On the Paradox of Reversible Processes in Thermodynamics

ABSTRACT

This paper discusses an argument by Norton (2014, 2016) to the effect that reversible processes in thermodynamics have paradoxical character, due to the infinite-time limit. For Norton, one can "dispel the fog of paradox" by adopting a distinction between idealizations and approximations, which he himself puts forward. Accordingly, reversible processes ought to be regarded as approximations, rather than idealizations. Here, we critically assess his proposal. In doing so, we offer a resolution of his alleged paradox based on the original work by Tatiana Ehrenfest-Afanassjeva on the foundations of thermodynamics.

1 Introduction

Reversible processes play a central role in orthodox thermodynamics. For one, their existence enables one to apply infinitesimal calculus to the description of thermodynamical systems, whereby crucial physical quantities, such as entropy, can be defined as state-functions in equilibrium state space. Furthermore, they are employed in the standard formulations of the Second Law, in order to derive an irreversible increase of entropy for thermodynamical processes. Nevertheless, in two recent papers Norton (2014, 2016) argues that reversible processes have paradoxical character. More to the point, it is usually presumed that they can obtain, at least approximately, when real thermodynamical processes proceed very slowly, and as such they are defined in the limit for time going to infinity. Norton claims that, due to the infinite-time limit, there arises a paradox and hence reversible processes are, strictly speaking, non-sense. That entails two closely related questions: how can the paradox be resolved? and what is the connection of such limit processes with real thermodynamical processes?

According to Norton (2014), one can "dispel the fog of paradox" by resorting to a distinction between idealizations and approximations, which applies to the description of physical systems by means of ideal systems constructed as mathematical limits. Indeed, he offers his analysis of reversible processes as an illustrative example of how limits can fail to yield idealizations: rather, in his view, such limit processes ought to be regarded as providing mere approximations of certain properties of extremely slow real processes. In his subsequent 2016 paper, he further develops his proposal by elaborating a characterization of reversible processes as sets of irreversible processes that is designed to resolve the alleged paradox as well as to secure a connection with real thermodynamical processes, with the aim of avoiding the standard mistakes and misconceptions that he imputes to various other authors. In the present paper, we contend that, his purported distinction does not really do much to resolve the paradox. Instead, we propose that a resolution can be given on the basis of the distinction between "quasi-processes" and "quasi-static processes" developed by Tatiana Ehrenfest-Afanassjewa in her seminal book on the foundations of thermodynamics (1956).

The paper is structured as follows. We begin in the next section by presenting Norton's general formulation of the *Paradox of Thermodynamically Reversible Processes*, together with its concrete application to the case of heat transfer. In section 3, we outline some possible ways out of the paradox. The distinction between idealizations and approximations is then discussed in section 4, where we also point out some critical aspects of Norton's proposal. We conclude in the last section by spelling out a way to resolve the paradox based on Ehrenfest-Afanassjewa's proposal and we defend it from Norton's own criticism.

2 The paradox

2.1 Equilibrium states and thermodynamical processes

Thermodynamics accounts for a wide range of physical processes, as diverse as the transfer of heat between bodies at different temperature and the expansion and compression of gases. It owes its great generality to the fact that it does not make any specific hypothesis about the microscopic constitution of the systems under investigation: rather, it only appeals to macroscopic variables, specifically pressure p, volume V and temperature T. Furthermore, its laws rest on empirically based principles, and as such they are conceived as universal laws of Nature: indeed, the Second Law is sometimes even regarded as grounding the arrow of time itself, thereby allowing us to distinguish between past and future. A crucial feature of the orthodox formulation of thermodynamics is that it is a theory of equilibrium states, whose formalism describes a thermal system just when it is at equilibrium. By definition, in such a state all the relevant variables remain constant, or at least approximately constant. Accordingly, the state space of an individual system is the space Γ_{eq} comprising all equilibrium states that the system can occupy: in general, that is a n-dimensional manifold $\mathbf{R}_n$ with real variables $x_1, x_2, ..., x_n$, wherein the relevant physical quantities and their mutual relations can be expressed in terms of the differentials $dx_1, dx_2, ..., dx_n$. This enables one to represent some thermodynamical quantities of interest, such as internal energy U and entropy S, as functions of state with their own exact differential. When the system is out of equilibrium, though, it cannot be described by any point in Γ_{eq}. In fact, non-equilibrium states have no place in the formalism of orthodox thermodynamics.

In order to describe thermodynamical processes, one needs to specify what is their driving mechanism, as well as whether there is any interaction with the external environment. Moreover, one would like to determine how the state of the system under investigation varies in the course of time. Indeed, a process is conceived as a change of state that begins at an initial time t_i and ends at a final time t_f. In principle, the initial state s_{t_i} and the final state s_{t_f} can be assumed to be equilibrium states, and hence they are represented as points in Γ_{eq}. In order for not to overburden the notation, let us refer to such states as s_i and s_f, respectively. Whether or not the intermediate states of the process can be given a state space description depends on whether they are states of equilibrium. In general, though, they are non-equilibrium states, and therefore the relevant process takes on the following form:

$$\mathcal{P} = \{s_i, s_f\} \tag{1}$$

Here, one does not really have any information about how the process unfolds between t_i and t_f. In fact, for general thermodynamical processes, there is no explicit description of the time-evolution of the system in state space, except for stipulating the distinction between the initial and the final states. Ideally, one would like to introduce an explicit reference to time t, so that the state space description of the process could be given by a parametrised continuous curve in Γ_{eq}, i.e.

$$\mathcal{P} = \{s_t \in \Gamma_{eq} : t_i \leq t \leq t_f\}, \tag{2}$$

whereby the system is assigned a different equilibrium state at any instant (cfr. Uffink 2001, p. 314). However, there are no equations of motion in orthodox thermodynamics; nor does time appear in any of the fundamental laws of the theory (not even the Second Law!). So, a connection with the variable t prescribed by the above form (2) could perhaps be established on the basis of some time-dependent law governing the driving mechanism of the process, but unfortunately the latter is not always available.

