0.001	**phi5**	0.527	0.490	0.011	0.006
phi6	0.505	0.471	0.001	0.002	**phi6**	0.527	0.492	0.009	0.007
phi7	0.502	0.473	0.001	0.001	**phi7**	0.525	0.497	0.013	0.012
phi8	0.492	0.474	0.000	0.002	**phi8**	0.525	0.508	0.030	0.028
phi9	0.447	0.449	0.035	0.000	**phi9**	0.514	0.517	0.103	0.065
phi10	0.317	0.341	0.224	0.001	**phi10**	0.429	0.453	0.328	0.096

viously adopted value of 0.8%. In this case the signals are processed without adding uncorrelated noise. The error estimation of these indexes has given values lower than 5% with a confidence level of 68% as like in Table 4. The values are basically the same as those reported in Table 4 with the only exception of the parameter ϕ_1, ϕ_9 and ϕ_{10}. These parameters show a dependency on damping ratio as proved by their sensitivity indexes, which means that the information about damping ratio is carried by fewer parameters with respect to those needed for the identification of the natural frequency. It is also worth observing that the parameters sensitive to damping ratio are those less sensitive to frequency, as shown in Fig. 2. This proves the low sensitivity of autoregressive models to any change in damping ratio because only a few parameters are necessary to carry this information even if the nominal damping ratio is higher. So, the conclusions about the sensitivity indexes of Table 4 were wrong. The effect of the change of damping ratio on the autoregressive model is always weak, not only in the case of lightly damped systems. It must be noticed that this outcome does not mean that the AR model cannot properly estimate the damping ratio

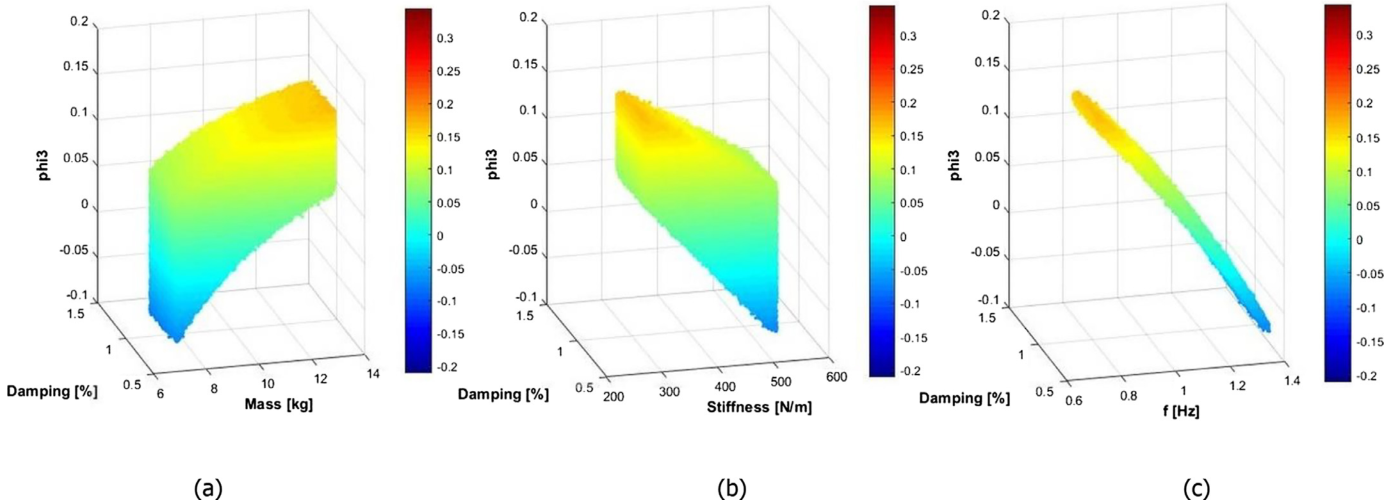

Fig. 3. Scatterplot of the autoregressive parameter φ_3 as function of: (a) damping ratio and mass; (b) damping ratio and stiffness; (c) damping ratio and frequency.

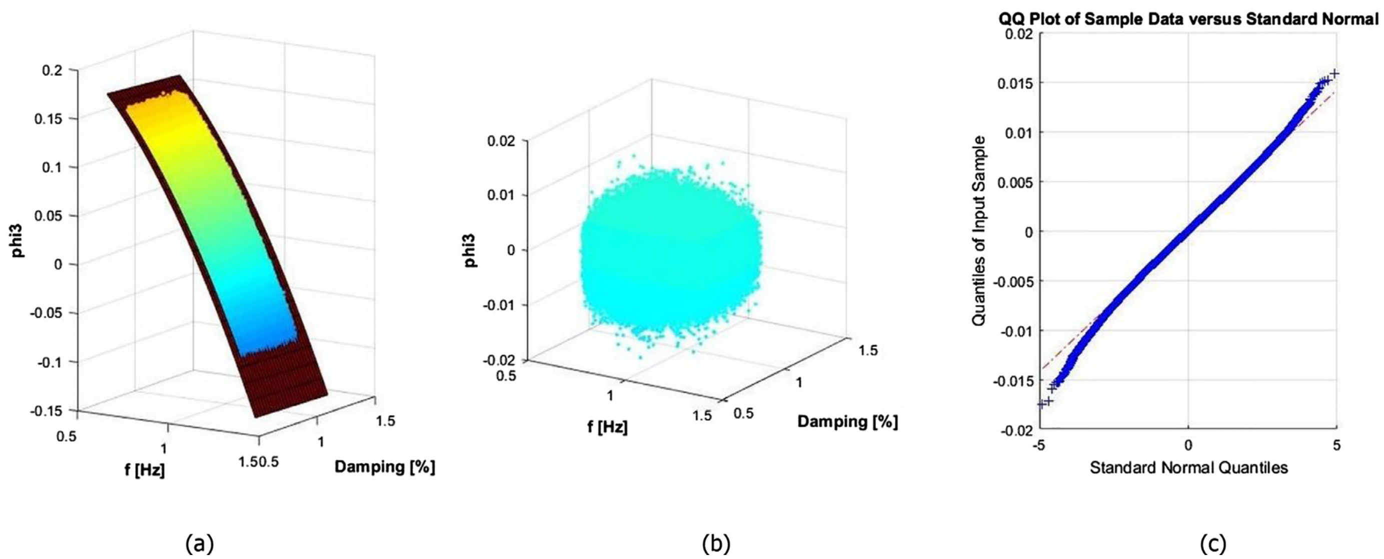

Fig. 4. Surface fitting of the autoregressive parameter φ_3 scatterplot as function of damping ratio and frequency: (a) fitting surface and scatterplot; (b) residues between scatterplot and fitting surface; (c) normality test of the residues.

but only that its influence on the whole model is weak. Moreover, it is reasonable to expect also the Mahalanobis Squared Distance to be less sensitive to damping ratio than frequency.

5.1.2. Uncertainty propagation results

The uncertainty propagation from the mechanical parameters to the AR parameters ϕ_i is crucial to understand the robustness of the autoregressive model fitting the signal. Moreover, the reliability of the chosen damage feature (which in our case is a function of the AR parameters) must be assessed. Indeed, their uncertainties are strictly linked together.

