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Evolution of superoscillations for Schrödinger equation in a uniform magnetic field

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Aharonov-Berry superoscillations are band-limited functions that oscillate faster than their fastest Fourier component. Superoscillations appear in several fields of science and technology, such as Aharonov's weak measurement in quantum mechanics, in optics, and in signal processing. An important issue is the study of the evolution of superoscillations using the Schrödinger equation when the initial datum is a weak value. Some superoscillatory functions are not square integrable, but they are real analytic functions that can be extended to entire holomorphic functions. This fact leads to the study of the continuity of a class of convolution operators acting on suitable spaces of entire functions with growth conditions. In this paper, we study the evolution of a superoscillatory initial datum in a uniform magnetic field. Moreover, we collect some results on convolution operators that appear in the theory of superoscillatory functions using a direct approach that allows the convolution operators to have non-constant coefficients of polynomial type. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4991489]

I. INTRODUCTION

Superoscillations are band-limited functions with the apparently paradoxical property that they can oscillate faster than their fastest Fourier component. They arise in a number of physical and mathematical contexts, for example, in signal processing and from Aharonov's weak-measurement in quantum mechanics, see Refs. 1, 9, and 10 and also the work of Ferreira and Kempf.[19,20,22,23] In optics and in other fields, Berry and some of his coauthors have given fundamental contributions, see, for example, Refs. 12, 11, 13, 15, and 16. We also quote the paper of Lindberg[25] for optical super-resolution. The case of superoscillatory functions in several variables has been considered by Berry[14] and a more systematic study of the mathematical theory of superoscillating sequences in several variables has been recently done in Ref. 6.

To provide the necessary background, we recall that a sequence of the form

$$Y_n(x,a) := \sum_{j=0}^{n} C_j(n,a) e^{ik_j(n)x}, \quad n \in \mathbb{N}, \tag{1}$$

where $a \in \mathbb{R}$ and $C_j(n,a)$ and $k_j(n)$ are real valued functions of the variables n, a, and j, is called the *generalized Fourier sequence*. A generalized Fourier sequence $Y_n(x,a)$ is said to be a *superoscillating sequence* if $|k_j(n)| \le 1$ and if there exists a compact subset of $\mathbb{R}$, on which Y_n converges uniformly to $e^{ig(a)x}$, where g is a continuous real valued function such that $|g(a)| > 1$.

The classical example of a superoscillating sequence that appears in the weak measurement is given by

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \sum_{j=0}^{n} C_j(n,a) e^{i(1-\frac{2j}{n})x}$$

with $a > 1$, $x \in \mathbb{R}$, and

$$C_j(n,a) = \binom{n}{j}\left(\frac{1+a}{2}\right)^{n-j}\left(\frac{1-a}{2}\right)^{j}. \tag{2}$$

This sequence converges on $\mathbb{R}$ locally uniformly to e^{iax}. The explanation of the superoscillatory phenomenon is based on the definition of weak values. A weak measurement of a quantum observable represented by the self-adjoint operator A, involving a pre-selected state ψ_0 and a post-selected state ψ_1, leads to the weak value

$$A_{weak} := \frac{(\psi_1, A\psi_0)}{(\psi_1, \psi_0)} = b + ib'.$$

The weak value A_{weak} is a complex number. Its real part b and its imaginary part b' can be interpreted as the shift b and the momentum b' of the pointer recording the measurement.

An important feature of the weak measurement is that, in contrast with the strong measurements of von Neumann (given by the expectation value of the operator A),

$$A_{strong} := (\psi, A\psi),$$

the real part b of A_{weak} can be very large with respect to A_{strong} because (ψ_1, ψ_0) can be very small when the states ψ_0 and ψ_1 are almost orthogonal. This is what produces the superoscillations.

In a series of papers[2–5,17] and the forthcoming monograph,[8] a method to study the evolution of superoscillations using the Schrödinger equation has been developed. It is based on Green functions and convolution operators acting on entire functions. Precisely we consider the Cauchy problem

$$i\frac{\partial \psi}{\partial t}(t,x) = H(x)\psi(t,x), \qquad \psi(0,x) = Y_n(x,a), \tag{3}$$

where H is the Hamiltonian operator. Once we obtain a closed form for the solution of the Cauchy problem, for example, using the fundamental solution, the problem that we face is to study the behavior of the solution $\psi_n(t,x)$ as n tends to infinity.

In most of the cases that we have treated, such as the free particle, the harmonic oscillator, and the particle in a uniform electric field, see Ref. 7, the convolution operators, which we have to consider, are of the form

$$\mathcal{U}_p\left(t, \frac{\partial}{\partial x}\right) := \sum_{m=0}^{\infty} a_m(t) \frac{\partial^{pm}}{\partial x^{pm}},$$

where $p \in \mathbb{N}$ and a_m are suitable complex numbers that depend on the Hamiltonian H and can depend on time. Using such operators, the solution to the Cauchy problem can be written as

$$\psi_n(t,x) = \mathcal{U}_p\left(t, \frac{\partial}{\partial x}\right) Y_n(x,a).$$

In order to show that superoscillations persist in time, we need to explicitly compute the limit

$$\lim_{n\to\infty} \psi_n(x,t)$$

and see if the limit function keeps the superoscillatory behavior. This problem is naturally studied in the complex setting.

Indeed, we replace the real variable x by the complex variable z so the operator $\mathcal{U}_p\left(t, \frac{\partial}{\partial x}\right)$ becomes $\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right)$. The functions $Y_n(x, a)$ extend to entire holomorphic functions with suitable growth conditions and on such class of functions the operator $\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right)$ acts continuously.

Once we have established the continuity of $\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right)$, the above limit can be computed as

$$\lim_{n\to\infty} \psi_n(z,t) = \mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right) \lim_{n\to\infty} Y_n(z,a)$$

in the complex setting and then we take the restriction to the real line. For the study of the evolution of superoscillatory functions under the Schrödinger equation with a singular potential, it turns out that we cannot consider just the class of operators $\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right)$ with constant coefficients with respect to z, but that we have to consider the case in which the coefficients a_m are analytic functions of z.

The general setting that allows us to guarantee that the operators $\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right)$ are continuous on a suitable space of entire functions uses Ehrenpreis' theory of Analytically Uniform spaces (AU-spaces for short),[18] see also Ref. 27. We will give the ideas of how it works in Sec. V. This theory is very

general and also allows the study of convolution operators with constant coefficients that act on the space of Sato-hyperfunctions.

In the case of operators

$$\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right) := \sum_{m=0}^{\infty} a_m(z) \frac{\partial^{pm}}{\partial z^{pm}}$$

with analytic coefficients $a_m(z)$, a direct approach is needed. We have to consider the problem of finding those entire functions $f(z)$ with suitable growth conditions such that

$$\sum_{m=0}^{\infty} a_m(z) \frac{\partial^{pm}}{\partial z^{pm}} f(z)$$

is an entire function.

In this paper, we study the operators $\mathcal{U}_p\left(t, \frac{\partial}{\partial z}\right)$ in the case in which the coefficients are polynomials $a_m(z)$. This problem is of independent interest. Then we apply these results to study the evolution of superoscillations in a uniform magnetic field in the homogeneous case. Precisely, we set $\mathbf{r} = (x, y, z)$ to be the coordinates of the space and consider the Schrödinger equation in a uniform magnetic field

$$i\frac{\partial}{\partial t}\psi_n(\mathbf{r}, t) = \left[-\frac{1}{2}\Delta - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{8}(x^2 + y^2)\right]\psi_n(\mathbf{r}, t)$$

with the superoscillatory initial datum given by

$$\psi_n(\mathbf{r}, 0) = \sum_{j=0}^{n} C_j(n, a) e^{i(1-\frac{2j}{n})(x+y+z)},$$

where the coefficients $C_j(n, a)$ are as in (2). Using the fundamental solution, we prove that

$$\begin{aligned}\psi_n(\mathbf{r}, t) = {} & \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)} \\ & \cdot \sum_{j=0}^{n} C_j(n, a) e^{i\left(1-\frac{2j}{n}\right)\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right) - i\left(1-\frac{2j}{n}\right)^2\left(\frac{t}{2}+2\tan\left(\frac{t}{2}\right)\right)}.\end{aligned} \tag{4}$$

Using the continuity of convolution operators, we then prove that as n tends to infinity, $\psi_n(\mathbf{r}, t)$ tends for any $t \in (0, \pi)$ uniformly on compact sets in $\mathbb{R}^3$ to $\phi_a(\mathbf{r}, t)$, the solution with initial datum $\phi_a(\mathbf{r}, t) = e^{ia(x+y+z)}$. Precisely

$$\lim_{n\to\infty} \psi_n(\mathbf{r}, t) = \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right) - ia^2\left(\frac{t}{2}+2\tan\left(\frac{t}{2}\right)\right) + ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}. \tag{5}$$

We point out that the superoscillatory behavior persists in time because ψ_n terms such as $e^{i\left(1-\frac{2j}{n}\right)\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}$, containing the band-limited factor $|1 - \frac{2j}{n}| \leq 1$, lead to the term $e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}$ with $a > 1$ in the limit function.

