

SEGMENTS AND HILBERT SCHEMES OF POINTS

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ABSTRACT. Using results obtained from the study of homogeneous ideals sharing the same initial ideal with respect to some term order, we prove the singularity of the point corresponding to a segment ideal with respect to the revlex term order in the Hilbert scheme of points in \mathbb{P}^n . In this context, we look inside properties of several types of “segment” ideals that we define and compare. This study led us to focus our attention also to connections between the shape of generators of Borel ideals and the related Hilbert polynomial, providing an algorithm for computing all saturated Borel ideals with the given Hilbert polynomial.

INTRODUCTION

The Hilbert scheme can be covered by some particular affine varieties [3, 7, 19, 23, 12] that have been called *Gröbner strata* in [12] because they are computed from a monomial ideal by Gröbner basis techniques. The behaviour of Gröbner strata can give interesting information on the Hilbert scheme itself. Very recently, in [22] M. Roggero showed that an open covering of the Hilbert scheme can be constructed from Borel ideals by avoiding to involve any term order, that is instead needed for Gröbner strata. This fact gives us further reasons to investigate Borel ideals and their very particular features.

Among Borel ideals there are several types of “segment” ideals, the definitions of which are already well known or arise from some interesting properties of Gröbner strata studied in [12] (Definitions 3.1 and 3.7). In section 3 we characterize the existence of this kind of ideals in terms of the corresponding Hilbert polynomial, in some cases. In this context we need to focus our attention also to the shape of admissible polynomials.

In [10] the coefficients of Hilbert polynomials are completely characterized by the numbers of components of certain subschemes defined by very particular ideals called *fans*. In [21] these numbers of components are described by the shape of minimal generators of Borel ideals. Although the geometric meaning is contained in the fans, in section 4 we observe that this connection between coefficients of Hilbert polynomials and minimal generators of Borel ideals can be described without using fans, but directly by the combinatorial properties of Borel ideals themselves. This study led us to project an algorithm to compute all saturated Borel ideals with a given Hilbert polynomial. In section 5 we describe this procedure.

In [20] and in [17] the smoothness of points of Hilbert schemes is studied by means of the dimension of the vector space of the global sections of the normal sheaf to the corresponding projective subscheme. In section 6, applying results of [12] about Gröbner strata, we make some new consideration (Theorem 6.2) on smoothness of points in the Hilbert scheme Hilb_d^n and, in particular, prove the main result of this paper, i.e. the point of Hilb_d^n corresponding to the segment ideal with respect to the revlex term order is singular (Theorem 6.4). In literature we have not found any proof of such a result.

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1. GENERAL SETTING

Let K be an algebraically closed field of characteristic 0, $S := K[x_0, \dots, x_n]$ the ring of polynomials over K in $n+1$ variables endowed so that $x_0 < x_1 < \dots < x_n$ and $\mathbb{P}_K^n = \text{Proj } S$ the n -dimensional projective space over K .

A *term* of S is a power product $x^\alpha := x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $\alpha_0, \alpha_1, \dots, \alpha_n$ are non negative integers. We set $\min(x^\alpha) := \min\{i : \alpha_i \neq 0\}$ and $\max(x^\alpha) := \max\{i : \alpha_i \neq 0\}$. We also let $\mathbb{T} := \{x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{n+1}\}$ be the monoid of all terms of S .

A *graded structure* on S is defined by assigning a *weight-vector* $w = (w_0, \dots, w_n) \in \mathbb{R}_+^{n+1}$ and imposing $v_w(x^\alpha) = \sum_{i=0}^n w_i \alpha_i$. For each non negative integer t , S_t is the K -vector space spanned by $\{x^\alpha \in \mathbb{T} : v_w(x^\alpha) = t\}$. The *standard grading* corresponds to $w = (1, \dots, 1)$ and, unless otherwise specified, we will always consider it.

For any $N \subseteq \mathbb{T}$, N_t is the set of the t -degree elements of N and $\lambda_{i,t}(N) := |\{x^\alpha \in N_t : i \leq \min(x^\alpha)\}|$ denotes the cardinality of the subset of terms of N_t which are not divided by x_0, \dots, x_{i-1} . For any homogeneous ideal $I \subseteq S$, I_t is the vector space of the homogeneous polynomials in I of degree t ; $I_{\leq t}$ and $I_{\geq t}$ are the ideals generated by the homogeneous polynomials of I of degree $\leq t$ and $\geq t$, respectively.