Fig. 3 shows the scatterplot of parameter ϕ_3 obtained by the system response signal without uncorrelated noise added to the output. The first picture from the left is the scatterplot of the chosen parameter as a function of *mass* and the second one as a function of *stiffness*, in the end, the dependency from frequency and damping ratio together is given in Fig. 3 (c). The dependency of ϕ_3 upon *mass* alone (Fig. 2 (a)) and *stiffness* alone (Fig. 2 (b)) is mainly linear but also the dependency from the frequency, i.e. the ratio of mass and stiffness, is almost linear. The effect of damping ratio is negligible as expected for lightly damped systems: varying change in damping ratio the value of ϕ_3 remains almost constant (Figs. 2 (c) and 3 (a) giving two different points of view). The shape of all the scatterplots is due to the important variation of the mechanical parameters m and k, so the surface is representative of the link between the AR parameter and these two physical quantities. The quantification of this link is possible using a fitting surface which is reported in Fig. 4 (a). The equation of the surface has no physical meaning, be- ing just an interpolating function. However, it is significant understand how the residues between the surface and the scatterplot points are distributed.

Fig. 4(b) shows the residues cloud and Fig. 4(c) its normality test results (Quantile-Quantile plot). The two charts prove the goodness of the interpolation since the residues are normally distributed around the fitting surface and they are homoscedastic. They can be used to quantify the uncertainty of the AR model estimation only related to the minimisation algorithm. Reiterating the same procedure for all the parameters ϕ_i, which have the same distribution but different values, it is possible to quantify the uncertainty estimation of the AR parameters ϕ_i in terms of $\pm 2\sigma$, where σ is the standard deviation of the residues around the fitting surface (cfr. Fig. 4 (b)). Fig. 5 shows for each parameter ϕ_i the uncertainty interval corresponding to the 95% of the residues around the mean value (blue intervals) and the variation of the same parameters changing the mechanical properties of the structure (red intervals).

Table 6
Percentage ratio between the uncertainty and the variation of the AR parameters due to the mechanical changes in the reference structure. Values are referred to simulations without noise, with SNR=20 dB and with SNR=10 dB.

	phi1	phi2	phi3	phi4	phi5	phi6	phi7	phi8	phi9	phi10
No noise	9.5	5.6	4.8	4.9	5.9	6.5	9.7	16.2	35.4	96.0
S/N=20 [dB]	30.3	19.2	16.6	16.7	19.9	21.8	32.0	52.6	126.9	233.2
S/N=10 [dB]	63.6	44.8	36.1	34.3	37.4	38.7	53.3	88.9	238.5	83.9

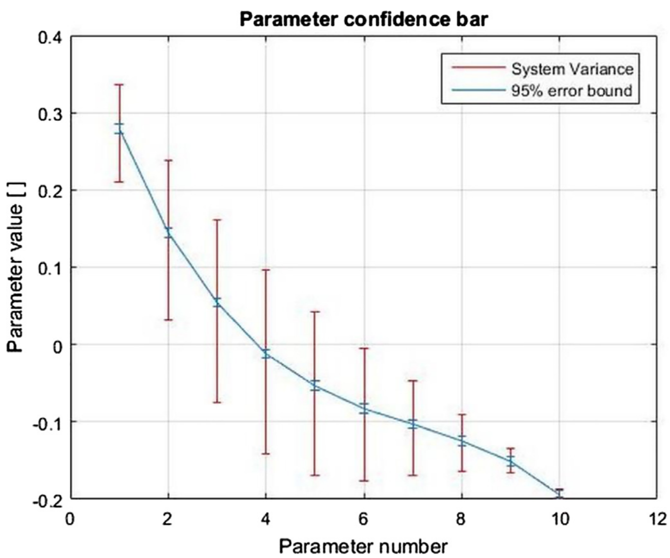

Fig. 5. Bands corresponding to the 95% percentile of the residues of each AR parameter (blue intervals) compared to the bands corresponding to the AR parameter variations due to the change of the mechanical parameters of the system (red intervals). In this case, no noise is added to the system response. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

It is easily seen that the ratio between the two bands is different for each parameter ϕ_i.

If the same AR model is built on the same response signals but adding uncorrelated noise, it is possible to see the effect in Fig. 6. The two charts have been obtained with a 20 dB SNR and a 10 dB SNR, respectively. As expected, the uncertainty estimation gets worse as the uncorrelated noise level increases.

All these statements can be summarised and quantified by Table 6. For the case without noise, the percentage ratio between uncertainty of the AR parameters ϕ_i and their change due to the system mechanical properties is in the range between 5 and 20%, except for the last two parameters ϕ_9 and ϕ_{10}. Since their correlation to the physical quantities of the structure is poor, the variation due to the change of these properties is comparable to the uncertainty estimation. For these two cases, the percentage uncertainty grows up to 35% and 96%, respectively.

When uncorrelated noise is added to the signal, the values reported in Table 6 confirm the same behaviour of Fig. 2. Increasing the noise, the relative uncertainty of parameters from ϕ_3 to ϕ_6 are almost the same – 15/20% for 20 dB SNR and 35/40% for 10 dB SNR – and theirs values are lower than the other AR parameters, where the percentage uncer- tainty exceeds 50%. The results seem to support the idea that the AR parameters are not equally depending on the physical quantities of the system. The SNR has a negative effect that could mask the variation of the AR parameters due to a physical change of the system.

5.2. Mahalanobis Squared Distance

The Mahalanobis Squared Distance is a feature that estimates the distance between a multivariate sample and a multivariate population. *DATAWHAT*, used in the previous Section 5.1 will be the group of un- known scenarios since the variability of the mechanical parameters is significant in this population. The healthy picture of the system will be provided by *DATAREF:* the database made by the AR parameters ob- tained by fitting the response functions of the simulated system when the mechanical parameters have a uniform distribution, as reported in Table 2. This means that the AR parameters corresponding to a mechan- ical system having mass and/or stiffness and/or damping ratio larger than three times the standard deviation of the corresponding nominal values, should be considered a damaged scenario.

5.2.1. GSA results

The first row of Table 7 reports the results of GSA when uncorrelated noise is added to the signal. The error estimation of these indexes and the other tables reported in this section has given values lower than 5% with a confidence level of 68%. Force amplitude and damping ratio seem to have a negligible effect on the output, while the values of the first order indexes state that mass and stiffness are the two main param- eters having a direct effect of the Mahalanobis distribution. However, their values are lower than 0.5; in fact, they are 0.23 and 0.14 respec- tively. This states that the distribution of MSD values not only depends on mass and stiffness but also on the joined effect of the two parame- ters together, as it can be seen by the second order index, which is close to 0.60. Since the second order index is large, the relationship between the Mahalanobis Squared Distance and the mechanical frequency of the system is expected to be multiplicative, as it will be shown in the next section. The second and the third row of Table 7 are the picture of what happens when uncorrelated noise is added to the signal. When the signal to noise ratio is 20 dB or 10 dB the relationship between the damage feature and the mechanical system parameters maintains its characteristics, especially for the second order index which is the most important. This means that even if there is a lot of noise in the signal, the Mahalanobis Squared Distance keeps its sensitivity to damage.