Still, even without a formal reference to time, one may hope to refine the state space description $\mathcal{P} = \{s_i, s_f\}$ in such a way to trace a (densely) continuous curve in equilibrium state space connecting the initial and the final states, rather than just fixing the two extremal points. In fact, the existence of equilibrium curves is related to the applicability of differential calculus to thermodynamics. For one, it guarantees that state-functions are well-defined in Γ_{eq} together with their exact differentials. Moreover, it allows one to find integrating divisors for the inexact differentials of those quantities of physical interest, such heat and work, which are not represented by state-functions. In particular, one can express the First Law in the differential form $dU = đQ - đW$. The inexact differential $đQ$ for heat Q has the temperature T as an integrating divisor, so that the exact differential for the entropy S of a thermal system can be written as $dS = \frac{đQ}{T}$. Likewise, for each point along a continuous curve in Γ_{eq} the inexact differential for work W can be written as $đW = \sum_i F_i dx_i$, where F_i are the generalized forces doing work on the system and dx_i are the displacements of each coordinate: for instance, in the case of the expansion of a gas it can take the simple form $đW = p \cdot dV$, with the pressure p of the gas being the integrating divisor and dV the displacement of volume V. In order to apply differential calculus, one thus needs to find equilibrium curves along which one can compute the integrals of such derivative quantities. The notion of reversible processes purports to give the sought-after state-space description.

Reversible processes are commonly defined in the literature as successions of equilibrium state. As such, a reversible process corresponds to an oriented continuous curve $\bar{\mathcal{P}}$ in equilibrium state space. This yields a more refined description of the process than $\mathcal{P} = \{s_i, s_f\}$, in that all intermediate states are represented as points in Γ_{eq} as well. Reversible processes owe their name "reversible" to a reversibility property which is often ascribed to them. Informally, it captures the idea that the process may as well proceed in the opposite direction too. To put it in more precise terms, the reversibility property implies that, given a process between the initial state s_i and the final state s_f described by an oriented continuous curve $\bar{\mathcal{P}}$ in Γ_{eq}, there is also another reverse process that is described by the same curve but oriented in the reversed direction, so that it goes from state s_f to state s_i after tracing the forward process back through all intermediate points. Although it is widely acknowledged that reversible processes cannot be realized in practice, all standard treatments of thermodynamics maintain that they would obtain, at least approximately, by making real processes proceed quasi-statically, namely so slowly and delicately that the system always remains close to equilibrium. This could be achieved by means of driving forces enacting the process. In the case of heat Q being exchanged between the system and its environment the driving forces are given by their temperature difference ΔT, whereas in the case of work W being exchanged the driving forces are given by the generalized forces F_i. Ideally, the process can be made to proceed "quasi-statically" as its driving forces become vanishingly small. In this vein, one feels entitled to suppose that quasi-static processes, and therefore also reversible processes, are constructed in such a quasi-static limit. More to the point, it is a standard presupposition that processes for which the driving forces approach zero would unfold infinitely slowly: accordingly, they are thought of as arising in the infinite-time limit $t \longrightarrow \infty$.

2.2 Norton's argument

Norton's paradox of thermodynamically reversible processes reveals an inconsistency in the understanding of reversible processes as ideal processes, which appears in their construction as infinite-time processes. Norton (2014) first outlines his argument in connection with the process of heat exchange between two bodies at different temperatures. He then develops a general formulation of the alleged paradox, which is presented in the following quotation from his 2016 paper:

> *Paradox of Thermodynamically Reversible Processes*
>
> 1. They are processes with a non-equilibrium imbalance of driving forces, such as non-zero temperature differences or unbalanced mechanical forces; for this imbalance is needed to move the system from one state to another.
> 2. At the same time they are sets of equilibrium states in which, by definition, there is no imbalance of forces; for then the forward and the reverse processes pass through the same set of equilibrium states and both can be represented by the same curve in equilibrium state space. [Norton (2016), p.43]

The paradox hinges on the asymmetric nature of equilibrium states in thermodynamics, which are such that a system out of equilibrium would spontaneously evolve into a unique equilibrium state (equilibration principle) but once the system is in equilibrium it will never spontaneously evolve into any other state. Statement 1 thus establishes a necessary condition for a thermodynamical process to take place: in fact, in order to undergo a change of state, a system needs to be set into non-equilibrium by some external disturbance, like an imbalance of driving forces. Instead, statement 2 depicts a reversible process as a succession of equilibrium states, each of which cannot spontaneously evolve into its successive state. Consequently, the absence of imbalance of forces entails that the system cannot be removed from any equilibrium state, thereby leading one to a contradiction with statement 1. In fact, in the quasi-static limit where the driving forces go to zero the limit process becomes a static process, where no change take place. Even more, if one associates the vanishing of driving forces with time increasing to infinity, the resulting infinite-time limit process presents contradictory properties. For Norton, that is ultimately what renders reversible processes paradoxical, as the example of heat transfer here below illustrates.

Norton (2014) focuses his analysis on the reversible process of heat trasfer between two bodies at different temperatures. Without loss of generality, let us consider a body at temperature T in thermal contact with a heat bath (with infinite capacity) at a different temperature T_{eq}. A non-zero quantity of heat Q is thus transferred from the hotter to the colder system. The process completes when the body is in thermal equilibrium with the bath, that is when it reaches the equilibrium temperature T_{eq}. The total heat exchanged during the process is proportional to the temperature difference $\Delta T = T_{eq} - T$: in fact, it is equal to $Q = C \cdot \Delta T$, where C is the heat capacity of the body. Norton argues that, if we try to convert this thermodynamical process into a reversible process by taking the infinite-time limit, we run into a non-sense. The connection with the variable time is established through the following formula

$$Q = -k \Delta T \Delta t \tag{3}$$

where the constant k is the heat transfer coefficient and Δt is the duration of the process. The formula expresses a time-dependent law governing the driving mechanism of the process, whereby the amount of heat exchanged Q is proportional not only with the driving forces, namely the non-zero temperature difference $\Delta T \neq 0$ but also with time elapsed during the exchange of heat. Accordingly, if one keeps the quantity Q fixed, temperature difference and time duration are inversely proportional. Therefore, when ΔT is made infinitesimally small, Δt grows to infinity so that the process would become infinitely slow. However, in the infinite-time limit the temperature difference becomes zero, which means that there cannot be any heat transfer at all. So, as Norton explains, the ideal process constructed in the limit exhibits the following two inconsistent properties:

- **(a)** A non-zero quantity of heat Q passes from one body to the other
- **(b)** The two bodies are at the same temperature

On the one hand, condition (a) requires that heat is transferred between the the body and the bath due to a non-zero temperature difference being the driving force of the process, just as in statement 1 in the general formulation of the paradox. On the other hand, condition (b) means that the limit process is such that the temperature difference is zero and hence, in accordance with statement 2, the body is in thermal equilibrium with the bath. This implies, in turn, that no heat passed from one to the other, in contradiction with condition (a). The reversible process of heat exchange defined in the infinite-time limit thus appears paradoxical.