Fixed any *term-order* \preceq on \mathbb{T} , each $f \in S$ has a unique ordered representation $f = \sum_{i=1}^s c(f, \tau_i) \tau_i$, where $c(f, \tau_i) \in K^*$, $\tau_i \in \mathbb{T}$, $\tau_1 \succ \dots \succ \tau_s$, $T(f) := \tau_1$ is the *maximal term* of f . For any $F \subset S$, $T\{F\} := \{T(f) : f \in F\}$, $T(F) := \{\tau T(f) : f \in F, \tau \in \mathbb{T}\}$ and $\mathcal{N}(F) := \mathbb{T} \setminus T(F)$. For any ideal $I \subset S$, $T\{I\} = T(I)$ and $\mathcal{N}(I)$ is an *order ideal*, often called *sous-escalier* or *Gröbner-escalier* of I . A subset $G \subset I$ is a *Gröbner-basis* of I if $T(G) = T(I)$ (see for instance [18]).

For a monomial ideal I , $G(I)$ denotes the unique set of minimal generators of I consisting of terms.

Remark 1.1. (1) In our setting, we consider on \mathbb{T} mainly (standard) graded term orders. In particular, given two terms x^α and x^β of \mathbb{T} of the same degree t , we say that x^α is less than x^β with respect to:

- (i) *lex* order if $\alpha_k < \beta_k$, where $k = \max\{i \in \{0, \dots, n\} : \alpha_i \neq \beta_i\}$;
- (ii) *revlex* order if $\alpha_h > \beta_h$, where $h = \min\{i \in \{0, \dots, n\} : \alpha_i \neq \beta_i\}$;
- (iii) a *reverse* order if $\alpha_0 > \beta_0$ or $\alpha_0 = \beta_0$ and $\frac{x^\alpha}{x_0^{\alpha_0}} \preceq \frac{x^\beta}{x_0^{\beta_0}}$, where \preceq is any graded term order

on $\mathbb{T} \cap K[x_1, \dots, x_n]$. Recall that a reverse order is well suited for the homogeneization of a Gröbner basis [5] and that revlex is a particular reverse order.

(2) With respect to the term orders introduced in (1), for every positive integers j and $n \geq 2$, \mathbb{T}_j can be decomposed in increasing order as follows, where we let $\mathbb{T}(n) := \mathbb{T} \cap K[x_0, \dots, x_{n-1}]$:

lex:

$$\mathbb{T}_j = \mathbb{T}(n)_j \sqcup \mathbb{T}_{j-1} x_n = \mathbb{T}(n)_j \sqcup x_n [\mathbb{T}(n)_{j-1} \sqcup \mathbb{T}(n)_{j-2} x_n] = \dots = \bigsqcup_{i=0}^j \mathbb{T}(n)_{j-i} x_n^i \quad (\star)$$

revlex:

$$\begin{aligned} \mathbb{T}_j &= x_0 \mathbb{T}_{j-1} \sqcup \{\tau \in \mathbb{T}_j : \min(\tau) \geq 1\} = x_0 \mathbb{T}_{j-1} \sqcup x_1 \{\tau \in \mathbb{T}_{j-1} : \min(\tau) \geq 1\} \sqcup \\ &\sqcup \{\tau \in \mathbb{T}_j : \min(\tau) \geq 2\} = \dots = \bigsqcup_{i=0}^n x_i \{\tau \in \mathbb{T}_{j-1} : \min(\tau) \geq i\} \end{aligned} \quad (\star\star)$$

reverse:

$$\mathbb{T}_j = x_0 \mathbb{T}_{j-1} \sqcup \{\tau \in \mathbb{T}_j : \min(\tau) \geq 1\} \quad (\star\star\star)$$

(3) For each positive integer j and $\omega \in \{1, \dots, \binom{n+j}{j} - 1\}$, let $\Lambda_\omega := \Lambda_{\omega,j}$ be the set of the ω smallest terms of \mathbb{T}_j w.r.t. revlex order. Thus, from $(\star\star)$ of (2) it follows straightforward that

$$\Lambda_\omega = \bigsqcup_{i=0}^{\gamma(\omega)} x_i \{\tau \in \mathbb{T}_{j-1} : \min(\tau) \geq i\} \sqcup x_{\gamma(\omega)+1} \{\tau_1, \dots, \tau_{\beta(\omega)}\}$$

where, $\gamma(\omega) + 1 = \min\{t \in \mathbb{N} : \omega \leq \sum_{\ell=0}^t \binom{j-1+n-\ell}{j-1}\}$, $\beta(\omega) := \omega - \sum_{\ell=0}^{\gamma(\omega)} \binom{j-1+n-\ell}{j-1}$ and $\tau_1, \dots, \tau_{\beta(\omega)}$ are the smallest $\beta(\omega)$ terms of \mathbb{T}_{j-1} satisfying $\min(\tau_i) \geq \gamma(\omega) + 1$, for every $1 \leq i \leq \beta(\omega)$.