As final statement, if the nominal damping ratio is set to 5%, the con- clusions drawn at the end of Section 5.1.1 are confirmed, see Table 8. Even if the nominal damping ratio is high, the Mahalanobis Squared Distance sensitivity to its change is low. The values of Table 8 show the dependence on damping ratio is inconsistent. Since the Mahalanobis Squared Distance is defined by Eq. (4) and only few autoregressive pa- rameters are sensitive to damping ratio, the final contribution of damping ratio to the Mahalanobis variation is negligible. Basically, it means this feature is suitable for detection of damages related to a change of natural frequencies rather than a change of damping ratio.

5.2.2. Uncertainty propagation results

Using data from *DATAWHAT* and *DATAREF,* it is possible to estimate the distribution of the Mahalanobis samples as the distance of each vector $\{\hat{\phi}\}$ - belonging to *DATAWHAT* - from all the vectors of *DATAREF*. The scatterplot of the damage feature shows the distribution of the samples as a function of the mechanical parameters, for instance the natural frequency and the damping ratio as it can be seen in Fig. 7. If the Maha-

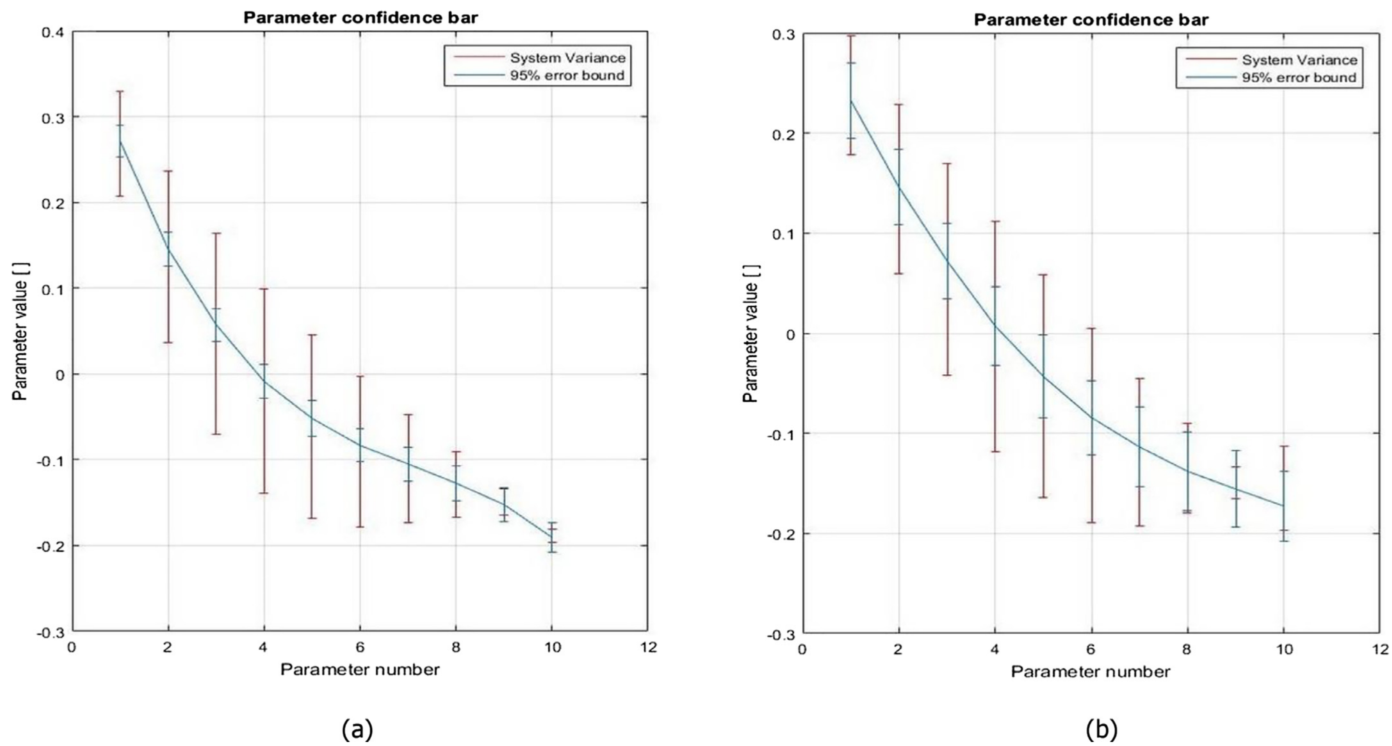

Fig. 6. Bands corresponding to the 95% percentile of the residues of each AR parameter (blue intervals) compared to the bands corresponding to the AR parameter variations due to the change of the mechanical parameters of the system (red intervals). (a) System response with a 20 dB SNR; (b) System response with a 10 dB SNR. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 7
GSA results of the Mahalanobis Squared Distance estimated by the autoregressive parameters modelling the response of the system without noise, SNR=20 dB and SNR=10 dB.

	I Order				II Order	Total Sens.Index			
	S_m	S_k	S_h	S_u	S_mk	St_m	St_k	St_h	St_u
No Noise	0.23	0.14	0.01	0.00	0.61	0.85	0.76	0.02	0.00
S/N=20 [dB]	0.24	0.14	0.01	0.00	0.61	0.85	0.76	0.02	0.01
S/N=10 [dB]	0.25	0.14	0.01	0.00	0.59	0.86	0.75	0.01	0.01

Table 8
GSA results of the Mahalanobis Squared Distance estimated by the autoregressive parameters modelling the response of the system without noise but higher damping ratio (5%).

	I Order				**II Order**	**Total Sens.Index**			
	S_m	**S_k**	**S_h**	**S_u**	**S_mk**	**St_m**	**St_k**	**St_h**	**St_u**
No Noise	0.19	0.12	0.06	0.001	0.58	0.81	0.74	0.10	0.005

lanobis estimation process were not affected by any type of uncertainty, the data should lay on a surface, in this case a paraboloid. Since the AR parameters ϕ_i are estimated through a minimisation algorithm, they are affected by uncertainty due to the tolerance of the solution used in the iteration, and the Mahalanobis Squared Distance is affected by uncertainty by itself. The samples are distributed around a fitting surface and the thickness of the cloud made by these points can be considered the uncertainty estimation of the Mahalanobis Squared Distance. Fig. 7 shows the scatterplot and the two surfaces enclosing the 95% of the data: the distance between the two surfaces corresponds to a Mahalanobis variation of 36.8 units. The trend is parabolic as expected in [14].

When uncorrelated noise is added to the signal, it is obvious to expect the growing of the Mahalanobis uncertainty. Table 9 summarises the results obtained when the signal to noise ratio is 20 dB and 10 dB: in the worst case, the uncertainty bandwidth has a variance about 70 of the Mahalanobis interpolating surface.

If the threshold to discriminate the undamaged and the damaged scenarios is estimated by using the extreme value statistics as proposed by Worden et al. [65], the classification of the scenarios are those reported in Fig. 8. In this case, the discriminating value between damaged and undamaged scenarios is estimated as the p percentile (p to be chosen) of the distribution of the maxima.

Table 9
Bands corresponding to the 95% percentile of the residues for the Mahalanobis Squared Distance.