3 Ways out of the paradox

Norton's argument aims to show that reversible processes have contradictory properties that arise when one takes the infinite-time limit. In his opinion, although the issue is addressed by almost all developments of the theory, none of the latter offers a satisfactory treatment (possibly with the sole exception of Duhem). In our discussion, we shall not enter into the subtleties of his critique of each of the very many authors he reviews. Rather, we wish to dwell on what it takes exactly to resolve the paradox of thermodynamically reversible process, without indulging at lenght on Norton's historical review and his elaborated taxonomy.

The resolution of the paradox requires one to give a consistent description of the curves in equilibrium state space representing reversible processes. In order to do so, one ought to dispense with one of the two horns of the paradox. Strictly speaking, statement 2 cannot be abandoned, since it contains the very idea that reversible processes are successions of equilibrium states. One is thus left with two alternative ways out of the paradox:

- either one relaxes statement 2 by admitting that, in compliance with statement 1, a reversible process can pass through non-equilibrium states, but their deviations from the states along an equilibrium curve are small enough to be negligible;
- or one drops statement 1 entirely, thereby denying that any reversible process represented by an equilibrium curve corresponds to a thermodynamical process.

According to the first solution, a reversible process is still regarded as a process since it retains the property of change of state characterizing real processes, yet it is an ideal process in the sense that it cannot be actualized in reality as it would take an infinite amount of time to complete. According to the second solution, instead, a reversible process is not a process at all, not even an ideal one: in fact, the alleged paradox would arise only if one mistakenly imposes upon a reversible process the property of change of state, which actually does not belong to it by definition. Let us evaluate both solutions in this order.

The idea underlying the first way out of the paradox is that the ideal process is composed of non-equilibrium states, which are however close enough to equilibrum that the transition from one state into another may as well be represented as a continuous curve in Γ_{eq}. Allowing for non-equilibrum states is necessary for a change of state to take place, and such a change is enacted by non-zero driving forces in the form of a disturbance from equilibrium. The disturbance becomes smaller and smaller as the driving forces diminish: supposedly, for the ideal process, where the driving forces go to zero, it becomes so small to be negligible. The problem is to determine what does it mean exactly for a non-equilibrium state to remain "close enough" to equilibrum. One can identify two main understandings of how disturbances from equilibrium can be deemed so small to be negligible, depending on whether they come as "insensible differences" or as "infinitesimal differences": the former notion presupposes that any difference from equilibrium is effectively too small to matter; the latter appeals to the notion of infinitesimals. Yet, Norton objects that in neither case can one construct an ideal process that proves consistent:

> An "insensible" or "infinitesimally small" disequilibrium is supposed to bring us a non-zero driving force, so that the state of the system is out of equilibrium and changes, while at the same time, the system remains in equilibrium. The disequilibrium that is essential to secure change is also supposed too small to matter. It is to no avail. Incantations of "infinitely slow", "insensible" and "infinitesimal" have no magical powers that overturn the law of the excluded middle. [Norton (2016), p.2]

In support of his thesis, he offers an extensive review of various proposals found in the literature and goes on to criticize all of them as inadequate. We summarize his main arguments here below.

The proposals invoking the notion of "insensible differences" are those that maintain that a reversible process is constructed as an ideal process when the driving forces differ insensibly from zero. In the same vein, some proposals even suggest that, due to the driving forces being so close to balance, a reversible process is such that it can be reversed by very small changes of force. The justification to adopt such proposals is that the intended differences are too small to matter, presumably because they remain always close to zero throughout the process. For Norton (2016, p.52), the difficulty here is that these non-zero differences are just the differences that ought to matter:

> If the deviation from equilibrium is so unimportant, it should be dropped. Of course, it cannot, for then there would be no change in time. It is a difference that makes a difference.

In other words, since the change of state is enacted by small but non-zero differences, if the latter were really so small to be negligible they could not even set the system out of equilibrum. The ideal process must then be a succession of equilibrium states, which cannot change. Yet, that is in contradiction with the assumption that the ideal process is a succession of non-equilibrium states (even though close to equilibrium). That leads back to the original paradox. Norton's concern about the understanding of small disturbances from equilibrium as "insensible differences" reveals the need to provide a more rigorous account of the idea that a non-equilibrium state is "close enough" to equilibrium. We will return to this important issue in the second part of the paper.

The proposals invoking the notion of "infinitesimal differences" attempt to refine the intuition that the differences can be close to zero and yet still non-zero. In fact, infinitesimals are conceived as quantities that are greater than zero but smaller than any positive real number. Treated as such, they appear to be a mathematical non-sense since, strictly speaking, there is no real number corresponding to the thus-defined quantities. Nevertheless, as a matter of fact, infinitesimal calculus is heavily employed in orthodox thermodynamics. Norton then concedes that infinitesimals can be consistently used in ordinary differential calculus as long as they are intended as surrogates of differential operators. What he laments is that "authors in thermodynamics, however, provide no account of unproblematic surrogates for their infinitesimals. Without such an account the contradictions internal to the account remain" (2016, p.54). Furthermore, according to Norton, nor can one hope to overcome the problem by appealing to the infinite-time limit, which is the complement of the infinitesimal driving forces: for, "[t]hese infinitely slow processes are just process in which no changes at all occur. They are frozen for all time in their initial states" (2016, p.55). Indeed, even if one assumes that the infinite slowness is brought about by the driving forces being arbitrarily close to perfect balance, the limit process can be shown to bear inconsistent properties, as in the example of heat transfer.

The second way out of the paradox of thermodynamically reversible processes, differently from the first one, does not simply try to reinterpret the content of statement 2, but it dispenses entirely from one horn of the paradox, namely statement 1. Rather than supposing the existence of ideal thermodynamical processes whose non-equilibrium states remain close enough to equilibrum, it entails that the equilibrium curves described in statement 2 do not correspond to thermodynamical processes, not even in the ideal sense. The paradox is therefore removed in that the condition of change of state introduced by statement 1 does not apply to reversible processes any more. Yet, in order for that to be a satisfactory solution, it ought to fulfill some basic requirements. To begin with, one needs to show that a reversible process can be consistently constructed as an equilibrum curve in the quasi-static limit. Then, one must also explain how such a construction connects with real thermodynamical processes changing in the course of time. In the rest of the paper, we present two possible accounts that deny that reversible processes are thermodynamical processes and we examine whether they fare well with respect to such requirements. The first account is Norton's own proposal to resolve the paradox, which is based on his distinction between idealizations and approximations. We discuss it in greater deal in section 4, where we point out some critical aspects of his argument. The second account bears on the interpretation of reversible processes as purely mathematical constructions. The particular proposal we focus on is based on Carathéodory's construction of infinite-time limit processes free from contradictions and the subsequent distinction between quasi-processes and quasi-static processes put forward by Tatjana Afanassjewa. We take up this proposal in section 5.