(4) Fixed any term order \preceq on \mathbb{T} and any weight vector w , the *weighted term order* \preceq_w is defined as follows:

$$x^\alpha \prec_w x^\beta \text{ if } v_w(x^\alpha) < v_w(x^\beta) \text{ or } v_w(x^\alpha) = v_w(x^\beta) \text{ and } x^\alpha \prec x^\beta.$$

Speaking of w -term order we understand \preceq to be the graded lex order.

Let $I \subset S$ be any homogeneous ideal. Then, $H_{S/I}(t)$ denotes the Hilbert function of the graded algebra S/I . It is well known that there are a polynomial $p_{S/I}(z) \in \mathbb{Q}[z]$, called *Hilbert polynomial*, and positive integers $\rho_H := \min\{t \in \mathbb{N} \mid H_{S/I}(t') = p_{S/I}(t'), \forall t' \geq t\}$, $\alpha_H := \min\{t \in \mathbb{N} \mid H_{S/I}(t) < \binom{n+t}{t}\}$ called respectively *regularity of the Hilbert function H* and *initial degree of H* (or also of I). For convenience, we will also say that $p_{S/I}(z)$ is the Hilbert polynomial for I or that I is an ideal *with* Hilbert polynomial $p_{S/I}(z)$. If I is not Artinian, set $\Delta H_{S/I}(t) := H_{S/I}(t) - H_{S/I}(t-1)$, for $t > 0$, and $\Delta H_{S/I}(0) := 1$; we use an analogous notation for Hilbert polynomials. If h is a linear form general on S/I , then it is easy to prove that $p_{S/(I,h)} = \Delta p_{S/I}$.

Polynomials $p(z) \in \mathbb{Q}[z]$ that are Hilbert polynomials of projective subschemes are called *admissible* and are completely characterized in [10] by the fact that they can be always written in a unique form of the following type (see [10, 13]), where ℓ is the degree of $p(z)$ and $m_0 \geq m_1 \geq \dots \geq m_\ell \geq 0$ are integers:

$$p(z) = \sum_{i=0}^{\ell} \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1}.$$

The saturation of a homogeneous ideal $I \subset S$ is $I^{sat} := \{f \in S \mid \forall i \in 0, \dots, n, \exists k_i : x_i^{k_i} f \in I\} = \cup_{h \geq 0} (I : \mathfrak{m}^h)$, where $\mathfrak{m} = (x_0, \dots, x_n)$, and I is saturated if $I = I^{sat}$.

If $X \subset \mathbb{P}_K^n$ is a projective subscheme, $reg(X)$ is its *Castelnuovo-Mumford regularity*, i.e. $reg(X) = \min\{t \in \mathbb{N} \mid H^i(\mathcal{I}_X(t' - i)) = 0, \forall t' \geq t\}$.

An ideal $I \subset S$ is m -regular if the i -th syzygy module of I is generated in degree $\leq m + i$ and the regularity $reg(I)$ of I is the smallest integer m for which I is m -regular. If I is saturated and defines a scheme X , then $reg(I) = reg(X)$ and we set $H_X(t) := H_{S/I}(t)$ and $p_X(z) := p_{S/I}(z)$.

For an admissible polynomial $p(z)$, the *Gotzmann number* r is the best upper bound for the Castelnuovo-Mumford regularity of a scheme having $p(z)$ as Hilbert polynomial and is computable by using the following unique form of an admissible polynomial:

$$p(z) = \binom{z+a_1}{a_1} + \binom{z+a_2-1}{a_2} + \dots + \binom{z+a_r-(r-1)}{a_r},$$

where $a_1 \geq a_2 \geq \dots \geq a_r \geq 0$. We refer to [8] for an overview of these arguments.

Example 1.2. If $p(z) = dz + 1 - g$ is an admissible polynomial, then its Gotzmann number is $r = \binom{d}{2} + 1 - g$. Indeed, we get

$$p(z) = \binom{z+1}{1} + \dots + \binom{z+1-(d-2)}{1} + \binom{z+0-(d-1)}{0} + \dots + \binom{z+0-\binom{d-2}{2}+g}{0}.$$

2. RESULTS ON BOREL IDEALS AND GRÖBNER STRATA

Definition 2.1. (1) For any $x^\alpha \in \mathbb{T}$ such that $\alpha_j > 0$, the terms obtained from x^α via a j -th elementary move are:

- (i) $e_j^+(x^\alpha) := x_0^{\alpha_0} \dots x_j^{\alpha_j-1} x_{j+1}^{\alpha_{j+1}+1} \dots x_n^{\alpha_n}$, for any $j \in \{0, \dots, n-1\}$;
- (ii) $e_j^-(x^\alpha) := x_0^{\alpha_0} \dots x_{j-1}^{\alpha_{j-1}+1} x_j^{\alpha_j-1} \dots x_n^{\alpha_n}$, for any $j \in \{1, \dots, n\}$,

and for each positive integer a we will denote by $(e_j^-)^a, (e_j^+)^a$ the corresponding elementary move applied a times.