	Two standard deviation of the Mahalanobis Squared Distance
	95% uncertainty band
No Noise	**36.8**
S/N=20 [dB]	**43.9 (36.8+20%)**
S/N=10 [dB]	**67.9 (36.8+80%)**

Previously, it has been seen that the variation of Mahalanobis Squared Distance is most correlated to the change of frequency, the damping ratio of the system has a negligible effect on it. Therefore, the

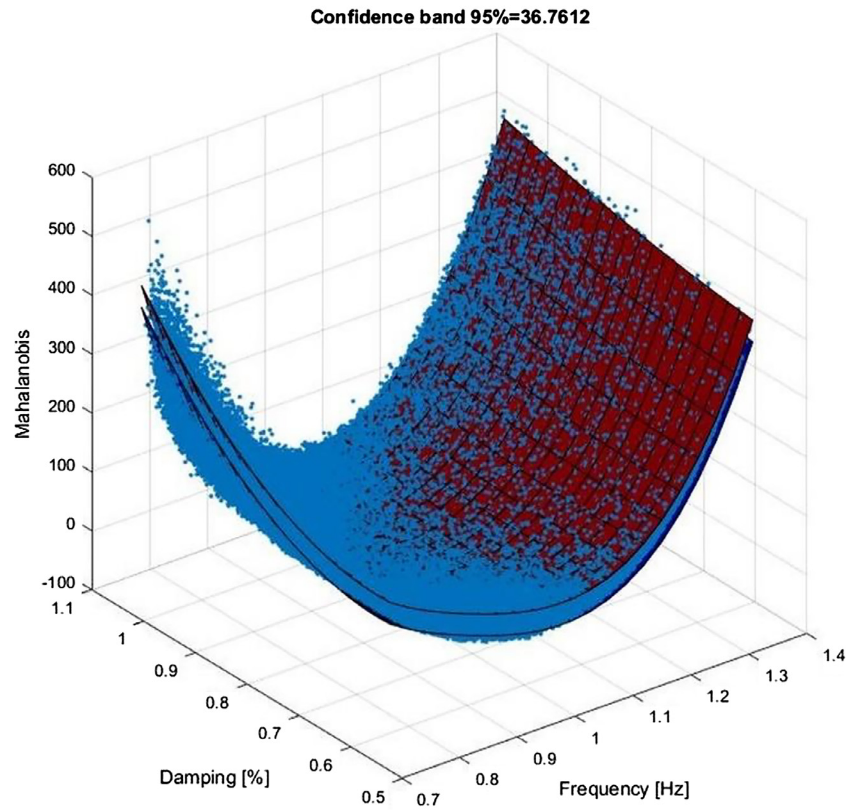

Fig. 7. Surface fitting of the Mahalanobis Squared Distance scatterplot as function of damping ratio and frequency: the two red surfaces correspond to the constant uncertainty bands of the DATAWHAT around the fitting surface. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

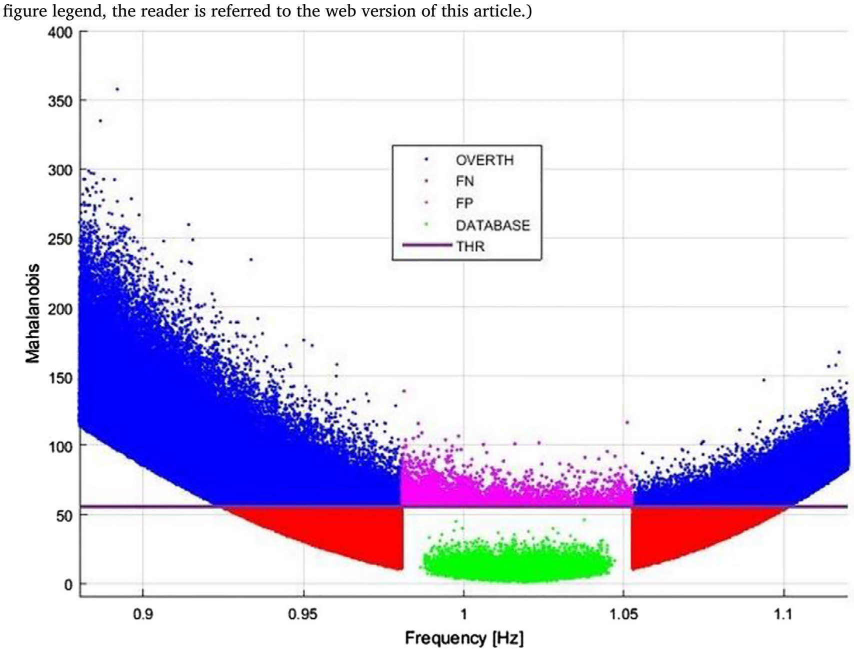

Fig. 8. Discriminant threshold as 1% outliers of the maxima distribution. The data refer to the case without adding uncorrelated noise to the signal.

authors focus their attention to frequency to describe the behaviour of the Mahalanobis Squared Distance.

In Fig. 8 the blue points are the data beyond the threshold - which are detected as damage scenarios - and the threshold is the horizontal violet line. The green points are the data denoting the reference condition, i.e. the scenarios belonging to *DATAREF*. Then the pink points are the false positives (I type error), which means the data recognised as damaged scenarios but their natural frequency belongs to the range of the reference *DATAREF*. The red points are the false negative (II type error) data, the scenarios recognised as undamaged but their natural frequency is out of the range described by *DATAREF*. Adding noise, it is reasonable to expect the raise of type I and II errors – pink and red points - as it will be show in Table 10. This illustrates the percentages of false positives and false negatives over the total amount of data belonging to *DATAWHAT* for the three conditions: no noise, 20 dB uncorrelated noise and 10 dB uncorrelated noise.

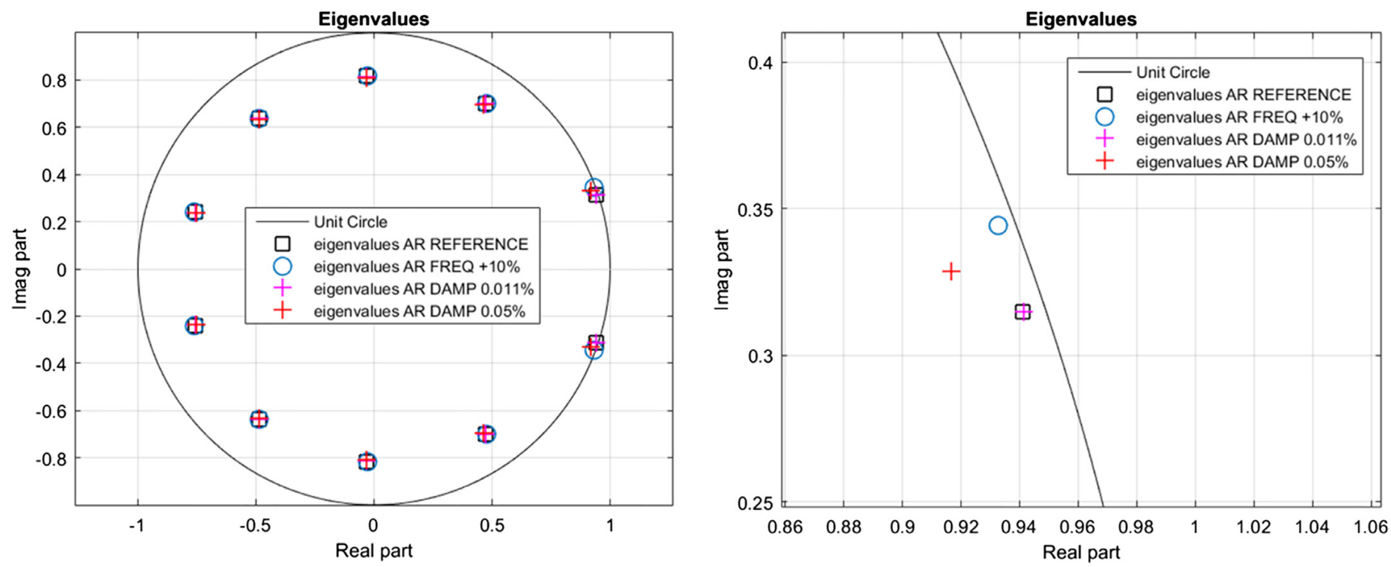

Fig. 9. Effects of structural changes on the FRF of the AR model: (a) frequency variation; (b) damping variation.