The plan of the paper is as follows: Sec. II contains the study of the convolution operators with polynomial coefficients. In Sec. III, we recall the fundamental solution of the Schrödinger equation in a uniform magnetic field and we give a preliminary result, which is necessary to study the superoscillatory behavior of the solutions. In Sec. IV, we consider the homogeneous case, and in Sec. V, we study the nonhomogeneous case. Here we need the theory of superoscillatory functions in several variables.

II. ENTIRE FUNCTIONS AND OPERATORS ASSOCIATED WITH SUPEROSCILLATIONS

The theory of AU-spaces is very general and it has been used in several cases when we considered convolution operators with constant coefficients. The aim of this section is to use direct methods to recover some results in the constant coefficient case that can be extended to the case of non-constant coefficients. We recall some well known definitions on entire functions, which will be useful to study the continuity of operators that appear in the theory of superoscillatory functions.

Let f be a non-constant entire function of a complex variable z. We define

$$M_f(r) = \max_{|z|=r} |f(z)|, \quad \text{for } r \geq 0.$$

The non-negative real number ρ defined by

$$\rho = \limsup_{r\to\infty} \frac{\ln \ln M_f(r)}{\ln r}$$

is called the order of f. If ρ is finite, then f is said to be of finite order, and if $\rho = \infty$, the function f is said to be of infinite order.

In the case f is of finite order, we define the non-negative real number

$$\sigma = \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho},$$

which is called the type of f. If $\sigma \in (0, \infty)$, we call f of normal type, while we say that f is of minimal type if $\sigma = 0$ and of maximal type if $\sigma = \infty$. The constant functions are said to be of minimal type of order zero.

Definition 2.1. Let $\rho \in (0, \infty)$ and $\sigma \in [0, \infty]$. We denote by $\mathcal{A}_{\rho,\sigma}$ the class of entire functions that are either of order less than ρ or of order ρ with type at most σ. We consider in $\mathcal{A}_{\rho,\sigma}$ the topology of relatively uniform convergence: for $\rho \in (0, \infty)$ and $\sigma \in [0, \infty)$, a sequence $f_n \in \mathcal{A}_{\rho,\sigma}$ converges to $f \in \mathcal{A}_{\rho,\sigma}$ in $\mathcal{A}_{\rho,\sigma}$ if

$$\|f_n - f\|_\varepsilon = \sup_{z\in\mathbb{C}} |f_n(z) - f(z)| e^{-(\sigma+\varepsilon)|z|^\rho} \to 0 \quad \textit{for all } \varepsilon > 0,$$

and for $\rho \in (0, \infty)$ and $\sigma = \infty$, a sequence $f_n \in \mathcal{A}_{\rho,\infty}$ converges to $f \in \mathcal{A}_{\rho,\infty}$ in $\mathcal{A}_{\rho,\infty}$ if

$$\|f_n - f\|_\varepsilon = \sup_{z\in\mathbb{C}} |f_n(z) - f(z)| e^{-|z|^{\rho+\varepsilon}} \to 0 \quad \textit{for all } \varepsilon > 0.$$

Remark 2.2. Note that convergence in $\mathcal{A}_{\rho,\sigma}$ implies locally uniform convergence. The topology on $\mathcal{A}_{\rho,\sigma}$ is however not the relative topology induced by the topology of locally uniform convergence but a finer topology: indeed, if $f(z) = \sum_{k=0}^{+\infty} a_k z^k$ is an entire function, then $f_n(z) = \sum_{k=0}^{n} a_k z^k$ converges locally uniformly to $f(z)$, but if $f \notin \mathcal{A}_{\rho,\sigma}$, then f_n obviously does not converge to f in $\mathcal{A}_{\sigma,\rho}$. Indeed, as pointed out in Ref. 28, in order for a sequence $f_n \in \mathcal{A}_{\rho,\sigma}$ to be a Cauchy sequence, it is necessary and sufficient that

(i) for each $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$\sup_{z\in\mathbb{C}} |f_n(z)| e^{-(\sigma+\varepsilon)|z|^\rho} < K$$

respectively if $\sigma = \infty$,

$$\sup_{z\in\mathbb{C}} |f_n(z)| e^{-|z|^{\rho+\varepsilon}} < K,$$

(ii) for every fixed $z \in \mathbb{C}$, the sequence $f_n(z) \in \mathbb{C}$ is a Cauchy sequence in $\mathbb{C}$.

There exists a well known characterization of functions in $\mathcal{A}_{\rho,\sigma}$ in terms of their Taylor coefficients, see Ref. 28.

Theorem 2.3. *Suppose that the entire function f has the Taylor expansion $f(z) = \sum_{n=0}^{+\infty} f_n z^n$, then $f \in \mathcal{A}_{\rho,\sigma}$ if and only if*

$$\limsup_{n\to\infty} \sqrt[n]{(n!)^{1/\rho} |f_n|} \leq (\rho\sigma)^{1/\rho}, \quad \text{for } \rho \in (0, \infty), \quad \sigma \in [0, \infty), \tag{6}$$

or

$$\limsup_{n\to\infty} \sqrt[n]{(n!)^{1/(\rho+\varepsilon)} |f_n|} = 0, \quad \text{for all } \varepsilon > 0, \quad \text{for } \rho \in [0, \infty) \text{ and } \sigma = \infty. \tag{7}$$

We introduce now a family of convolution operators $\mathcal{U}_p(\frac{\partial}{\partial z})$ depending on a parameter $p \in \mathbb{N}$, which appear in the theory of superoscillatory functions.

Definition 2.4. Given a sequence of complex numbers $(a_n)_{n\in\mathbb{N}_0}$ and a number $p\in\mathbb{N}$, we define the formal operator

$$\mathcal{U}_p\left(\frac{\partial}{\partial z}\right) := \sum_{n=0}^{+\infty} a_n \frac{\partial^{pn}}{\partial z^{pn}}, \tag{8}$$

where $\frac{\partial}{\partial z}$ is the derivative with respect to z. We say that the operator $\mathcal{U}_p(\frac{\partial}{\partial z})$ can be applied to an entire function f if the series

$$\mathcal{U}_p\left(\frac{\partial}{\partial z}\right) f(z) := \sum_{n=0}^{+\infty} a_n f^{(pn)}(z) \tag{9}$$

converges for at least one point $z\in\mathbb{C}$.

Operators of form (8) play an important role in the theory of superoscillations. Setting

$$b_k = \begin{cases} a_n, & \text{if } k = np \\ 0, & \text{otherwise} \end{cases},$$

one has

$$\mathcal{U}_p\left(\frac{\partial}{\partial z}\right) = \sum_{k=0}^{+\infty} b_k \frac{\partial^k}{\partial z^k}. \tag{10}$$

Operators of this form have been studied in Ref. 28. Since easy modifications of the proofs therein show the following results, we only prove the first lemma explicitly in order to present the techniques applied. Actually, instead of adapting the proofs in Ref. 28, the following lemmas can even be obtained by applying the results in Ref. 28 to (10) taking into account the definition the coefficients b_k.

The following three lemmas correspond to Lemmas 2.1–2.3 in Ref. 28 and characterize operators of form (8) that are applicable to a space $\mathcal{A}_{\rho,\sigma}$.

Lemma 2.5. Let $\rho\in(0,+\infty)$ and $\sigma\in(0,\infty)$ and let $p\in\mathbb{N}$ be a given fixed number. If for every $f\in\mathcal{A}_{\rho,\sigma}$ series (9) converges for $z=0$, then

$$\limsup_{n\to\infty} \sqrt[pn]{((pn)!)^{1-1/\rho}\,|a_n|} < (\rho\sigma)^{-1/\rho}. \tag{11}$$

Conversely, suppose that (11) is satisfied. Then (9) converges absolutely for every $f\in\mathcal{A}_{\rho,\sigma}$ and every $z\in\mathbb{C}$.