4 Idealizations vs Approximations

Norton's (2012) distinction between idealizations and approximations aims to clarify the sense in which ideal systems arising as mathematical limits can provide a description of real systems. To this extent, he defines an approximation as an inexact description of a target system: specifically, it is given by a proposition expressing some property of the target system within a certain margin of inaccuracy, where the degrees of "inexactness" that one may tolerate depends on one's specific purposes. Instead, rather than being propositional, an idealization corresponds to a (possibly ideal) system whose properties yield inexact descriptions, and thus approximations, of some properties of the target system. The question, then, is whether a limit system obtained by letting some parameter go to infinity yields just an approximation of a real system, for which the given parameter is very large but still necessarily finite, or whether it can actually be promoted to the status of idealization. A necessary condition for an idealization is that it should not exhibit contradictory properties. In fact, if the infinite system turns out to be inconsistent, then it cannot exist at all[1].

In Norton (2014), the paradigm example of failure of idealization is given by thermodynamically reversible processes. Indeed, if one tries to construct them as ideal processes in the infinite-time limit, one would run into his alleged paradox. In the particular case of heat exchange discussed in section 2.2, as the temperature difference between the body and the heat bath diminishes one has a sequence of thermodynamical processes $\mathcal{P}_{\Delta t}$ taking an increasingly longer time Δt to unfold: the limit process $\mathcal{P}_\infty$ would then exhibit the inconsistent properties expressed by conditions (a) and (b), i.e.

$$\text{Properties of } \mathcal{P}_\infty : \text{(a) } Q \neq 0 \text{ and (b) } \Delta T = 0$$

In fact, if one takes the limit for $\Delta t \longrightarrow \infty$, the time-dependent law (3) returns the indeterminate form $Q = 0 \cdot \infty$. It follows that the thus-defined infinitely slow process does not exist, and hence it cannot constitute an idealization of any real processes $\mathcal{P}_{\Delta t}$ of heat exchange, no matter how large Δt may be. Allegedly, that would put one in position to resolve the paradox plaguing reversible processes, as Norton (2014) announces in the following quote:

> Whatever [reversible processes] may be, they are not idealizations. Recognizing that fact helps us dispel the fog of paradox that surrounds them (p.200)... How is it possible that they figure so centrally in thermodynamic theory? The answer is that they figure as approximations and not as idealizations (p. 202).

So, in order to "dispel the fog of paradox", one ought to regard reversible processes as mere approximations of real thermodynamical processes that proceed extremely slowly, instead of granting them the status of idealizations. For instance, in the example of heat exchange, the unrealizable limit property of "transferring heat... with no temperature difference", that is the conjuction of the two properties (a) $Q \neq 0$ and (b) $\Delta T = 0$ ascribed to the

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[1] For completeness, let us mention that, for Norton, another necessary condition is that crucial properties of the finite system be preserved in the limit. Here, we shall not be concerned with it, except for noticing that what properties ought to be regarded as crucial is mostly a pragmatic matter, depending on the purpose of the description at stake, thereby making Norton's notion of idealization become context-dependent.

limit process $\mathcal{P}_\infty$, must be understood as yielding an approximation of the property of "transferring heat... with very small temperature difference" that is exhibited by real processes for which Δt is very large but still finite. The latter is the combination of property (a) $Q \neq 0$ and another property which is different from (b), whereby the non-zero temperature difference $\Delta T \neq 0$ can be set arbitrarily small. The notion of approximation thus allows one to recover limit properties without the need to posit an ideal process with contradictory properties. Moreover, it draws a connection with real thermodynamical processes. That is how, in Norton's account, one should reform the talk of reversible processes in order to avoid to fall into a non-sense.

This proposal is further developed and made more precise in Norton's 2016 paper. There, he gives an explicit characterization of reversible processes free of paradox, which is designed to overcome the difficulty related to the notion of "close enough" to equilibrium discussed in the previous section. In the setting envisaged by Norton, a reversible process is defined as a continuous set of irreversible processes, each of which exchanges work or heat with its surroundings due to an imbalance of driving forces. Such irreversible processes are made entirely of non-equilibrium states that may get arbitrarily close to equilibrium states. The defining set can be further subdivided into a forward set and a reverse set, so that the total heat gained and the total work done have opposite signs. In the limit when the net driving forces go to zero, the non-equilibrium states of the processes in both sets approach the same succession of equilibrium states. The differential relations $đW = \sum_i X_i dx_i$ and $đQ = dU + đW$ can thus be integrated over the corresponding curve in Γ_{eq}. If one does so, the values of the integrals correspond to the limit values of work done W_f and heat gained Q_f in the forward process and to the limit values of work done $W_r = -W_f$ and heat gained $Q_r = -Q_f$ in the reverse process, respectively. Here, the reversibility property is implemented by the existence of both forward and reverse processes: as such, it is no more attached to the vague idea of a single process being so close to equilibrium that it may as well be reversed a suitably small change of the driving forces. In fact, the concept of "close enough" to equilibrium is now shifted to those irreversible processes for which the net driving forces are arbitrarily small. Supposedly, such processes have the property of completing in a finite amount of time, just like real thermodynamical processes effectively do. Within this framework, Norton thus generalizes the notion of approximation we discussed above for the example of heat exchange: the limit properties obtained in the quasi-static limit, far from belonging to an ideal process, are just inexact descriptions of properties of real target processes changing in the course of time.

To summarize, Norton's proposed resolution of the paradox of thermodynamically reversible processes appeals to his distinction between idealizations and approximations, which is then enforced by a definition of reversible processes as sets of irreversible processes grounding an allegedly unproblematic notion of “close enough” to equilibrium. But does that really “dispel the fog of paradox”? In the conclusive part of this section we take on this issue, and we pinpoint some critical aspects of Norton’s proposal.