Table 10
Damage detection performances fixing the threshold by Mahalanobis tail percentage estimation.

	NO NOISE	20 dB NOISE	10 dB NOISE
POSITIVE	62.74%	62.97%	57.79%
NEGATIVE	37.26%	37.03%	42.21%
FALSE NEGATIVE	15.23%	14.94%	19.17%
FALSE POSITIVE	0.93%	0.86%	0.05%
TOTAL ERROR	**16.16%**	**15.80%**	**19.22%**

The results show the threshold estimation by using the maxima distribution of the Mahalanobis Squared Distance is very effective in reducing the I type error. Basically, since the threshold value is quite large, it is difficult to make this type of mistake. However, the drawback is the large number of false negatives. Since the Mahalanobis Squared Distance is a paraboloid as a function of the mechanical parameters of the system –at least for the 1 dof system- it is difficult to avoid the type II error using just a constant threshold. The false negatives are rather dangerous for structural health monitoring since this means that incipient damage conditions are not always detected.

It could also be noted that the performance of the Mahalanobis Squared Distance with data corrupted by 20 dB of uncorrelated noise is comparable to the case without noise. This confirms the robustness of the Mahalanobis Squared Distance as a damage feature thanks to its noise rejection property. The 10 dB noise condition indeed is an extreme case that should not happen in real practice when the measurement setup is properly settled down.

6. Comments to the results in a SHM prospective

The previous sections have reported an extended analysis of AR models in the SHM context on the base of a Global Sensitivity Analysis framework. Some results corroborate and enhance knowledges already reported in the literature, such as the good noise rejection property of the MSD or the independency of AR models from the excitation amplitude. Other results can be said unexpected and original.

When a signal representing the dynamic response of a system is modelled by a pure AR model, the AR parameters are not equally depending on the mechanical parameters of the structure. This means it could be possible to strengthen those parameters, which are more linked to the physic of the system, to improve a damage detection process. For instance, in the case of the MSD, it could be possible to explored the possibility to defined a Weighted Mahalanobis Squared Distance where different weights are given to the variables. Some papers have proposed the use of a Weighted Mahalanobis Distance (WMD) in the field of Gaussian classification but its application to SHM is rather uncommon as to the authors' knowledge [66–68].

Another important and unexpected outcome is that AR modelling is much more sensitive to the frequency of the system rather than its damping. To better understand this behaviour, Fig. 9 shows what happens to the eigen values of the AR model with respect to the nominal system in three cases: a frequency change of 10%, a damping ratio of 10% and a damping ratio of 500%. Fig. 9 (a) shows that the poles of the discrete systems are equally distributed in the complex plane as declared in [14].

Among all the poles, only the ones close to the physical poles of the system show a clear change according to the system they are representing, as shown in Fig. 9 (b). Varying the frequency around 10% the pole of the model is subjected to a strong variation. On the contrary, in order to see a variation of the same magnitude due to the damping ratio, it must vary about 500%. From a dynamic point of view, it could be said that the AR model is just representing what is already well-known in the dyamics theory: the poles of a mechanical system are more depending on frequency than damping ratio. The direct consequence is that AR modelling can be efficiently used to detect a damage related to a change of the frequency, a mass or stiffness loss for instance; however, if the structure is subjected to a significant change of its damping ratio, the model will signal this occurrence by a slight alteration.

The final important outcome of this work is the dependency of the AR model on the SNR of the recorded signal. This effect is much stronger as the AR parameter is less correlated to the mechanical property of the system. In practical situation, once the order of a model is fixed, the SNR ratio of the signal could vary from one acquisition to another. Supposing the nature of the forcing source would not change, i.e. the shape of the input spectrum remains stable, this means the AR model could experience a variation due to the operational conditions. Changing the level of the input excitation, the SNR in the signal might be influenced, thus also the AR model could manifest a variation which is indirectly due to the operational conditions. For instance, the AR model representing the response of a civil structure could show a variation from day to night accordingly with the impact of the traffic around the building.

7. Conclusion

This work has proved that the AR parameters depend in different ways on the dynamic parameter of the system itself. Most of the AR parameters depend on the natural frequency of the system and only few parameters on the damping ratio when its value is consistent. This means that the autoregressive model is more sensitive to a change of natural frequency since most of the AR parameters strongly depends on it. Since the Mahalanobis Squared Distance is a function of the AR parameters, also this damage feature shows more sensitivity to a change of natural frequency of the system than damping ratio. The damage which can be detected by the Mahalanobis Squared Distance more easily is the one can cause a frequency change. Moreover, adding uncorrelated noise to the signal it has been proved that the sensitivity of the AR parameters to a change of natural frequency of the system is worsened but the negative effect is not the same for all the parameters. Some of them maintain a strong relationship with the mechanical parameters of the system and this positive property seems to be the reason why the Mahalanobis Squared Distance has a good noise rejection. Indeed, the sensitivity analysis on this damage feature shows that the relationship between the damage feature and the parameters of the system do not change when uncorrelated noise is added to the signal.

The conclusions of this paper should be extended to the methods using AR parameters to predict the reliability of aging structures. A further work should be a wide analysis of the effects of the outcomes of this work on the uncertainty of remaining lifetime prediction.

Appendix A. Brief GSA analysis on a MDOF system

This appendix shows briefly the results obtained from a variance-based GSA applied to a 3 dof system (Fig. A.1). The system used to perform the analysis is fixed by its modal parameters accordingly to Tables A.1 and A.2.

The GSA simulations are made by varying the modal natural frequencies around 10% and the damping ratios around 50%. The database DATAWHAT is generated following the approach of Section 4. Modal residues are kept fixed since they are related to the zeros and the ampli-

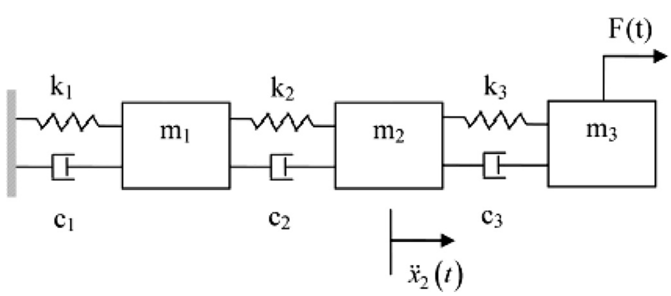

Fig. A.1. Simulated 3 dof system

Table A.1
Parameters of 3 dof system related to the nominal condition.

3 dof system	1st dof	2nd dof	3rd dof
mass [kg]	1	1	1
stiffness [N/m]	500	1000	5000
damping [Ns/m]	1.78	3.5	6.7

Table A.2
Modal parameters of 3 dof system related to the nominal condition.