Proof. If (11) is not satisfied, then there exists an increasing sequence of natural numbers n_k such that

$$|a_{n_k}| > \left((\rho\sigma)^{-1/\rho} - \frac{1}{k}\right)^{pn_k} ((pn_k)!)^{-1+1/\rho},$$

where $k = k_0, k_0+1, k_0+2, \ldots$ with $k_0 > (\rho\sigma)^{1/\rho}$. By (6), the function

$$f(z) := \sum_{k=k_0}^{+\infty} \frac{1}{(pn_k)!\,a_{n_k}} z^{pn_k}$$

then belongs to $\mathcal{A}_{\rho,\sigma}$ but

$$\sum_{n=0}^{+\infty} a_n f^{(pn)}(0) = \sum_{k=k_0}^{+\infty} a_{n_k} f^{(pn_k)}(0) = \sum_{k=k_0}^{+\infty} 1 = +\infty,$$

which contradicts the assumption that (9) converges at $z=0$ for any function in $\mathcal{A}_{\rho,\sigma}$.

Now assume that (11) holds true. Then there exists τ with $0<\tau<(\rho\sigma)^{-1/\rho}$ and $L>0$ such that

$$|a_n| < L\tau^{pn}((pn)!)^{-1+1/\rho} \quad \text{for all } n\in\mathbb{N}_0.$$

Observe that for $f\in\mathcal{A}_{\rho,\sigma}$ and $\xi\in\mathbb{C}$, the function

$$f_\xi(z) = f(z+\xi) = \sum_{k=0}^{+\infty} \frac{f^{(n)}(\xi)}{n!} z^n$$

also belongs to $\mathcal{A}_{\rho,\sigma}$. Thus, by (6), for every $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that

$$|f^{(n)}(\xi)| < K_\varepsilon \left((\rho\sigma)^{1/\rho} + \varepsilon\right)^n (n!)^{1-1/\rho} \quad \text{for all } n \in \mathbb{N}_0.$$

If in particular we choose ε small enough, such that $\beta := \left((\rho\sigma)^{1/\rho} + \varepsilon\right)\tau < 1$, then

$$\sum_{n=0}^{+\infty} \left|a_n f^{(pn)}(\xi)\right| \le K_\varepsilon L \sum_{n=0}^{+\infty} \beta^n = \frac{K_\varepsilon L}{1-\beta},$$

which concludes the proof. □

Lemma 2.6. Let $\rho \in (0, +\infty)$. If (9) converges for every $f \in \mathcal{A}_{\rho,0}$ at $z = 0$, then

$$\limsup_{n\to\infty} \sqrt[pn]{((pn)!)^{1-1/\rho}\,|a_n|} < +\infty. \tag{12}$$

Conversely, suppose that (12) is satisfied. Then (9) converges absolutely for every $f \in \mathcal{A}_{\rho,\sigma}$ and every $z \in \mathbb{C}$.

Lemma 2.7. Let $\rho \in [0, +\infty)$. If (9) converges for every $f \in \mathcal{A}_{\rho,\infty}$ at $z = 0$, then

$$\limsup_{n\to\infty} \sqrt[pn]{((pn)!)^{1-1/(\rho+\varepsilon)}\,|a_n|} < \infty \tag{13}$$

for some $\varepsilon > 0$. Conversely, if (13) holds true for some $\varepsilon > 0$, then (9) converges absolutely for every $f \in \mathcal{A}_{\rho,\sigma}$ and every $z \in \mathbb{C}$.

In order to investigate the superoscillatory behavior, it is in certain situations necessary to consider differential operators of form (8), but with non-constant coefficients. Although we consider in this paper only operators with constant coefficients, we state the following results, which consider the case of polynomial coefficients, for the sake of completeness.

Definition 2.8. Given q sequences of complex numbers $(a_{n,0})_{n\in\mathbb{N}_0}, \dots, (a_{n,q})_{n\in\mathbb{N}_0}$ and we set $P_n(z) = \sum_{j=0}^{q} a_{n,j} z^j$. For $p \in \mathbb{N}$, we define the formal operator

$$\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right) := \sum_{n=0}^{+\infty} P_n(z) \frac{\partial^{pn}}{\partial z^{pn}}, \tag{14}$$

whose coefficients are polynomials of degree lower or equal to q. We say that $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ is applicable to the class $\mathcal{A}_{\rho,\sigma}$ if the series

$$\sum_{n=0}^{+\infty} P_n(z) f^{(pn)}(z)$$

converges for any $f \in \mathcal{A}_{\rho,\sigma}$ and any $z \in \mathbb{C}$.

Convolution operators with polynomial coefficients are also studied in Ref. 28. As before, the following result can be obtained by simple adaptations of the techniques used therein. Again, one can also consider $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ as an operator of the form (10), where b_k are polynomial coefficients that equal P_n if $k = nj$ and 0 otherwise. Then one can directly apply Theorem 2.5 in Ref. 28 in order to obtain the following lemma.

Lemma 2.9. The operator $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ given by (14) is applicable to $\mathcal{A}_{\rho,\sigma}$ if and only if

(i) in the case $0 < \rho, \sigma < \infty$, we have

$$\limsup_{n\to\infty} \sqrt[pn]{((pn)!)^{1-1/\rho}|a_{n,j}|} < (\rho\sigma)^{-1/\rho} \quad \text{for } j \in \{0, \dots, q\},$$

(ii) in the case $0 < \rho < \infty$ and $\sigma = 0$, the operator $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ is applicable to some class $\mathcal{A}_{\rho,\sigma'}$ with $\sigma' > 0$,

(iii) *in the case* $0 \le \rho < \infty$ *and* $\sigma = \infty$, *the operator* $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ *is applicable to some class* $\mathcal{A}_{\rho',\sigma}$ *with* $\rho' > \rho$.

Finally, the next result, which corresponds to Theorem 3.4 and Corollary 3.4 in Ref. 28, gives information about the range and some continuity properties of the considered operators.

Theorem 2.10. *Let the operator* $\mathcal{U}_{p,q}(D_z)$ *defined in* (14) *be applicable to the class* $\mathcal{A}_{\rho,\sigma}$.

(i) *If* $\rho \in (0, 1]$ *and* $\sigma \in (0, \infty)$, *then* $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ *maps* $\mathcal{A}_{\rho,\sigma}$ *continuously into itself.*

(ii) *If* $\rho \in (1, \infty)$ *and* $\sigma \in (0, \infty)$, *then* $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ *maps* $\mathcal{A}_{\rho,\sigma}$ *continuously into* $\mathcal{A}_{\rho,\theta}$ *with*

$$\theta := \sigma\left(1 - d^{\rho/(\rho-1)}\right)^{1-\rho},$$

where

$$d := \gamma(\sigma\rho)^{1/\rho} \quad \text{and} \quad \gamma := \max_{j\in\{0,\dots,q\}} \limsup_{n\to+\infty} \sqrt[pn]{(pn!)^{1-1/\rho}|a_{n,j}|}.$$

(iii) *If* $\rho \in (0, \infty)$ *and* $\sigma = 0$ *or* $\sigma = \infty$, *then* $\mathcal{U}_{p,q}\left(z, \frac{\partial}{\partial z}\right)$ *maps* $\mathcal{A}_{\rho,\sigma}$ *continuously into itself.*