4.1 Critique of Norton’s proposal

To begin with, the question whether thermodynamically reversible processes should be regarded as idealizations or not arguably depends on what is the specific notion of idealization one adopts. After all, the literature abounds with so many diverse stipulations of what the term could mean. Nevertheless, for Norton that is more than just a matter of nomenclature:

> It is accepted that thermodynamically reversible processes involve physical impossibilities. The awkwardness is then excused by calling them idealizations... But merely declaring something an idealization produced by taking a limit is no guarantee that the result is well-behaved. The result need not be... Thermodynamically reversible processes turn out to be just such a troublesome case in which the uncritical taking of limits brings disaster. [p.46]

As examples of authors who made the asserted disastrous mistake of regarding reversible processes as idealizations Norton quotes Goodenough (1911) and Zemansky (1968). From our own perspective, we are not quite concerned with whether the latter, or even other authors sharing the same intuition, may have actually intended the notion of idealization in the way that was subsequently defined by Norton himself and, if so, whether they would deserve any blame for their mistake. What we wish to emphasize, instead, is that the resolution of the paradox of reversible processes rests just on the recognition that, exactly because they are physically impossible, reversible processes are not thermodynamical processes at all, independently from what nomenclature one adopts to characterize them once they are constructed in the infinite-time limit.

More to the point, the infinite-time process $\mathcal{P}_\infty$ does not need to coincide with a reversible process $\bar{\mathcal{P}}$ represented by a curve in equilibrium state space. In fact, to return to example of heat exchange, even though $\bar{\mathcal{P}}$ also obtains in the quasi-static limit when the temperature difference vanishes, it just proves to be a static process during which, contrary $\mathcal{P}_\infty$, no heat is exchanged at all. Norton (2014) himself explicitly acknowledges its esistence: “Once we set $\Delta T = 0$ we must have $Q = 0$ as well, no matter what the time... That limit process consists of two bodies at the same temperature passing no heat between them" (p. 203). Indeed, strictly speaking, the quantity of heat Q exchanged between the bodies in thermal contact is supposed to diminish together with their temperature difference, and hence it ought to become zero in the limit for $\Delta T = 0$ where the reversible process is defined. The mismatch between $\mathcal{P}_\infty$ and $\bar{\mathcal{P}}$ is due to the fact that the infinite-time limit is constructed on the basis of formula (3) and Norton requires that, rather than diminishing, the quantity $Q = -k\Delta T \Delta t \neq 0$ remains fixed while the temperature difference ΔT is made smaller and smaller. For one, this example shows that the infinite-time limit does not always yield a reversible process: in particular, that depends on how one takes the limit in the relevant time-dependent law one appeals to. Furthermore, and more importantly for our purposes here, it means that the fact that the infinite-time limit process has contradictory properties does not entail that a reversible process turns out to be inconsistent as well. For instance, the construction $\bar{\mathcal{P}}$ in the case of heat transfer fulfills the equilibrium condition (b), but it does not satisfy condition (a). In other words, it exhibits the following mutually compatible properties:

$$\text{Properties of } \bar{\mathcal{P}} : \text{not-(a) } Q = 0 \text{ and (b) } \Delta T = 0$$

As such, the reversible process $\bar{\mathcal{P}}$ does not really fail to be an idealization according to Norton's own criterion. Instead, given that condition (a) is necessary for heat exchange to take place, the thus-constructed limit process just fails to be an actual process. In fact, it does not undergo any change of state, since the two systems put in thermal contact are always in equilibrium. Accordingly, one can give up one horn of Norton's paradox, namely statement 1, while still maintaining that a reversible processes traces an equilibrium curve, as prescribed by statement 2. Therefore, adjudicating whether one may or may not regard reversible processes as idealizations does not quite seem to be what helps one "dispel the fog of paradox" surrounding them.

As for Norton's claim that reversible processes yield mere approximations of properties of real thermodynamical processes, in our view his proposal has some merits, but it is also subject to a severe formal limitation. Certainly, defining a reversible process by means of a set of irreversible processes whose quasi-static limit is a curve in equilibrium state space enables one to remove the reference to a single process with contradictory properties. Moreover, such irreversible processes bear a connection with real thermodynamical processes, in that they also pass through non-equilibrium states and complete in a finite amount of time. In this vein, extremely slow processes for which the imbalance of driving forces is sufficiently small can have properties that are approximated by the corresponding limit quantities computed over the equilibrium curve. Nonetheless, there arises an issue concerning Norton's intended notion of "close enough" to equilibrium. That is that, in order for one to provide a rigorous formalization of such a notion, one ought to settle the question whether, and when, a non-equilibrium state is sufficiently close to an equilibrium state on a topological ground. However, since orthodox thermodynamics is a theory of equilibrium states, its state-space description does not include non-equilibrium states. This means that no metric in the state space of the theory can tell us how close an equilibrium point is to any non-equilibrium state, given that the latter is not represented by a point in Γ_{eq}. Formally, a topological definition of closeness to equilibrium, and with it even a rigorous notion of approximation, can only be given by embedding Γ_{eq} into a larger non-equilibrium state space endowed with a suitable metric. Yet, such a description is not available within orthodox thermodynamics. Therefore, the irreversible processes Norton posits in his purported definition of a reversible process lack a formal representation in state-space, in that they consist of non-equilibrium states: consequently, the sense in which some of them may be approximated by an equilibrium curve cannot have any topological underpinning.

To circumvent this limitation, Norton tries to establish the concept of "close enough" to equilibrium on a different ground. In his view, given that the irreversible processes defining a reversible process form a continuous set, one can always find some processes for which the values of the relevant thermodynamical quantities are arbitrarily close to the values computed by integration along an equilibrium curve. The choice of such irreversible processes is then justified by assuming that they correspond to real thermodynamical processes that, in practice, remain as close to equilibrium as one wishes. Nevertheless, here one could object that this assumption is somewhat too strong, in that it excludes the typical behaviour of thermodynamical processes. Let see why by examining the nature of the irreversible processes posited by Norton more carefully. As he describes them,

> [w]hile they may come arbitrarily close, none of the states of the irreversible processes are exactly equilibrium states. For otherwise the processes cannot complete in any finite time. [Norton (2016), p.45]

Presumably, these processes could be thought of as real processes because they have the following two properties: they take a finite amount of time to complete and they go through non-equilibrium states. However, Norton's definition also imposes a more stringent condition, namely that all states of an irreversible process be non-equilibrium states. As a matter of fact, this property is typically not shared by real processes: for, typical thermodynamical processes $\mathcal{P}$, whether they are force-driven or spontaneous, begin in some initial state s_i of equilibrium, move into a non-equilibrium state due to some external disturbance, but eventually ends in a final state s_f when equilibrium is reached. Accordingly, their extremal states ought to be points in equilibrium state space Γ_{eq}, quite contrary to Norton's irreversible processes, which instead are never in any equilibrium state. In follows that his definition leaves out a large class of thermodynamical processes, namely those that complete when the system undegoing the process attains equilibrium. So, in this respect the sought-after connection with real processes, which is required in order to obtain a satisfactory resolution of the paradox of reversible processes, is less well-grounded within Norton's proposal than it may perhaps appear at first.