Modal parameters	
1st Natural Frequency [Hz]	2.03
2nd Natural Frequency [Hz]	19.17
3rd Natural Frequency [Hz]	51.77
1st damping ratio	0.045
2nd damping ratio	0.045
3nd damping ratio	0.045

Table A.3
GSA results of the autoregressive parameters modelling the response of the 3dof system without noise.

	Omega 1	Omega 2	Omega 3	h 1	h 2	h 3
Phi 1	0.01	0.08	0.35	0.01	0.03	0.53
Phi 2	0.02	0.2	0.24	0.06	0.01	0.46
Phi 3	0.1	0.08	0.26	0.31	0.1	0.12
Phi 4	0.13	0.32	0.01	0.14	0.26	0.12
Phi 5	0.2	0.59	0	0.09	0.07	0.02
Phi 6	0.44	0.46	0.01	0.07	0	0.01
Phi 7	0.73	0	0.01	0.05	0.1	0.11
Phi 8	0.48	0.36	0.01	0.01	0.11	0.03
Phi 9	0.31	0.63	0	0	0.05	0.01
Phi 10	0.34	0.65	0	0	0	0
Phi 11	0.67	0.26	0	0	0.04	0
Phi 12	0.54	0.28	0	0.01	0.14	0.01
Phi 13	0.22	0.71	0	0.01	0.05	0
Phi 14	0.22	0.76	0	0	0.01	0
Phi 15	0.46	0.5	0	0.01	0.01	0
Phi 16	0.81	0.06	0	0	0.11	0.01
Phi 17	0.35	0.59	0	0	0.06	0
Phi 18	0.22	0.76	0	0	0.02	0
Phi 19	0.25	0.74	0	0	0	0
Phi 20	0.62	0.22	0	0	0.11	0.01
Phi 21	0.22	0.65	0	0	0.11	0.01
Phi 22	0.07	0.88	0	0	0.04	0
Phi 23	0.07	0.91	0	0	0.01	0
Phi 24	0.2	0.78	0	0	0	0
Phi 25	0.68	0.18	0	0.01	0.12	0
Phi 26	0.31	0.61	0	0.01	0.07	0
Phi 27	0.14	0.81	0	0.01	0.03	0
Phi 28	0.18	0.78	0	0.02	0	0
Phi 29	0.25	0.64	0	0.03	0.02	0
Phi 30	0.36	0.3	0	0.05	0.23	0

tude of the FRF of the system; the AR model is an only-pole function and is not influenced by these quantities. The AR model used to represent the response of the system in correspondence of mass 2 is made by 30 parameters. No noise is added to the signals.

The results confirm the analysis conducted on the 1dof model. The AR parameters do not depend on the modal parameters equally. The relationship is stronger with the natural frequencies and negligible with the damping. Among the natural frequencies, the dependency of the AR parameters is not equally distributed but a general rule cannot be found since this behaviour depends on the specific mechanical system, i.e. the number of natural frequencies and their value Table A.3.

Appendix B. PAWN results

This appendix briefly reports the results obtained by applying PAWN method [61]. The reader is asked to refer to [60] for a complete descrip- tion of the approach. The moment-invariant technique has been applied to the AR parameters of the model fitting the dynamic response of the

Table B.1
PAWN results of the AR parameters modelling the response of the system without noise.

	Kolmogorov-Smirnov statistic			
	Mass	**Stiffness**	**Damping**	**Force**
phi1	0.48	0.50	0.07	0.01
phi2	0.48	0.50	0.03	0.01
phi3	0.48	0.50	0.02	0.01
phi4	0.49	0.50	0.02	0.02
phi5	0.49	0.50	0.02	0.02
phi6	0.49	0.50	0.02	0.01
phi7	0.48	0.49	0.01	0.01
phi8	0.48	0.49	0.02	0.01
phi9	0.45	0.45	0.05	0.01
phi10	0.23	0.27	0.21	0.02

Table B.2
PAWN results of the autoregressive parameters modelling the response of the system with noise.

Kolmogorov-Smirnov statistic								
	20SNR				10SNR			
	Mass	Stiffness	Damping	Force	Mass	Stiffness	Damping	Force
phi1	0.48	0.49	0.04	0.02	0.38	0.42	0.07	0.03
phi2	0.50	0.50	0.04	0.02	0.45	0.46	0.03	0.02
phi3	0.50	0.50	0.04	0.02	0.47	0.47	0.05	0.02
phi4	0.51	0.49	0.03	0.02	0.47	0.47	0.05	0.02
phi5	0.49	0.50	0.03	0.02	0.46	0.47	0.04	0.02
phi6	0.50	0.50	0.02	0.02	0.46	0.46	0.02	0.02
phi7	0.50	0.49	0.02	0.02	0.43	0.42	0.04	0.02
phi8	0.45	0.45	0.04	0.02	0.33	0.34	0.06	0.02
phi9	0.26	0.29	0.06	0.02	0.12	0.15	0.08	0.03
phi10	0.12	0.08	0.09	0.02	0.34	0.25	0.05	0.02

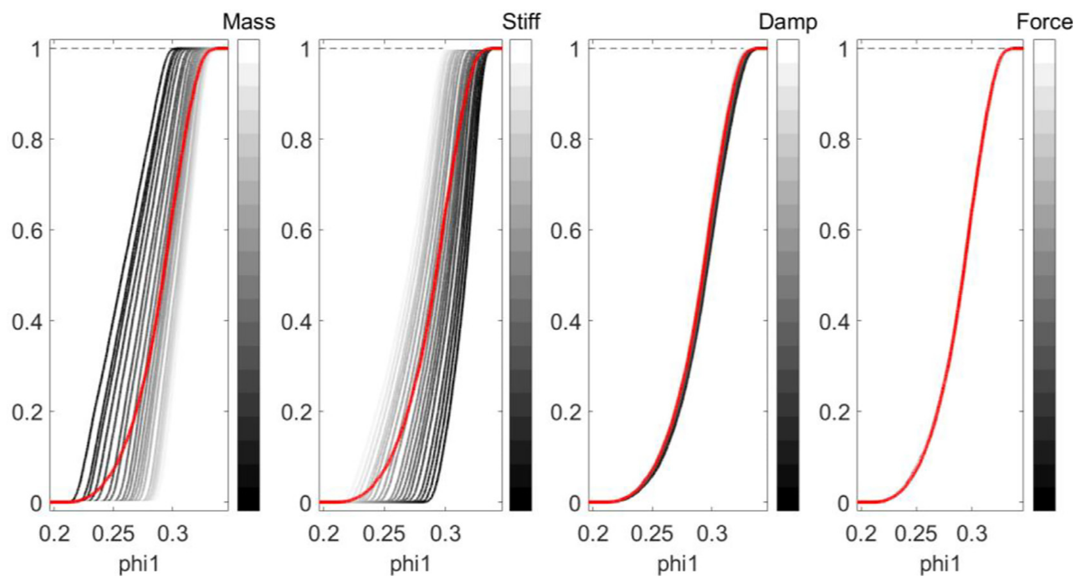

Fig. B.1. Unconditional CDF and the conditional CDFs of ϕ_1 with variation of the inputs accordingly to Table 3.

1dof system described in Table 1. The results should be compared with those reported in Tables 4, 5 and Fig. 2.