III. PRELIMINARIES ON SUPEROSCILLATIONS IN UNIFORM MAGNETIC FIELD

We recall the fundamental solution of the Schrödinger equation in a uniform magnetic field from the paper of Sondheimer and Wilson.[26] The Hamiltonian H of a free electron in a constant magnetic field $\mathcal{H}$ is given by

$$\mathcal{H} = -\frac{h^2}{8\pi^2 m}\Delta + \frac{qh}{2\pi imc}\mathbf{A}\cdot\nabla + \frac{q^2\mathbf{A}^2}{2mc^2},$$

where $\mathbf{A} = \frac{1}{2}\mathbf{H}\times\mathbf{r}$ is the vector potential, $-q$ is the charge, and m is the mass of the electron and the remaining symbols have the usual meaning. The Schrödinger equation studied in Ref. 26 is

$$-\frac{\partial}{\partial\gamma}\psi(\mathbf{r},\gamma) = \mathcal{H}\psi(\mathbf{r},\gamma),$$

where $\mathbf{r} = (x, y, z)$ are the coordinates of the space. If the magnetic field is taken along the z direction, then the vector potential is $(-\frac{1}{2}Hy, \frac{1}{2}Hx, 0)$ and the equation for the wave function $\psi(\mathbf{r},\gamma)$ turns into

$$\partial_\gamma\psi(\mathbf{r},\gamma) = \left[\frac{h^2}{8\pi^2 m}\Delta - \frac{qhH}{4\pi imc}(x\partial_y - y\partial_x) - \frac{q^2H^2}{8mc^2}(x^2+y^2)\right]\psi(\mathbf{r},\gamma),$$

which has the fundamental solution

$$K(\mathbf{r},\mathbf{r}',\gamma) = \left(\frac{2\pi m}{h^2\gamma}\right)^{\frac{3}{2}} \frac{\mu_0 H\gamma}{\sinh(\mu_0 H\gamma)} \exp\Bigg[-\frac{2\pi^2 m}{h^2\gamma}\cdot \Big(2i\mu_0 H\gamma(x'y - y'x) + \mu_0 H\gamma\coth(\mu_0 H\gamma)\left[(x-x')^2 + (y-y')^2\right] + (z-z')^2\Big)\Bigg],$$

where μ_0 is the Bohr magneton,

$$\mu_0 = \frac{qh}{4\pi mc}.$$

With the position

$$\gamma = \frac{t}{-i\hbar} \qquad \text{with} \qquad \hbar = \frac{h}{2\pi},$$

we have

$$\frac{\partial}{\partial\gamma} = -i\hbar\frac{\partial}{\partial t}.$$

The Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = \mathcal{H}\psi(r,T)$$

has therefore the fundamental solution

$$K(\mathbf{r},\mathbf{r}',t)=\left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}}\frac{\mu_0 Hit}{\hbar\sinh\left(\mu_0 Hi\frac{t}{\hbar}\right)}\exp\Bigg[-\frac{m}{2\hbar it}\cdot$$
$$\left(-2\mu_0 H\frac{t}{\hbar}(x'y-y'x)+\mu_0 H\frac{it}{\hbar}\coth\left(\mu_0 H\frac{it}{\hbar}\right)\left((x-x')^2+(y-y')^2\right)+(z-z')^2\right)\Bigg].$$

For the sake of simplicity, we set

$$c=\hbar=m=H=q=1$$

and obtain the following theorem.

Theorem 3.1. *The fundamental solution of the Schrödinger equation*

$$i\frac{\partial}{\partial t}\psi(\mathbf{r},t)=\left[-\frac{1}{2}\Delta-\frac{i}{2}\left(x\frac{\partial}{\partial_y}-y\frac{\partial}{\partial_x}\right)+\frac{1}{8}(x^2+y^2)\right]\psi(\mathbf{r},t)$$

when the magnetic field is taken along the z direction and the vector potential is $(-\frac{1}{2}y,\frac{1}{2}x,0)$ *is given by*

$$K(\mathbf{r},\mathbf{r}',t)=\left(\frac{1}{2\pi it}\right)^{\frac{3}{2}}\frac{t}{2\sin\left(\frac{t}{2}\right)}\cdot$$
$$\exp\left[\frac{i}{2t}\left(-t(x'y-y'x)+\frac{t}{2}\cot\left(\frac{t}{2}\right)\left[(x-x')^2+(y-y')^2\right]+(z-z')^2\right)\right].$$

We now use Theorem 3.1 to find the solution of the following Cauchy problem.

Theorem 3.2. *Let* $\mathbf{b}=(b_1,b_2,b_3)\in\mathbb{R}^3$ *and write* $\mathbf{r}=(x,y,z)$. *Then the solution of the Cauchy problem*

$$i\frac{\partial}{\partial t}\phi_{\mathbf{b}}(\mathbf{r},t)=\left[-\frac{1}{2}\Delta-i\left(x\frac{\partial}{\partial_y}-y\frac{\partial}{\partial_x}\right)+\frac{1}{8}\left(x^2+y^2\right)\right]\phi_{\mathbf{b}}(\mathbf{r},t),$$
$$\phi_{\mathbf{b}}(\mathbf{r},0)=e^{i(b_1x+b_2y+b_3z)}$$

is given by

$$\phi_{\mathbf{b}}(\mathbf{r},t)=\frac{1}{\cos(\frac{t}{2})}e^{-\frac{i}{4}\tan\left(\frac{t}{2}\right)\left(x^2+y^2\right)}e^{\mathcal{E}_{\mathbf{b}}(t,x)}\tag{15}$$

with

$$\mathcal{E}_{\mathbf{b}}(t,x):=ib_1x+ib_2y+ib_3z-i\tan\left(\frac{t}{2}\right)(b_1^2+b_2^2)-i\frac{t}{2}b_3^2-i\tan\left(\frac{t}{2}\right)(b_2x-b_1y).$$

Proof. We represent the solution using the fundamental solution from Theorem 3.1 and obtain

$$\phi_{\mathbf{b}}(\mathbf{r},t)=\left(\frac{1}{2\pi it}\right)^{3/2}\frac{t}{2\sin\left(\frac{t}{2}\right)}$$
$$\times\int_{\mathbb{R}^3}e^{\frac{i}{2t}\left(-t(x'y-y'x)+\frac{t}{2}\cot\left(\frac{t}{2}\right)\left((x-x')^2+(y-y')^2\right)+(z-z')^2\right)}e^{i(b_1x'+b_2y'+b_3z')}d\mathbf{r}'.$$

For the sake of simplicity, we set

$$M(t):=\left(\frac{1}{2\pi it}\right)^{3/2}\frac{t}{2\sin(\frac{t}{2})}$$

so that we obtain

$$\phi_{\mathbf{b}}(\mathbf{r},t)=M(t)\int_{\mathbb{R}^3}e^{\mathcal{E}(\mathbf{r},\mathbf{r}',t)}d\mathbf{r}'$$

with

$$\mathcal{E}(\mathbf{r},\mathbf{r}',t) := \frac{i}{2t}\left(-t(x'y-y'x)+\chi(t)\left[(x-x')^2+(y-y')^2\right]+(z-z')^2\right)+i(b_1x'+b_2y'+b_3z'),$$

where $\chi(t) := \frac{t}{2}\cot\left(\frac{t}{2}\right)$. The exponent can be rewritten as

$$\begin{aligned}\mathcal{E}(\mathbf{r},\mathbf{r}',t) = {}& \frac{i}{2t}\left[\chi(t)[x^2+y^2]+z^2\right]\\ &+\frac{i}{2t}\left[(z'+(tb_3-z))^2-(tb_3-z)^2\right]\\ &+\frac{i}{2t}\chi(t)\left[\left(x'+\frac{1}{\chi(t)}\left(tb_1-x\chi(t)-\frac{ty}{2}\right)\right)^2-\frac{1}{\chi(t)^2}\left(tb_1-x\chi(t)-\frac{ty}{2}\right)^2\right]\\ &+\frac{i}{2t}\chi(t)\left[\left(y'+\frac{1}{\chi(t)}\left(tb_2-y\chi(t)+\frac{tx}{2}\right)\right)^2-\frac{1}{\chi(t)^2}\left(tb_2-y\chi(t)+\frac{tx}{2}\right)^2\right].\end{aligned}$$

Some simple manipulations yield

$$\begin{aligned}\mathcal{E}(\mathbf{r},\mathbf{r}',t) = {}& \frac{i}{2t}\left[\chi(t)[x^2+y^2]+z^2-(tb_3-z)^2\right]\\ &+\frac{i}{2t}\chi(t)\left[-\frac{1}{\chi(t)^2}\left(tb_1-x\chi(t)-\frac{ty}{2}\right)^2-\frac{1}{\chi(t)^2}\left(tb_2-y\chi(t)+\frac{tx}{2}\right)^2\right]\\ &+\frac{i}{2t}(z'+(tb_3-z))^2\\ &+\frac{i}{2t}\chi(t)\left(x'+\frac{1}{\chi(t)}\left(tb_1-x\chi(t)-\frac{ty}{2}\right)\right)^2\\ &+\frac{i}{2t}\chi(t)\left(y'+\frac{1}{\chi(t)}\left(tb_2-y\chi(t)+\frac{tx}{2}\right)\right)^2.\end{aligned}$$