5 Reversible processes as "quasi-processes"

This section aims to outline a way out of the paradox of thermodynamically reversible processes that draws from the work of Tatiana Ehrenfest-Afanassjewa. According to this proposal, one denies that reversible processes are thermodynamical processes. Differently from Norton's proposal, though, one does not need to posit any set of irreversible processes for the sake of constructing a reversible process. Rather, a reversible process is defined just as a continuous curve in equilibrium state space: as such, it is a mere mathematical construction, which serves as a tool for differential calculus. In order to prevent one from interpreting reversible processes as actual thermodynamical processes, Ehrenfest-Afanassjewa (1956) proposes to refer to them as *quasi-processes*, and she carefully distinguishes the intended notion from that of quasi-static processes. The relevant definition is elaborated in a section titled "Quasiprozesse und Quasistatische Prozesse" of the opening chapter of her book on the foundations of thermodynamics, which develops an earlier 1925 paper on the axiomatization of the theory:

> [C]ontinuous sequences of equilibrium states that connect two given equilibrium states... are represented graphically by curves in $\mathbf{R}_n$. They have have always been given the name of "processes" and in particular "reversible" processes. We will prefer to call them "*quasi-processes*," because - evidently - one cannot realize them through a real process; in this regard however we will delib-

> erately omit the adjective “reversible”. [Ehrenfest-Afanassjewa (1956) p.13[2]]

Ehrenfest-Afanassjewa’s nomenclature is meant to emphasize the fact that a quasi-process is not physically realizable and as such it cannot be a thermodynamical process. It is simply an oriented equilibrium curve $\bar{\mathcal{P}}$ along which one can integrate derivative operators. In addition, the proposed name excises the misleading reference to the reversibility property: since quasi-processes are not real changes of state, there is no forward or backward direction in which they take place and hence, strictly speaking, they cannot be reversible. In fact, the reversibility property does not have anything to do with the direction of thermodynamical processes. There is no physical meaning attached to the idea that an equilibrium curve $\bar{\mathcal{P}}$ can be traversed in both directions, in that there is no underlying process taking place. In the same vein, in the Preface to the English translation of the 1912 seminal book on the foundations of statistical mechanics, Ehrenfest-Afanassjewa (1959) reiterates that the existence of an integrating divisor T for the inexact differential $đQ$ “has to do only with the mathematical espression of $đQ = dU - đW$ in terms of the differentials $dx_1, dx_2, ..., dx_n$ of the *equilibrium* parameters $x_1, x_2, ..., x_n$ and is completely independent of the direction in time of the development of the natural processes” (p.ix)[3]. As a quasi-process, a reversible process just enables one to integrate the exact differential $dS = \frac{đQ}{T}$ along the corresponding curve in Γ_{eq}, but it does not refer to any thermodynamical process that can be reversed: in fact, it does not refer to any process at all.

The intepretation of reversible processes as mathematical constructions can only provide a satisfactory resolution of Norton’s paradox if both their construction proves consistent and a connection with real thermodynamical processes is established. Let us examine how Afanassjewa’s proposal can fulfill these requirements.

An example of how one can construct reversible processes free of paradox is found in Carathéodory’s (1909) axiomatic approach to thermodynamics, where time t appears as an explicit variable. Carathéodory specializes his account to simple systems, which he defines on the basis of precise assumptions. First of all, for simple systems any state in the n-dimensional equilibrium space Γ_{eq} is fixed by one independent thermal coordinate, e.g. the internal energy U of the system, plus $n-1$ deformation coordinates $x_1(t), ..., x_{n-1}(t)$

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[2] The author is grateful to Lena Zuchowski for kindly helping with the English translations of Afanassjewa’s quotations from the original text in German.

[3] More to the point, she warns that thermodynamically reversible processes should not be related to the reversibility of the microscopic laws of motion in statistical mechanics. Cfr. Uffink (2001, p.218 and p.317) for a similar claim, which we also endorse.

continuosly changing in the course of time, which in turn determine other thermodynamical quantities such as volume V. Moreover, he assumes that from any fixed state one can reach any other state with different deformation coordinates by means of more than one adiabatic process, where different processes differ on the amount of work done on the system[4]. In fact, during the expansion of a gas from a given volume at the initial time t_i to a greater volume at the final time t_f the amount of work depends on the rate at which the process unfolds in the course of time, that is on how slowly it proceeds. A "quasi-static change of state" is thus defined in the limit when the derivates of $x_1(t), ..., x_{n-1}(t)$ with respect to t converge uniformly to zero. Finally, Carathéodory states that the total work done during a reversible process constructed in such a quasi-static limit is uniquely determined by the time-dependent expression

$$W(t) = \int_{t_i}^{t_f} đW \qquad (4)$$

where the inexact differential takes on the Pfaffian form $đW = p_1 dx_1 ..., p_n dx_n$, with the p_n's being functions of the state variables. In virtue of (4), one can then show that the existence of an integrating divisor T for the differential form $đQ = dU - đW$ is equivalent to Carathéodory's principle[5]. We do not need to review this principle here, except for noticing that it expresses the Second Law by entailing that for some adiabatic processes, such as the irreversible free expansion of an isolated gas, the entropy difference $S(s_f) - S(s_i)$ between the final and the initial time must be different from zero. What is important to stress for our purposes is that Carathéodory demonstrates that reversible processes can be consistently constructed as equilibrium curves, and that one can do so by taking a rigorously defined infinite-time limit.

Norton himself recognizes that Carathéodory's description of reversible quasi-static processes is free of paradox. At the same time, though, he contests that, notwithstanding its formal rigor, it is inadequate in that it "proved too terse to fix a definite meaning for quasi-static processes in the subsequent literature" (2016, p.57). According to him, there are at least three distinct readings of the intended notion, which he attributes to different authors: that is, bare curve (Ehrenfest-Afanassjewa 1956, Redlich 1968, Pauli 1973), force-balanced curve either in the strict formulation (Wilson 1957, Reiss 1965, Pippard 1966, Buchdahl 1966, Linder 2004) or in the qualified formulation (Zemansky 1968), and iterated equilibria (Ehrenfest-Afanassjewa

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[4] In addition, he also supposes that the values of work for all such adiabatic processes form a connected interval.

[5] That is: in every arbitrarily close neighbourhood of a given state s in Γ_{eq}, there are states s' that cannot be approached arbitrarily closely by any adiabatic process.