Pawn method is simple. The approach characterises the conditional and unconditional distributions by their Cumulative Distribution Functions (CDFs). The sensitivity is than measure by estimating the variations between the unconditional CDF and the conditional CDFs. As a measurement of the distance among the CDFs, the method applies the Kolmogorov-Smirnov statistic, i.e. the absolute value of the maximum distance between unconditional CDF and the conditional CDFs.

As an example, Fig. B.1 reports the comparison of the unconditional CDF and the conditional CDFs of the first AR parameter ϕ_1 obtained by the system described in Table 1 and letting the inputs varying accord- ing to Table 3. The number of simulations used is 10 ^{6}. As it can be seen from the plots, the red lines correspond to the unconditional CDF of ϕ_1, whereas the black lines are the conditional CDFs obtained by fixing the input variables at different values of their range. From the plots, it is
clear that mass and stiffness have a strong effect on the probability dis- tribution, whereas damping and force amplitude basically do not change anything.

If this analysis is repeated for each AR parameter and the Kolmogorov-Smirnov statistic is estimated, it is possible to obtained moment-independent sensitivity indexes that can be compared with those reported in Tables 4, 5 and Fig. 2.

Table B.1 reports the results of moment-independent GSA applied on the AR parameters modelling the dynamic response without adding uncorrelated noise. Pawn's sensitivity indexes show the same conclusion obtained from the variance-based GSA. Mass and stiffness have a strong effect on all the autoregressive model, whereas damping and force am- plitude are negligible.

Table B.2 reports the results obtained when uncorrelated noise is added to the response of the mechanical system. In this case, the correspondence with the results reported in Fig. 2 is less strong, especially when the level of noise is low. However, it is possible to see the same trend. When uncorrelated noise is added to the signal, the sensitivity of some indexes decreases. For instance, when SNR is 20 dB, the last three AR parameters ϕ_8, ϕ_9 and ϕ_{10} show lower sensitivity values for mass and stiffness. When SNR is 10 dB the correspondence with Fig. 2 is clearer. All the AR parameters show lower sensitivity indexes, especially the first two AR parameters ϕ_1, ϕ_2 and the last three ϕ_8, ϕ_9 and ϕ_{10}.

Finally, Table B.3 shows the results when the nominal value of damping is higher. Also in this case it is possible to recognise a correspondence with the results obtained by a variance-based GSA. The sensitivity of the AR parameters to damping in general is still low. Only ϕ_1, ϕ_2, ϕ_9 and ϕ_{10} shows an increased dependency on damping, especially the first and the last one.

Table B.3
PAWN results of the AR parameters modelling the response of the highly damped system without noise.

	Kolmogorov-Smirnov statistic			
	Mass	Stiffness	Damping	Force
phi1	0.45	0.39	0.38	0.02
phi2	0.48	0.48	0.16	0.01
phi3	0.48	0.49	0.08	0.02
phi4	0.48	0.48	0.07	0.01
phi5	0.48	0.49	0.05	0.02
phi6	0.48	0.48	0.04	0.02
phi7	0.48	0.48	0.02	0.02
phi8	0.48	0.48	0.04	0.01
phi9	0.47	0.46	0.13	0.01
phi10	0.25	0.24	0.33	0.02