Introducing the notations

$$\begin{aligned}\mathcal{E}_0(\mathbf{r},t) := {}& \frac{i}{2t}\left[\chi(t)[x^2+y^2]+z^2-(tb_3-z)^2\right]\\ &-\frac{i}{2t\chi(t)}\left[\left(tb_1-x\chi(t)-\frac{ty}{2}\right)^2+\left(tb_2-y\chi(t)+\frac{tx}{2}\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_1(\mathbf{r},\mathbf{r}',t) := {}& \frac{i}{2t}(z'+(tb_3-z))^2\\ &+\frac{i}{2t}\chi(t)\left(x'+\frac{1}{\chi(t)}\left(tb_1-x\chi(t)-\frac{ty}{2}\right)\right)^2\\ &+\frac{i}{2t}\chi(t)\left(y'+\frac{1}{\chi(t)}\left(tb_2-y\chi(t)+\frac{tx}{2}\right)\right)^2,\end{aligned}$$

we therefore have

$$\mathcal{E}(\mathbf{r},\mathbf{r}',t) = \mathcal{E}_0(\mathbf{r},t)+\mathcal{E}_1(\mathbf{r},\mathbf{r}',t).$$

Applying Ref. 21, Eq. (3.323-2), we compute the regularized integral

$$\int_{\mathbb{R}} e^{i\alpha x^2}\,dx = \lim_{\beta\to 0^+}\int_{\mathbb{R}} e^{-x^2(\beta-i\alpha)}\,dx = \left(\frac{i\pi}{\alpha}\right)^{1/2}.$$

For $a, b, c \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}$, we therefore obtain

$$\int_{\mathbb{R}^3} e^{i\alpha(x'+a)^2} e^{i\beta(y'+b)^2} e^{i\gamma(z'+c)^2} dx'dy'dz'$$
$$= \int_{\mathbb{R}^3} e^{i\alpha(x')^2} e^{i\beta(y')^2} e^{i\gamma(z')^2} dx'dy'dz' = \frac{(i\pi)^{3/2}}{(\alpha\beta\gamma)^{1/2}}$$

and thus

$$\int_{\mathbb{R}^3} e^{\mathcal{E}_1(\mathbf{r},\mathbf{r}',t)} d\mathbf{r}' = \frac{(i\pi)^{3/2}}{\frac{\chi(t)}{2t}(\frac{1}{2t})^{1/2}} = \frac{(2t\pi i)^{3/2}}{\chi(t)}.$$

Altogether we have

$$\phi(\mathbf{r}, t) = M(t)e^{\mathcal{E}_0(\mathbf{r},t)} \int_{\mathbb{R}^3} e^{\mathcal{E}_1(\mathbf{r},\mathbf{r}',t)} d\mathbf{r}' = M(t)\frac{(2t\pi i)^{3/2}}{\chi(t)} e^{\mathcal{E}_0(\mathbf{r},t)} = \frac{1}{\cos\left(\frac{t}{2}\right)} e^{\mathcal{E}_0(\mathbf{r},t)}$$

and some simple calculations show that

$$\mathcal{E}_0(\mathbf{r}, t) = -\frac{i}{4}\tan\left(\frac{t}{2}\right)(x^2 + y^2) + \mathcal{E}_{\mathbf{b}}(t, x).$$

So we get the statement. □

We will use Theorem 3.2 to study the evolution of superoscillations for two different initial data. In the homogeneous case, we consider the evolution of superoscillations of the form

$$\psi_n(\mathbf{r}, 0) = \sum_{j=0}^{n} C_j(n, a) e^{i(1-\frac{2j}{n})(x+y+z)}.$$

This case can be treated using the theory of superoscillations in one variable and it is the case of weak values. Since all space variables have the same exponential coefficient, namely, $i\left(1 - \frac{2j}{n}\right)$, we can see the effect of the magnetic field on the evolution of the initial datum better than in the nonhomogeneous case. For the nonhomogeneous case, we consider the initial datum

$$\psi_n(\mathbf{r}, 0) = \sum_{j=0}^{n} C_j(n, a)\, e^{i\left(x\left(1-\frac{2j}{n}\right)^{q_1} + y\left(1-\frac{2j}{n}\right)^{q_2} + z\left(1-\frac{2j}{n}\right)^{q_3}\right)},$$

where q_1, q_2, and q_3 are suitable real numbers. In this case, we need the theory of superoscillations in several variables recently introduced in Ref. 5.

IV. SUPEROSCILLATIONS IN THE HOMOGENEOUS CASE

In this section and also in Sec. V, we are working with the space variable (x, y, z). When we consider convolution operators such as

$$\sum_{m=0}^{\infty} a_m(t)\frac{\partial^{pm}}{\partial x^{pm}}, \quad \sum_{m=0}^{\infty} b_m(t)\frac{\partial^{pm}}{\partial y^{pm}}, \quad \sum_{m=0}^{\infty} c_m(t)\frac{\partial^{pm}}{\partial z^{pm}}, \quad \textit{for} \;\; p \in \mathbb{N},$$

with an abuse of notation we use the same notation also when (x, y, z) become complex variables. We now use the results of Sec. III to study the homogeneous case, that is, the case in which the coefficients in the exponential function of the initial datum satisfy $b_j = a > 1$ for $j = 1, 2, 3$. We shall therefore write ϕ_a instead of $\phi_{\mathbf{b}} = \phi_{(a,a,a)}$ in this section for the sake of simplicity.

Lemma 4.1. Let $b_j = a$ for $j = 1, 2, 3$. The solution $\phi_a(\mathbf{r}, t)$ given by (15) can be written as

$$\phi_a(\mathbf{r}, t) = \frac{1}{\cos(\frac{t}{2})}\, e^{-\frac{i}{4}(x^2+y^2)\tan(\frac{t}{2})}$$
$$\cdot \sum_{m=0}^{\infty} \frac{i^m}{m!}\left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right)^m \frac{\partial^{2m}}{\partial z^{2m}} e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}.$$

Proof. Since $b_j = a$ for $j = 1, 2, 3$, the exponent $\mathcal{E}_{\mathbf{b}}(\mathbf{r}, t)$ in (15) is

$$\mathcal{E}_{\mathbf{b}}(\mathbf{r}, t) = ia\left(x + y + z - (x - y)\tan\left(\frac{t}{2}\right)\right) - ia^2\left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right). \tag{16}$$

Now observe that

$$\begin{aligned}
&e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)-ia^2\left(\frac{t}{2}+2\tan\left(\frac{t}{2}\right)\right)}\\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\left(-ia^2\left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right)\right)^m e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}\\
&= \sum_{m=0}^{\infty} \frac{i^m}{m!}\left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right)^m (ia)^{2m} e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}\\
&= \sum_{m=0}^{\infty} \frac{i^m}{m!}\left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right)^m \frac{\partial^{2m}}{\partial z^{2m}} e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)}.
\end{aligned}$$

Putting these pieces together, we get the statement. □

Theorem 4.2. *Let $a > 1$ and set $\mathbf{r} = (x, y, z)$. Then the solution of the Cauchy problem*

$$\begin{aligned}
i\frac{\partial}{\partial t}\psi_n(\mathbf{r}, t) &= \left[-\frac{1}{2}\Delta - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{8}\left(x^2 + y^2\right)\right]\psi_n(\mathbf{r}, t),\\
\psi_n(\mathbf{r}, 0) &= \sum_{j=0}^{n} C_j(n, a) e^{i\left(1-\frac{2j}{n}\right)(x+y+z)},
\end{aligned} \tag{17}$$

where the coefficients $C_j(n, a)$ are as in (2), is given by

$$\begin{aligned}
\psi_n(\mathbf{r}, t) = &\frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)}\\
&\cdot \sum_{j=0}^{n} C_j(n, a) e^{i\left(1-\frac{2j}{n}\right)\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)-i\left(1-\frac{2j}{n}\right)^2\left(\frac{t}{2}+2\tan\left(\frac{t}{2}\right)\right)}.
\end{aligned} \tag{18}$$

Moreover, $\lim_{n\to\infty} \psi_n(\mathbf{r}, t) = \phi_a(\mathbf{r}, t)$ uniformly on compact sets in $\mathbb{R}^3$ for any $t \in (0, \pi)$, i.e.,

$$\lim_{n\to\infty} \psi_n(\mathbf{r}, t) = \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)-ia^2\left(\frac{t}{2}+2\tan\left(\frac{t}{2}\right)\right)+ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)} = \phi_a(\mathbf{r}, t). \tag{19}$$

Proof. First of all we observe that, because of the linearity of the problem, the solution of (17) is given by the superposition

$$\psi_n(\mathbf{r}, t) = \sum_{j=0}^{n} C_j(n, a)\phi_{1-2j/n}(\mathbf{r}, t).$$

Writing the above identity explicitly using Theorem 3.2 and (16), we get (18).