1956, Landberg 1978, Callen 1985, O'Connell and Haile 2005). Norton then goes on to criticize in some detail all such readings of the notion of quasi-static process and rejects them as inadequate. Here, we just focus on how the proposal based on Ehrenfest-Afanassjewa's distinction between quasi-processes and quasi-static processes settles the ambiguity in Carathéodory's construction. In doing so, we subsequently offer a defense of Ehrenfest-Afanassjewa's proposal from Norton's criticism (section 5.1).

The alleged ambiguity in the interpretation of Carathéodory's approach stems from an assumption that he makes in his derivation: that is that "[d]uring any quasistatic reversible process, the external work... can be so determined as if the forces that produce it were equal to those required to maintain equilibrium" (p.238). Clearly, this provision is analogous to the equilibrium condition (b) in the example of heat exchange, whereby the temperature difference vanishes. In fact, if there is no imbalance of forces, no work can be done at all. For instance, when the pressure p of a gas equals the pressure of its environment, the gas cannot expand, and yet one can compute the total work done in the process by integrating $đW = pdV$ over the equilibrium curve, just as in formula (4). On the other hand, Carathéodory also adds that, if a process takes place at a sufficiently slow rate, the difference between the work done on the system and $W(t)$ becomes "smaller than the uncertainty of our measurements" (p.238). The ambiguity comes from the fact that he seems to refer to a quasi-static reversible process both as the ideal process constructed in the limit and as a real thermodynamical process that remains close enough to equilibrium. Ehrenfest-Afanassjewa's distinction sharply separates these two cases: the former corresponds to a quasi-process, whereas the latter ought to be regarded as a quasi-static process. To this extent, she elaborates a definition of quasi-static processes that purportedly draws a connection with real thermodynamical processes. Let us spell it out here below.

Although quasi-processes cannot be realized, Ehrenfest-Afanassjewa explains that, by means of an appropriate "forced change of state", the system can be made to assume any possible equilibrium state. She then goes on to argue that an equilibrium curve representing a quasi-process can be approximated by a "quasi-static process", which is in turn defined as

> a dense discrete sequence of equilibrium states which belong to [a quasi-process], in such a way that during the process that connects two consecutive equilibrium states, the state of the system is also close to an equilibrium state. [p.13]

This is a conception of quasi-static processes as iterated equilibria, whereby the process goes from one equilibrium state to the next along the given discrete sequence of points in Γ_{eq} by passing through intermediate non-equilibrium states. An *elementary quasi-static process* is just the process that connects two consecutive equilibrium states in a quasi-static process. Hence, a quasi-static process amounts to a sequence of elementary quasi-static processes. During each stage, the change of state proceeds quasi-statically in the sense that, while it is forced by an imbalance between some parameters of the system and its environment, such parameters still remain close enough to the equilibrium conditions. Supposedly, when the imbalance vanishes, that is when the driving forces go to zero, the dense discrete sequence of equilibrium points representing a quasi-static process will approach the continuous curve in Γ_{eq} representing a quasi-process. The two concepts are, however, sharply distinct: for, quasi-processes are not thermodynamical processes but purely mathematical objects, whereas quasi-static processes are supposed to be thermodynamical processes, albeit idealized ones. But how is the sought-after connection with real processes made?

For one, real processes complete in a finite amount of time. Yet, since a quasi-static process is a sequence of elementary quasi-static processes, its duration increases with the number of such elements and thus, when the latter goes to infinity, the whole quasi-static process would become infinitely slow. Here, Ehrenfest-Afanassjewa is careful to note that, however, the notions of quasi-static processes and infinitely slow process should not get conflated. More to the point, she warns that the infinite-time limit must not be taken literally: the idea of infinitely slow simply means that "the process at issue is sufficiently slow that in each of its states we may thereby neglect certain deviations from equilibrium states" (p.11). The connection with real thermodynamical processes taking a finite time to unfold is instead made through the concept of elementary quasi-static processes. Indeed, the duration of a quasi-static change of state is connected with the driving forces in the following way:

> the average velocity of the parameter exchange - $\frac{q-q^1}{\Delta t}$ - where q is the initial and q^1 is the final value of the parameter at issue (...) and Δt the duration of the process, will be the smaller, the smaller the parameter difference... $q - q^1$ is. [p. 12]

This means that any given elementary quasi-static process has fixed finite duration Δt. Even in the limit when the difference $q - q^1$ between the initial and final values of the parameters vanishes, the velocity of change goes to zero without entailing that Δt increases to infinity. Indeed, elementary quasi-static processes are processes that complete in a finite amount of time. Moreover, just like real thermodynamical processes, they also go through non-equilibrium states. Accordingly, just as the processes in the continuous set defining a reversible process in Norton's account, elementary quasi-static processes are irreversible processes. However, differently from the irreversible processes posited by Norton, they exhibit another important property of real processes, namely that their initial and final states are equilibrium states, which as such are represented by points in Γ_{eq}. In fact, this property is built into the very idea of iterated equilibria.

5.1 Defense of Ehrenfest-Afanassjewa's proposal

Ehrenfest-Afanassjewa's treatment of quasi-static processes thus provides the means to establish a connection of equilibrium curves with real thermodynamical processes, which assuage some of the difficulties of Norton's proposal. More to the point, her distinction between quasi-processes and quasi-static processes serves the purpose of elucidating how to solve the paradox of thermodynamically reversible processes in a way that, in our opinion, appears more perspicuous than Norton's distinction between idealizations and approximations. Indeed, Ehrenfest-Afanassjewa's characterization of reversible processes as quasi-processes directly removes the ambiguity that stems from regarding them as thermodynamical processes changing in the course of time, as well as the misconception that they could also proceed in the reverse direction. Granted, the two proposals do not seem to differ too much from each other. But it is worth to noticing that the basic intuition underlying Norton's attempt to solve the paradox was already contained in the original work by Ehrenfest-Afanassjewa on the foundations of thermodynamics.

Nonetheless, Norton objects that Ehrenfest-Afanassjewa's account is still unsatisfactory. As he puts it, "[q]uasi-static processes as iterated equilibria are real processes unfolding in time. They depend on just the troublesome notions we have seen in other treatments" (p. 59). He offers two reasons in support of his claim. One is that the concept of "close enough" to equilibrium Ehrenfest-Afanassjewa adopts suffers from the standard difficulties of other proposals. The other is that the alleged jump from one equilibrium point to the next characterizing the account based on iterated equilibria cannot really complete. We address the two points of criticism in this order, before concluding our analysis.