References

[1] Yang L, Zhang J, Guo Y, Wang P. A bayesian-based reliability estimation approach for corrosion fatigue crack growth utilizing the random walk. Qual Reliab Eng Int 2016;32:2519–35.
[2] Sankararaman S, Daigle MJ, Goebel K. Uncertainty quantification in remain- ing useful life prediction using first-order reliability methods. IEEE Trans Reliab 2014;63:603–19.
[3] Siegel D, Ly C, Lee J. Methodology and framework for predicting helicopter rolling element bearing failure. IEEE Trans Reliab 2012;61:846–57.
[4] Ceravolo R, Pescatore M, De Stefano A. Symptom-based reliability and generalized repairing cost in monitored bridges. Reliab Eng Syst Saf 2009;94:1331–9.
[5] Okasha NM, Frangopol DM, Orcesi AD. Automated finite element updating using strain data for the lifetime reliability assessment of bridges. Reliab Eng Syst Saf 2012;99:139–50.
[6] Rabiei M, Modarres M. A recursive Bayesian framework for structural health man- agement using online monitoring and periodic inspections. Reliab Eng Syst Saf 2013;112:154–64.
[7] Farrar CR, Worden K. An introduction to structural health monitoring. New Trends Vib Based Struct Heal Monit 2010:1–17.
[8] Farrar CR, Worden K. Structural health monitoring: a machine learning perspective. Struct Heal Monit A Mach Learn Perspect. 2013. doi:10.1002/9781118443118.
[9] Sohn H, Fugate ML, Farrar CR, Alamos L. Damage diagnosis using statistical process control. Conf. Recent Adv. Struct. Dyn. 2000.
[10] Peter Carden E, Brownjohn JMW. ARMA modelled time-series classification for structural health monitoring of civil infrastructure. Mech Syst Signal Process 2008;22:295–314.
[11] Sohn H, R Farrar C. Damage diagnosis using time series analysis of vibration signals. Smart Mater. Struct 1995.
[12] Sohn H, Allen DW, Worden K, Farrar CR. Statistical damage classification using se- quential probability ratio tests. Struct Heal Monit 2003;2:57–74.
[13] Pakzad SN, Yao R. Autoregressive statistical pattern recognition algorithms for dam- age detection in civil structures. Mech Syst Signal Process 2012;31:355–68.
[14] Yao R, Pakzad SN. Damage and noise sensitivity evaluation of autoregressive fea- tures extracted from structure vibration. Smart Mater Struct 2014;23:25007.
[15] Roy K, Bhattacharya B, Ray-Chaudhuri S. ARX model-based damage sensitive fea- tures for structural damage localization using output-only measurements. J Sound Vib 2015;349:99–122.
[16] Pandit SM. Modal and spectrum analysis: data dependent system in state space. Wiley-Interscience; 1991.
[17] Ewins DJ. Modal testing: theory, practice and application. Research studies press LTD; 2000.
[18] Van HS, Vandewalle J. The total least squares problem: computational aspects and analysis. Soc Ind Appl Math 1991:300.
[19] Weyrick RC. (1975) Fundamentals of automatic control. Glencoe/McGraw Hill
[20] Ljung L. (1999) System identification: theory for the user (second ed.). Prentice Hall. doi: 10.1016/S0005-1098(01)00214-X
[21] Bailey RA. Design of comparative experiments. Empir Econ. 2013. doi:10.1017/CBO9780511611483.
[22] Worden K, Farrar CR, Manson G, Park G. The fundamental axioms of structural health monitoring. In: Proc. R. Soc. A Math. Phys. Eng. Sci.; 2007. p. 1639–64.
[23] Worden K, Sohn H, Farrar CR. Novelty detection in a changing environment: regres- sion and interpolation approaches. J Sound Vib 2002;258:741–61.
[24] Mahalanobis PC. On the generalized distance in statistics. In: Proc. Natl. Inst. Sci. India; 1936. p. 49–55.
[25] Saltelli A, Ratto M, Andres T, Campolongo F, Cariboni J, Gatelli D. et al. (2008) Global sensitivity analysis: the primer.
[26] Rubinstein R, Kroese D. Simulation and the monte carlo method. New York: Wiley; 2007.
[27] Blatman G, Sudret B. Adaptive sparse polynomial chaos expansion based on least angle regression. J Comput Phys 2011;230:2345–67.
[28] Blatman G, Sudret B. An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis. Probabilistic Eng Mech 2010;25:183–97.
[29] Sudret B. Global sensitivity analysis using polynomial chaos expansions. Reliab Eng Syst Saf 2008;93:964–79.
[30] Mukhopadhyay T, Chakraborty S, Dey S, Adhikari S, Chowdhury R. A critical as- sessment of kriging model variants for high-fidelity uncertainty quantification in dynamics of composite shells. Arch Comput Methods Eng 2016:1–24.
[31] Biswas S, Chakraborty S, Chandra S, Ghosh I. Kriging based approach for estimation of vehicular speed and passenger car units on an urban arterial. J Transp Eng Part A Syst 2016;143:1–11.
[32] Deheeger F, Sudret B. Metamodel-based importance sampling for structural reliabil- ity analysis. Probabilistic Eng. Mech. 2011.
[33] Dubourg V. (2011) Adaptive surrogate models for reliability analysis and reliability-based design optimization. 309
[34] Chakraborty S, Chowdhury R. Modelling uncertainty in incompressible flow simulation using Galerkin based generalized ANOVA. Comput Phys Commun 2016;208:73–91.
[35] Chakraborty S, Chowdhury R. Sequential experimental design based generalised ANOVA. J Comput Phys 2016;317:15–32.
[36] Chakraborty S, Chowdhury R. Polynomial correlated function expansion. Model Simul Tech Struct Eng 2017:348–73.
[37] Chatterjee T, Chakraborty S, Chowdhury R. A bi-level approximation tool for the computation of FRFs in stochastic dynamic systems. Mech Syst Signal Process 2016;70–71:484–505.
[38] Chakraborty S, Chowdhury R. A hybrid framework for the estimation of rare failure event probability. J Eng Mech. 2016 1943-7889.0001223. doi:10.1061/(ASCE)EM.
[39] Chakraborty S, Chowdhury R. Towards "h-p adaptive" generalised ANOVA. Comput Methods Appl Mech Eng. 2017. doi:10.1016/j.cma.2017.03.028.
[40] Shu SX, Gong WH. An artificial neural network-based response surface method for reliability analyses of c- φ slopes with spatially variable soil. China Ocean Eng 2016;30:113–22.
[41] Manzhos S, Carrington T. A random-sampling high dimensional model repre- sentation neural network for building potential energy surfaces. J Chem Phys 2006;125:84109.
[42] Bhattacharyya B. A critical appraisal of design of experiments for uncertainty quan- tification. Arch Comput Methods Eng. 2017. doi: 10.1007/s11831-017-9211-x.
[43] Ruano MV, Ribes J, Ferrer J, Sin G. Application of the Morris method for screen- ing the influential parameters of fuzzy controllers applied to wastewater treatment plants. Water Sci Technol 2011;63:2199.
[44] Cropp RA, Braddock RD. The new morris method: an efficient second-order screen- ing method. Reliab Eng Syst Saf 2002;78:77–83.
[45] Ketema AA, Langergraber G. Sensitivity analysis of the CLARA simplified planning tool using the morris screening method. Water Sci Technol 2015;71:234.
[46] Gan Y, Liang X-Z, Duan Q, Choi H Il, Dai Y, Wu H. Stepwise sensitivity analysis from qualitative to quantitative: Application to the terrestrial hydrological modeling of a Conjunctive Surface-Subsurface Process (CSSP) land surface model. J Adv Model Earth Syst 2015;7:648–69.
[47] Zhang L, Lu Z, Cheng L, Hong D. Moment-independent regional sensitivity anal- ysis of complicated models with great efficiency. Int J Numer Methods Eng 2015;103:996–1014.
[48] Wei P, Lu Z, Ruan W, Song J. Regional sensitivity analysis using revised mean and variance ratio functions. Reliab Eng Syst Saf 2014;121:121–35.
[49] Joint Comitee for Guides in Metrology (2008) Evaluation of measurement data — Guide to the expression of uncertainty in measurement.
[50] Sobol' IM. Sensitivity estimates for nonlinear mathematical models. Math Model Comput Exp 1993;1:407–14.
[51] Cannavó F. Sensitivity analysis for volcanic source modelling quality assessment and model selection. Comput. Geosci 2012.
[52] Sobol IM. The distribution of points in a cube and the approximate evaluation of integrals. Zh Vychisl Mat i Mat Fiz 1967;7(7):784–802.
[53] Borgonovo E, Plischke E. Sensitivity analysis: A review of recent advances. Eur J Oper Res. doi: http://dx.doi.org/10.1016/j.ejor.2015.06.032
[54] Borgonovo E, Castaings W, Tarantola S. Model emulation and moment-indepen- dent sensitivity analysis: An application to environmental modelling. Environ Model Softw 2012;34:105–15.
[55] Borgonovo E, Tarantola S. Moment independent and variance-based sensitivity anal- ysis with correlations: An application to the stability of a chemical reactor. Int J Chem Kinet 2008;40:687–98.
[56] Caniou Y, Sudret B. Distribution-based global sensitivity analysis in case of corre- lated input parameters using polynomial chaos expansions. 11th Int. Conf. Appl. Stat. Prob. Civ. Eng; 2011.
[57] Chakraborty S, Chowdhury R. Moment independent sensitivity analysis: PCFE–based approach. J Comput Civ Eng 2017;31:6016001.
[58] Chakraborty S, Chowdhury R. A hybrid approach for global sensitivity analysis. Re- liab Eng Syst Saf 2016;158:50–7.
[59] Caniou Y, Sudret B. Distribution-based global sensitivity analysis using polynomial chaos expansions. Procedia - Soc Behav Sci 2010;2:7625–6.
[60] Pianosi F, Wagener T. A simple and efficient method for global sensitivity analysis based on cumulative distribution functions. Environ Model Softw 2015;67:1–11.
[61] Pianosi F, Sarrazin F, Wagener T. A matlab toolbox for global sensitivity analysis. Environ Model Softw 2015;70:80–5.
[62] Kaliyaperumal G, Imam B, Righiniotis T. Advanced dynamic finite element analysis of a skew steel railway bridge. Eng Struct 2011;33:181–90.
[63] Malm R, Andersson A. Field testing and simulation of dynamic properties of a tied arch railway bridge. Eng Struct 2006;28:143–52.
[64] Bracewell RN. The fourier transform and its applications. McGraw-Hill; 2000.
[65] Worden K, Manson G, Fieller NRJ. Damage detection using outlier analysis. J Sound Vib 2000;229:647–67.

[66] Han YJ, He Z, Song GF. Research for multidimensional systems diagnostic analysis based on improved Mahalanobis distance. In: IE EM 2009 - Proc. 2009 IEEE 16th Int. Conf. Ind. Eng. Eng. Manag.; 2009. p. 213–17.
[67] Wang D, Yeung DS, Tsang EC. Weighted Mahalanobis squared distance kernels for supported vector machines. IEEE Trans Neural Netw 2007;18:1453–63.
[68] Wang Y, Ni H, Liu P, Li W. Improved SDA based on mixed weighted Mahalanobis distance. Signal, Image Video Process 2016;10:65–74.