To prove (19), we apply Lemma 4.1. We define the operator

$$\mathcal{U}\left(t, \frac{\partial}{\partial z}\right) := \sum_{m=0}^{+\infty} \frac{i^m}{m!}\left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right)^m \frac{\partial^{2m}}{\partial z^{2m}}$$

and have

$$\begin{aligned}
\psi_n(\mathbf{r}, t) &= \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)} \mathcal{U}\left(t, \frac{\partial}{\partial z}\right) \sum_{j=0}^{n} C_j(n, a) e^{i\left(1-\frac{2j}{n}\right)\left(x+y+z-(x-y)(\tan\left(\frac{t}{2}\right)\right)}\\
&= \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)} \mathcal{U}\left(t, \frac{\partial}{\partial g}\right) \sum_{j=0}^{n} C_j(n, a) e^{i\left(1-\frac{2j}{n}\right)g(\mathbf{r},t)}
\end{aligned}$$

with

$$g(\mathbf{r}, t) = x + y + z - (x - y)\tan(t/2).$$

By Ref. 8, the sequence $\sum_{j=0}^{n} C_j(n,a)e^{i\left(1-\frac{2j}{n}\right)g}$ converges to e^{iag} in $\mathcal{A}_{2,0}$ and the operator $\mathcal{U}(t,\partial/\partial g)$ acts continuously on this space. Indeed, denoting $C(t) := \frac{t}{2} + 2\tan\left(\frac{t}{2}\right)$, we have

$$\mathcal{U}(t,\partial/\partial g) = \sum_{m=0}^{+\infty} \frac{i^m C(t)^m}{m!} \frac{\partial^{2m}}{\partial g^{2m}},$$

and using Stirling's approximation formula $n! \sim \sqrt{2\pi n}(n/e)^n$, we obtain

$$\begin{aligned}\limsup_{m\to+\infty} \sqrt[2m]{((2m)!)^{\frac{1}{2}} \frac{C(t)^m}{m!}} &= \sqrt{C(t)} \limsup_{m\to+\infty} \frac{\left(\sqrt{4\pi m}(\frac{2m}{e})^{2m}\right)^{\frac{1}{4m}}}{\left(\sqrt{2\pi m}(\frac{m}{e})^m\right)^{\frac{1}{2m}}} \\ &= \sqrt{C(t)} \limsup_{m\to+\infty} 2^{\frac{1}{2}+\frac{1}{8m}} (2\pi m)^{-\frac{1}{8m}} \\ &= \sqrt{2C(t)} < +\infty.\end{aligned}$$

The continuity of the operator $\mathcal{U}(t,\partial/\partial z)$ on $\mathcal{A}_{2,0}$ therefore follows from Lemma 2.6 and Theorem 2.10. We can thus exchange the limit and the operator such that

$$\begin{aligned}&\lim_{n\to+\infty} \psi_n(\mathbf{r},t) \\ &= \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)} \mathcal{U}\left(t, \frac{\partial}{\partial g}\right) e^{iag(t,\mathbf{r})} \\ &= \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)} \sum_{m=0}^{+\infty} \frac{i^m}{m!} \left(\frac{t}{2} + 2\tan\left(\frac{t}{2}\right)\right)^m \frac{\partial^{2m}}{\partial z^{2m}} e^{ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)} \\ &= \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right) - ia^2\left(\frac{t}{2}+2\tan\left(\frac{t}{2}\right)\right) + ia\left(x+y+z-(x-y)\tan\left(\frac{t}{2}\right)\right)} \\ &= \phi_a(t,x).\end{aligned}$$

Finally, observe that $\mathbf{r} \mapsto g(\mathbf{r},t)$ is uniformly continuous and maps bounded sets to bounded sets. Thus, for any subset $K \subset \mathbb{R}^3$, there exists a compact subset $K' \subset \mathbb{C}^3$ such that $g(K) \subset K'$. Since convergence in $\mathcal{A}_{2,0}$ implies uniform convergence on compact sets, the sequence ψ_n converges uniformly to ϕ_a on K. □

V. SUPEROSCILLATIONS IN THE NON HOMOGENEOUS CASE

Denote by z the m-tuple of complex numbers $z := (z_1, \ldots, z_m)$, let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ be a multi-index of length $|\alpha| = \sum_{j=1}^{m} \alpha_j$, and set $z^\alpha = z_1^{\alpha_1} \ldots z_m^{\alpha_m}$. The space $\mathcal{H}(\mathbb{C}^m)$ of entire functions in the variable z is equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}^m$.

For the continuity of the convolution operators in several variables, we apply Ehrenpreis' theory of Analytically Uniform spaces (AU-spaces, for short), see Ref. 18. The theory is quite involved. We recall that a topological vector space (usually containing generalized functions) $\mathcal{X}$ is said to be an AU-space if its dual $\mathcal{X}'$ is isomorphic (usually via some variation of the Fourier-Borel transform) to a space $\mathcal{A}$ of entire functions with suitable growth at infinity. The general theory of AU-spaces assures that a continuous multiplier on $\mathcal{A}$, i.e., an entire function F such that the product by F acts continuously on $\mathcal{A}$, can be interpreted as the symbol of a convolution operator on $\mathcal{X}$.

The spaces we are interested in are of two types, see Refs. 18, 24, and 27.

Definition 5.1. Let p be any positive real number. The space

$$\mathcal{A}_p(\mathbb{C}^m) := \{f \in \mathcal{H}(\mathbb{C}^m) \ : \exists\ A,\ B > 0\ :\ |f(z)| \le A\exp(B|z|^p)\}$$

is said to be the space of functions of order p and finite type.

Definition 5.2. The space

$$\mathcal{A}_{p,0}(\mathbb{C}^m) := \{f \in \mathcal{H}(\mathbb{C}^m)\ :\ \forall\ \varepsilon > 0,\ \exists\ A_\varepsilon > 0\ :\ |f(z)| \le A_\varepsilon \exp(\varepsilon|z|^p)\}$$

is said to be the space of functions of order p and minimal type.

We also consider on these spaces the topology of relatively uniform convergence, cf. Definition 2.1.

The following theorem, which can be found in Ref. 27, is of crucial importance to show the persistence of superoscillations.

Theorem 5.3. *For $p > 1$, there exists a topological isomorphism between the space $\mathcal{A}_p(\mathbb{C}^m)$ and the dual space of the space $\mathcal{A}_{p',0}(\mathbb{C}^m)$ with*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Conversely, there exists a topological isomorphism between the space $\mathcal{A}_{p,0}(\mathbb{C}^m)$ and the dual space of $\mathcal{A}_{p'}(\mathbb{C}^m)$.

To investigate the nonhomogeneous case, we need to recall some results from the theory of superoscillating sequences in several variables. We limit ourselves to the physical case of three variables. For further details, we refer to the recent paper.[6]

Definition 5.4 (Generalized Fourier sequence). *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ be a multi-index of length $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and let*

$$P(u_1, u_2, u_3) = \sum_{|\alpha| \le h} a_\alpha u_1^{\alpha_1} u_1^{\alpha_2} u_3^{\alpha_3}, \quad \text{with} \quad a_\alpha \in \mathbb{C},$$

be a polynomial of degree h. Let $k_{\ell,j}(n)$ for $\ell = 1, 2, 3$, $n \in \mathbb{N}$, and $0 \le j \le n$ be real numbers and set

$$Z_{\ell,j}(x_\ell) := e^{ix_\ell k_{\ell,j}(n)}, \quad \ell = 1, 2, 3.$$

We call the generalized Fourier sequence in three variables *a sequence of the form*

$$f_n(x_1, x_2, x_3) = \sum_{j=0}^{n} \mathcal{K}_j(n, a) P(Z_{1,j}(x_1), Z_{2,j}(x_2), Z_{3,j}(x_3)), \tag{20}$$

where $a \in \mathbb{R}$ and $\mathcal{K}_j(n, a) \in \mathbb{R}$ for $j = 0, \ldots, n$ and $n \in \mathbb{N}$.