(2) For any positive integer t , $<_B$ denotes the partial order on \mathbb{T}_t given by the transitive closure of the relation: $e_j^-(x^\beta) < x^\beta$, i.e. $x^\alpha <_B x^\beta$ if x^α is gotten from x^β via a finite sequence of elementary moves e_j^- .

(3) A set $B \subset \mathbb{T}_t$ is a *Borel set* if, for every x^α of B and x^β of \mathbb{T}_t , $x^\alpha <_B x^\beta$ implies that x^β belongs to B .

(4) A monomial ideal $J \subset S$ is a *Borel ideal* if, for every degree t , $J \cap \mathbb{T}_t$ is a Borel set.

The combinatorial property by which Borel ideals are defined is also called *strong stability*. It has been first introduced in [9] and later in [10], where the ideals satisfying it are called *balanced*. In characteristic 0 it is equivalent to the property for an ideal J of being fixed by lower triangular matrices, from which the name Borel ideals derives.

From the definition it follows immediately that, if $B \subset \mathbb{T}_t$ is a Borel set, then the set $N := \mathbb{T}_t \setminus B$ has the property that for every $x^\gamma \in N$ and $x^\delta \in \mathbb{T}_t$, with $x^\delta <_B x^\gamma$ it holds $x^\delta \in N$, that is N is closed w.r.t. elementary moves e_j^- . In particular, if J is a Borel ideal, then for every integer $t \geq 0$, $\mathcal{N}(J)_t$ is closed w.r.t. elementary moves e_j^- and J_t is closed w.r.t. elementary moves e_j^+ .

Remark 2.2. Note that, for every term order \preceq , if $x^\alpha, x^\beta \in \mathbb{T}_t$ satisfy $x^\alpha <_B x^\beta$ then $x^\alpha \prec x^\beta$. Namely, as $x^\alpha <_B x^\beta$ means that there is a finite number of elementary moves e_j^- connecting x^β to x^α , assuming that $x_j \mid x^\beta$ for a suitable $0 \leq j \leq n$, we can verify our contention for $x^\alpha = e_j^-(x^\beta)$. Setting $\tau := \frac{x^\beta}{x_j}$ and writing $x^\alpha = e_j^-(x^\beta) = x_{j-1}\tau$, $x^\beta = x_j\tau$, we get $x^\alpha \prec x^\beta$ as $x_{j-1} \prec x_j$.

Proposition 2.3. For a Borel ideal $J \subset S$,

- (i) in our notation J^{sat} is obtained by setting $x_0 = 1$ in the minimal generators of J ;
- (ii) the Krull dimension of S/J is equal to $\min\{\max(x^\alpha) : x^\alpha \in J\} = \min\{i \in \{0, \dots, n\} : x_i^t \in J, \text{ for some } t\}$;
- (iii) the regularity of J is equal to the maximum degree of its minimal generators.

Proof. (i) For example, see [21, Property 2].

(ii) This result follows straightforward from Lemma 3.1(a) of [11] or from Corollary 4, section 5, chapter 9 of [5]. Thus, if J is saturated and ℓ is the degree of the Hilbert polynomial of J , we get $\ell = \min\{i \in \{0, \dots, n\} : x_i^t \in J, \text{ for some } t\} - 1$.

(iii) See [2, Proposition 2.9]. \square

Remark 2.4. Let $B \subset \mathbb{T}_t$ be a non-empty Borel set, $N := \mathbb{T}_t \setminus B$ and $J = (B)$ the Borel ideal generated by the terms of B , so that $\mathcal{N}(J)_t = N$. Thus

$$\begin{aligned} \mathcal{N}(J)_{t+1} &= x_0 N \sqcup x_1 \{x^\alpha \in N : 1 \leq \min(x^\alpha)\} \sqcup \\ &\sqcup x_2 \{x^\alpha \in N : 2 \leq \min(x^\alpha)\} \sqcup \dots \sqcup x_{n-1} \{x^\alpha \in N : n-1 \leq \min(x^\alpha)\} \end{aligned}$$