In Norton's view, what makes Ehrenfest-Afanassjewa's notion of quasi-static processes troublesome is the fact that "[t]he states through which the process passes are not equilibrium states, but they are so close that they are 'as good as' or 'well-nigh' (p.59)". Indeed, as a quasi-static process proceeds extremely slowly its deviations from equilibrium can be neglected. Norton's concern is that her concept of "close enough" to equilibrium just runs into the same difficulties as the other proposals he criticized, which we reviewed in section 2.1. Nevertheless, it should be stressed that, as we explained there, in such proposals the property of being "close enough" to equilibrium is attributed to ideal processes represented by equilibrium curves: allegedly, that renders their definition unsuitable inasmuch as one cannot formally determine the sense in which their deviation from equilibrium is sufficiently small for not to matter, thereby leading one to the original paradox of reversible processes. Yet, according to Ehrenfest-Afanassjewa's nomenclature quasi-static processes are not ideal processes at all, and hence Norton's objection does not apply to them. Rather, a quasi-static process is supposed to be a real thermodynamical process, and as such it can get as close as one likes to equilibrium conditions without ever becoming an inconsistent process. Specifically, what she means by "close enough" to equilibrium is described in the following quotation:

> the state of the system throughout the process is close to an equilibrium state; the values of the intensity parameters, through which the system is coupled with the outer world, can be set equal to those of the outer world without perceptible errors. [p. 12]

Arguably, the idea of closeness to equilibrium intended here is reminiscent of the notion of "insensible differences", in that the imbalance between the parameters is assumed to be too small to matter. Differently from the troublesome accounts examined in section 3.1, though, Ehrenfest-Afanassjewa gives an empirical underpinning to the claim that a difference is insensible: that is, the imbalance between parameters ought to be such that it falls beneath "perceptible errors". That has a two-fold advantage, since it clearly specifies with respect to what the small differences at stake are supposed not to matter as well as when can they be deemed sufficiently small for not to matter. Indeed, within the present proposal, the differences turn out to be insensible with respect to our means of observation, and as such they can be neglected just when they are too small for one to be able to detect them. One thus has an empirically grounded criterion to establish the extent to which deviations from equilibrium become unimportant, which readily applies to real thermodynamical processes such as Ehrenfest-Afanassjewa's quasi-static processes. In our view, given the impossibility of introducing a topology determining the distance between non-equilibrium and equilibrium states within the formalism of thermodynamics, the thus-defined concept of "close enough" to equilibrium appears to be rather appealing: in fact, it is possibly the best one can ask for within a theory where a rigorous state-space representation can be given solely for equilibrium states.

Norton's second objection to Ehrenfest-Afanassjewa contends that in a quasi-static process a change cannot really complete. His argument goes as follows. Let us consider again a body with heat capacity C in thermal contact with a heat bath, and let us suppose that its temperature T is higher than the temperature T_e of the bath. The body loses heat to the bath until it reaches the same temperature. Yet, if one assumes that the rate of heat transfer is $k(T - T_e)$, the time-evolution of the process would be governed by the differential equation $\frac{d}{dt}C(T - T_e) = -k(T - T_e)$. Since the solution of the latter for the variable T approaches the limit T_e only asymptotically, the final temperature may never be reached. It then follows that the change of temperature of the body would not complete in any finite amount of time. Norton's argument rests on the application of the time-dependent law $Q = -k(T-T_e)\Delta t$ for heat transfer, which makes temperature difference and time duration inversely proportional for any fixed Q. To be sure, his conclusion is formally correct. Yet, it seems to pose a challenge for all accounts based on orthodox thermodynamics, wherein differential equations apply only to equilibrium states, rather than just for the iterated equilibria approach: indeed, the argument does not depend on any feature that is peculiar to the latter. More to the point, it reveals a tension between the formalism of the theory, which admits no autonomous equation of motion and time does not appear as a fundamental parameter, and the establishment of independent laws governing the time-evolution of a process.

Here, we suggest two possible solutions to cope with Norton's conclusion, which can both be adopted within Ehrenfest-Afanassjewa's proposal. For one, we could remove any formal dependence of the temperature difference with respect to time. In this vein, in accordance with the above quotation by Ehrenfest-Afanassjewa about the average velocity of parameter exchange, the change of state in an elementary quasi-static process from the initial temperature T of the body to the final equilibrium temperature T_e takes place at velocity $\frac{T-T_e}{\Delta t}$: then, if Δt is kept fixed, the velocity of change diminishes with the temperature difference and goes to zero when $T - T_e$ vanishes, but that would happen without the time duration of the process increasing to infinity. Of course, this has the drawback that one ought to dispense from the time-dependent formula (3) for heat transfer, notwithstanding the fact that such a law has important applications in thermal physics. Alternatively, we could adopt a less radical solution, which resorts to Ehrenfest-Afanassjewa's notion of "close enough" to equilibrium. That is, one can maintain the time-dependent law (3) for heat transfer, so that the solution of the above differential equation for the variable T is still asymptotic to the equilibrium temperature T_{eq}. Yet, while this means that the latter cannot be attained in any finite time, when the process proceeds extremely slowly (namely for very large times Δt), the body will eventually reach a temperature that is so close to that of the thermal bath with which it is in contact that their temperature difference cannot be detected by our means of observation. In other words, the actual temperature T will be empirically indistiguishable from T_{eq}, even though strictly speaking one can never obtain thermal equilibrium. One may perhaps not find the present proposal fully satisfactory. However, in this respect, Ehrenfest-Afanassjewa's account appears no less wanting than Norton's own notion of approximation, contrary to what he contends in his objection.

6 Conclusion

To conclude, Ehrenfest-Afanassjewa's proposal offers the prospect to resolve Norton's paradox of thermodynamically reversible processes, and it does so in a way that appears more perspicuous than Norton's distinction between idealizations and approximations. Accordingly, reversible processes are not inconsistent since they are quasi-processes, which means that they correspond just to continuous curves in equilibrium state-space Γ_{eq}: as such, they should not be interpreted as thermodynamical processes, not even ideal ones. The connection with real thermodynamical processes is then established thanks to her notion of quasi-static processes as iterated equilibria, which hinges on an empirically grounded concept of "close enough" to equilibrium. Admittedly, her account does not provide a complete formal description of how real thermodynamical processes evolve throughout non-equilibrium states in the course of time. Yet, that seems an unavoidable price to pay when dealing with a theory such as orthodox thermodynamics where the variable time t does not appear explicitly in the fundamental equations and there lacks a metric determining how close non-equilibrium states are to equilibrium.

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