An important example of functions $Z_{\ell,j}(x_\ell) = e^{ix_j k_{\ell,j}(n)}$ is given when $k_{\ell,j}(n) = (1 - 2j/n)^{p_\ell}$ with $p_\ell \in \mathbb{N}$. To emphasize the dependence on j and p_ℓ, we write

$$z_{j,p_\ell}(x_\ell) := e^{ix_\ell (1 - \frac{2j}{n})^{p_\ell}}, \quad j = 1, \ldots, m, \quad p_\ell \in \mathbb{N}.$$

Definition 5.5 (Superoscillating sequence). *A generalized Fourier sequence $f_n(x_1, x_2, x_3)$ is said to be a superoscillating sequence if*

$$\lim_{n \to \infty} f_n(x_1, x_2, x_3) = Q\left(e^{ig_1(a)x_1}, e^{ig_2(a)x_2}, e^{ig_3(a)x_3}\right),$$

where $Q(u_1, u_1, u_3)$ is a polynomial and

- *$|k_j(n)| \le 1$ for $j = 1, 2, 3$,*
- *$a \in \mathbb{R}$,*
- *there exists a compact subset of $\mathbb{R}^3$, which will be called a superoscillation set, on which f_n converges uniformly to $Q(e^{ig_1(a)x_1}, e^{ig_2(a)x_2}, e^{ig_3(a)x_3})$, where the functions g_j are continuous, real valued and satisfy $|g_j(a)| > 1$ for $j = 1, 2, 3$.*

We recall the following important result that will be used in the sequel.

Theorem 5.6. *Let $q_j \in \mathbb{N}$ for $j = 1, \ldots, m$ be even numbers and let*

$$z_{k,q_j}(x_j) := e^{ix_j (1 - \frac{2k}{n})^{q_j}}, \qquad j = 1, 2, 3.$$

Assume that there exist $r_\ell \in \mathbb{N}$, $\ell = 2, 3$, *such that*

$$q_1 = r_2 q_2 + r_3 q_3$$

and consider the polynomial of degree h in 3 *variables that is given by*

$$P(u_1, \ldots, u_m) = \sum_{|\alpha| \le h} a_\alpha u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3},$$

where $a_\alpha \in \mathbb{C}$ *and* α *is a multi-index. We define*

$$f_n(x_1, \ldots, x_m) = \sum_{j=0}^{n} C_j(n, a) P(z_{k,q_1}(x_1), z_{k,q_2}(x_2), z_{k,q_3}(x_3)).$$

Then $f_n(x_1, x_2, x_3)$ *is superoscillating, that is,*

$$\lim_{n\to\infty} f_n(x_1, x_2, x_3) = P\left(e^{ix_1(-ia)^{q_1}}, e^{ix_2(-ia)^{q_2}}, e^{ix_3(-ia)^{q_3}}\right).$$

Definition 5.7. Let $a > 1$ *and let* $q_j \in \mathbb{N}$, $j = 1, 2, 3$, *be even numbers and assume that there exist* $r_\ell \in \mathbb{N}$, $\ell = 2, 3$, *such that*

$$q_1 = r_2 q_2 + r_{3m} q_3.$$

We define the sequence

$$\mathcal{Y}_n(x_1, x_2, x_3) = \sum_{j=0}^{n} C_k(n, a)\, e^{ix_1\left(1-\frac{2k}{n}\right)^{q_1}} e^{ix_2\left(1-\frac{2k}{n}\right)^{q_2}} e^{ix_3\left(1-\frac{2k}{n}\right)^{q_3}}.$$

As a consequence, we have the following particular case of Theorem 5.6.

Corollary 5.8. Let $\mathcal{Y}_n(x_1, x_2, x_3)$ *and* q_j, $j = 1, 2, 3$, *be as in Definition 5.7. Then* $\mathcal{Y}_n(x_1, x_2, x_3)$ *is superoscillating and*

$$\lim_{n\to\infty} \mathcal{Y}_n(x_1, x_2, x_3) = e^{ix_1(-ia)^{q_1}} e^{ix_2(-ia)^{q_2}} e^{ix_3(-ia)^{q_3}}.$$

Theorem 5.9. *Let* $\mathbf{b} := (b_1, b_2, b_3) \in \mathbb{R}^3$ *and* $\mathbf{r} = (x, y, z)$ *and define the differential operators*

$$\mathcal{A}\left(t, \frac{\partial}{\partial z}\right) := \sum_{m\ge 0} \frac{1}{m!} \left(\frac{it}{2}\right)^m \frac{\partial^{2m}}{\partial z^{2m}} \quad \text{and} \quad \mathcal{B}\left(t, \frac{\partial}{\partial x}\right) := \sum_{n\ge 0} \frac{1}{n!} \left(i \tan(t/2)\right)^n \frac{\partial^{2n}}{\partial x^{2n}}.$$

Then the solution of the Cauchy problem

$$i\frac{\partial}{\partial t} t\phi(\mathbf{r}, t) = \left[-\frac{1}{2}\Delta - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{1}{8}(x^2 + y^2)\right]\phi(\mathbf{r}, t),$$
$$\phi(\mathbf{r}, 0) = e^{i(b_1 x + b_2 y + b_3 z)},$$

can be written as

$$\phi_{\mathbf{b}}(\mathbf{r}, t) = \frac{1}{\cos\left(\frac{t}{2}\right)} e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)} \mathcal{A}\left(t, \frac{\partial}{\partial z}\right) \mathcal{B}\left(t, \frac{\partial}{\partial \xi}\right) \mathcal{B}\left(t, \frac{\partial}{\partial \eta}\right) e^{ib_1\xi(x,y,t) + ib_2\eta(xy,t) + ib_3 z}$$

with $\xi(x, y, t) = x + \tan(t/2)y$ *and* $\eta(x, y, t) = y - \tan(t/2)x$.

Proof. We recall that the solution is given by (15) and we rewrite the exponent $\mathcal{E}_{\mathbf{b}}(\mathbf{r}, t)$ as

$$\mathcal{E}_{\mathbf{b}}(\mathbf{r}, t) = ib_1\left(x + \tan\left(\frac{t}{2}\right)y\right) - i\tan\left(\frac{t}{2}\right)b_1^2$$
$$+ ib_2\left(y - \tan\left(\frac{t}{2}\right)x\right) - i\tan\left(\frac{t}{2}\right)b_2^2 + ib_3 z - i\frac{t}{2}b_3^2.$$

Now observe that

$$e^{ib_3 z - \frac{it}{2}b_3^2} = \sum_{m\ge 0} \frac{1}{m!}\left(\frac{it}{2}\right)^m \frac{\partial^{2m}}{\partial z^{2m}} e^{ib_3 z} = \mathcal{A}\left(t, \frac{\partial}{\partial z}\right) e^{ib_3 z}.$$

Moreover, we have

$$e^{ib_2\left(y-\tan\left(\frac{t}{2}\right)x\right)-i\tan\left(\frac{t}{2}\right)b_2^2}=\sum_{n=0}^{+\infty}\frac{1}{n!}\left(i\tan\left(t/2\right)\right)^n\left(ib_2\right)^{2n}e^{ib_2\left(y-\tan\left(\frac{t}{2}\right)x\right)}$$
$$=\sum_{n\geq 0}\frac{1}{n!}\left(i\tan\left(t/2\right)\right)^n\frac{\partial^{2n}}{\partial\xi^{2n}}e^{ib_1\xi(x,y,t)}=\mathcal{B}\left(t,\frac{\partial}{\partial\xi}\right)e^{ib_1\xi(x,y,t)}$$

and with similar computations we get

$$e^{ib_1\left(x+\tan\left(\frac{t}{2}\right)y\right)-i\tan\left(\frac{t}{2}\right)b_1^2}=\mathcal{B}\left(t,\frac{\partial}{\partial\eta}\right)e^{ib_2\eta(x,y,t)}.$$

We thus have

$$e^{\mathcal{E}_{\mathbf{b}}(\mathbf{r},t)}=\left(\mathcal{B}\left(t,\frac{\partial}{\partial\xi}\right)e^{ib_1\xi(x,y,t)}\right)\left(\mathcal{B}\left(t,\frac{\partial}{\partial\eta}\right)e^{ib_2\eta(x,y,t)}\right)\left(\mathcal{A}\left(t,\frac{\partial}{\partial z}\right)e^{ib_3z}\right)$$
$$=\mathcal{A}\left(t,\frac{\partial}{\partial z}\right)\mathcal{B}\left(t,\frac{\partial}{\partial\xi}\right)\mathcal{B}\left(t,\frac{\partial}{\partial\eta}\right)e^{ib_1\xi(x,y,t)+ib_2\eta(xy,t)+ib_3z}$$

and obtain the statement. □

Theorem 5.10. *Let $|a|>1$ and $\mathbf{r}=(x,y,z)$ and let $q_i\in\mathbb{N}$ for $j=1,2,3$ be even numbers such that $q_1=r_2q_2+r_3q_3$ for some $r_2,r_3\in\mathbb{N}$ as in Definition 5.7. Then the solution of the Cauchy problem*

$$i\frac{\partial}{\partial t}\mathcal{Y}_n(\mathbf{r},t)=\left[-\frac{1}{2}\Delta-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+\frac{1}{8}\left(x^2+y^2\right)\right]\mathcal{Y}_n(\mathbf{r},t),\tag{21}$$
$$\mathcal{Y}_n(\mathbf{r},0)=\sum_{j=0}^{n}C_j(n,a)e^{i\left(x\left(1-\frac{2k}{n}\right)^{q_1}+y\left(1-\frac{2k}{n}\right)^{q_2}+z\left(1-\frac{2k}{n}\right)^{q_3}\right)}$$

is given by

$$\mathcal{Y}_n(\mathbf{r},t)=\frac{1}{\cos\left(\frac{t}{2}\right)}e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)}\sum_{j=0}^{n}C_j(n,a)e^{i\left(1-\frac{2k}{n}\right)^{q_1}x+i\left(1-\frac{2k}{n}\right)^{q_2}y+i\left(1-\frac{2k}{n}\right)^{q_3}z}$$
$$\cdot e^{-i\tan\left(\frac{t}{2}\right)\left(\left(1-\frac{2k}{n}\right)^{2q_1}+\left(1-\frac{2k}{n}\right)^{2q_2}\right)-i\frac{t}{2}\left(1-\frac{2k}{n}\right)^{2q_3}-i\tan\left(\frac{t}{2}\right)\left(\left(1-\frac{2k}{n}\right)^{q_2}x-\left(1-\frac{2k}{n}\right)^{q_1}y\right)},\tag{22}$$

where $\mathcal{F}(\mathbf{r},t)$ is defined as in Theorem 5.9. Moreover $\mathcal{Y}_n(\cdot,t)$ converges for any $t\in[0,\pi)$ locally uniformly to $\phi_{\mathbf{a^q}}(\mathbf{r},t)$ with $\mathbf{a^q}=((ia)^{q_1},(ia)^{q_2},(ia)^{q_3})$, i.e.,

$$\lim_{n\to+\infty}\mathcal{Y}_n(\mathbf{r},t)=\frac{1}{\cos\left(\frac{t}{2}\right)}e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)}$$
$$\cdot e^{ix(ia)^{q_1}+iy(ia)^{q_2}+iz(ia)^{q_3}-i\tan\left(\frac{t}{2}\right)\left((ia)^{2q_1}+(ia)^{2q_2}\right)-i\frac{t}{2}(ia)^{2q_3}-i\tan\left(\frac{t}{2}\right)\left((ia)^{q_2}x-(ia)^{q_3}y\right)}.$$

Proof. The identity (22) follows from Theorem 3.2 and the linearity of Eq. (21). In order to show the convergence of the sequence $\mathcal{Y}_n(\cdot,t)$ for $t\in[0,\pi)$, we apply Theorem 5.9 and write $\mathcal{Y}_n(\mathbf{r},t)$ with the position

$$\mathbf{b_{j,\,n}}=((1-2j/n)^{q_1},(1-2j/n)^{q_2},(1-2j/n)^{q_3})$$

as

$$\mathcal{Y}_n(\mathbf{r},t)=\sum_{j=0}^{n}C_j(n,a)\phi_{\mathbf{b_{j,n}}}(\mathbf{r},t)$$
$$=\frac{1}{\cos\left(\frac{t}{2}\right)}e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)}\mathcal{A}\left(t,\frac{\partial}{\partial z}\right)\mathcal{B}\left(t,\frac{\partial}{\partial\xi}\right)\mathcal{B}\left(t,\frac{\partial}{\partial\eta}\right)\mathcal{G}_n(\xi(x,y,t),\mu(x,y,t),z)$$

with $\xi(x,y,t)$ and $\mu(x,y,t)$ defined as in Theorem 5.9 and

$$\mathcal{G}_n(\xi,\mu,z):=\sum_{j=0}^{n}C_j(n,a)e^{i(1-2j/n)^{q_1}\xi+i(1-2j/n)^{q_2}\eta+i(1-2j/n)^{q_3}z}.$$

By Corollary 5.8, the sequence $\mathcal{G}_n(\xi,\eta,z)$ tends to $\mathcal{G}(\xi,\eta,z) := e^{i(ia)^{q_1}\xi+i(ia)^{q_2}\eta+i(ia)^{q_3}z}$ in $\mathcal{A}_{2,0}(\mathbb{C}^3)$, and the theory of AU-spaces and Theorem 5.3 imply that the operators $\mathcal{A}\left(t,\frac{\partial}{\partial z}\right)$, $\mathcal{B}\left(t,\frac{\partial}{\partial \xi}\right)$, and $\mathcal{B}\left(t,\frac{\partial}{\partial \eta}\right)$ act continuously on this space. We thus have for

$$\mathcal{R}_n(\xi,\mu,z) = \mathcal{A}\left(t,\frac{\partial}{\partial z}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \xi}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \eta}\right)\mathcal{G}_n(\xi,\eta,z)$$

that

$$\begin{aligned}\lim_{n\to+\infty}\mathcal{R}_n(\xi,\eta,z) &= \lim_{n\to+\infty}\mathcal{A}\left(t,\frac{\partial}{\partial z}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \xi}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \eta}\right)\mathcal{G}_n(\xi,\mu,z)\\ &= \mathcal{A}\left(t,\frac{\partial}{\partial z}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \xi}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \eta}\right)\mathcal{G}(\xi,\mu,z)\\ &=: \mathcal{R}(\xi,\eta,z)\end{aligned}$$

in $\mathcal{A}_{p,0}(\mathbb{C}^3)$. Convergence in this space however implies uniform convergence on compact subsets of $\mathbb{C}^3$. Now observe that $\mu(\cdot,\cdot,t)$ and $\eta(\cdot,\cdot,t)$ map bounded sets to bounded sets. Thus, for any compact subset $K\subset\mathbb{R}^3$, there exists a compact subset $K'\subset\mathbb{C}^3$ such that

$$\{(\xi(x,y,t),\eta(x,y,t),z) : (x,y,z)\in K\}\subset K'.$$

The uniform convergence of $\mathcal{R}_n$ to $\mathcal{R}$ on K' implies that the functions

$$(x,y,z)\mapsto \mathcal{R}_n(\xi(x,y,t),\eta(x,y,t),z)$$

converge uniformly to $(x,y,z)\to\mathcal{R}(\xi(x,y,t),\eta(x,y,t),z)$ on K because

$$\begin{aligned}&\sup_{(x,y,z)\in K}|\mathcal{R}_n(\xi(x,y,t),\eta(x,y,t),z)-\mathcal{R}(\xi(x,y,t),\eta(x,y,t),z)|\\ &\qquad\le \sup_{(\xi,\eta,z)\in K'}|\mathcal{R}_n(\xi,\eta,z)-\mathcal{R}(\xi,\eta,z)| \overset{n\to+\infty}{\longrightarrow} 0.\end{aligned}$$

Thus we also have

$$\begin{aligned}\lim_{n\to+\infty}\mathcal{Y}_n(\mathbf{r},t) &= \lim_{n\to+\infty}\frac{1}{\cos\left(\frac{t}{2}\right)}e^{\mathcal{F}(\mathbf{r},t)}\mathcal{R}_n(\xi(x,y,t),\eta(x,y,t),z)\\ &= \frac{1}{\cos\left(\frac{t}{2}\right)}e^{-\frac{i}{4}(x^2+y^2)\tan\left(\frac{t}{2}\right)}\mathcal{A}\left(t,\frac{\partial}{\partial z}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \xi}\right)\mathcal{B}\left(t,\frac{\partial}{\partial \eta}\right)e^{i(ia)^{q_1}\xi(x,y,t)+i(ia)^{q_2}\eta(x,y,t)+i(ia)^{q_3}z}\\ &= \phi_{\mathbf{a}^{\mathbf{q}}}(\mathbf{r},t).\end{aligned}$$

□

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