and $\mathbb{T}_{t+1} \setminus \mathcal{N}(J)_{t+1}$ is a Borel set. In particular, if J is a Borel ideal and $N := \mathcal{N}(J)_t$, for every degree t we have (see [16, Theorem 3.7]):

$$\begin{aligned} \mathcal{N}(J)_{t+1} &\subseteq \mathcal{N}(J_{\leq t})_{t+1} = x_0 N \sqcup x_1 \{x^\alpha \in N : 1 \leq \min(x^\alpha)\} \sqcup \\ &\sqcup x_2 \{x^\alpha \in N : 2 \leq \min(x^\alpha)\} \sqcup \dots \sqcup x_{n-1} \{x^\alpha \in N : n-1 \leq \min(x^\alpha)\} \end{aligned}$$

from which $|\mathcal{N}(J)_{t+1}| \leq |\mathcal{N}(J_{\leq t})_{t+1}| = \sum_{i=0}^{n-1} \lambda_{i,t}(N(J_{\leq t}))$ and $G(J)_{t+1} = \mathcal{N}(J_{\leq t})_{t+1} \setminus \mathcal{N}(J)_{t+1}$.

Definition 2.5. For each Borel subset $A \subset \mathbb{T}_t$ the *minimal* elements of A , w.r.t. $<_B$, are the terms $x^\alpha \in A$ such that $e_j^-(x^\alpha) \notin A$ for every $j > 0$ with $\alpha_j > 0$ and the *maximal* elements outside A , w.r.t. $<_B$, are the terms $x^\beta \notin A$ such that $e_j^+(x^\beta) \in A$ for every $j > 0$ with $\beta_j > 0$.

Remark 2.6. Let $B \subset \mathbb{T}_t$ be a Borel subset, if $x^\alpha \in B, x^\beta \notin B$ are respectively a minimal element of B and a maximal element outside B w.r.t. $<_B$, then both $B \setminus \{x^\alpha\}$ and $B \cup \{x^\beta\}$ are Borel subsets of \mathbb{T}_t as by definition both are closed w.r.t. elementary moves e_j^\pm .

Proposition 2.7. Let $p(z)$ be an admissible polynomial with Gotzmann number r and let $J \subset S$ be a Borel ideal with $p(z)$ as Hilbert polynomial. Then, for each $s > r$, a minimal term of J_s w.r.t. $<_B$ is divided by x_0 .

Proof. As $s > r \geq \text{reg}(J)$, for each $x^\alpha \in J_s$ it holds $x^\alpha = \tau \cdot x^\gamma$ for some $x^\gamma \in G(J)$ and $\tau \in \mathbb{T}$ with $\deg(\tau) > 0$. If $x_0 \nmid x^\alpha$, let $j > 0$ be such that $x_j \mid \tau$, then $x^{\alpha'} := \tau' \cdot x^\gamma$, with $\tau' = \frac{\tau \cdot x_0}{x_j}$, satisfies $x^{\alpha'} \in J_s$ and $x^{\alpha'} <_B x^\alpha$, contradicting the minimality of x^α . \square

Given an admissible polynomial $p(z)$, a term order \preceq and a monomial ideal J with $p(z)$ as Hilbert polynomial, the *Gröbner stratum* $\text{St}(J, \preceq)$ in the Hilbert scheme $\mathcal{Hilb}_{p(z)}^n$ of subschemes of \mathbb{P}^n with Hilbert polynomial $p(z)$ is an affine variety that parameterizes the family of ideals having the same initial ideal J with respect to \preceq [23, 7, 12, 3, 19]. When only homogeneous ideals are concerned, we write $\text{St}_h(J, \preceq)$. Now, we recall briefly the construction of $\text{St}(J, \preceq)$, and hence of $\text{St}_h(J, \preceq)$, referring to Definition 3.4 of [12], although here we omit many details that make the procedure more efficient.

For any term x^α of $G(J)$ we set $F_\alpha := x^\alpha + \sum_{\{x^\beta \in \mathcal{N}(J) : x^\beta < x^\alpha\}} c_{\alpha\beta} x^\beta$, considering $c_{\alpha\beta}$ as new variables. Then, we reduce all the S -polynomials $S(F_\alpha, F_{\alpha'})$ with respect to $\{F_\alpha\}_{x^\alpha \in J}$. The ideal $\mathcal{A}(J)$ generated in $K[c_{\alpha\beta}]$ by the x -coefficients of the reduced polynomials is the defining ideal of $\text{St}(J, \preceq)$ and does not depend on the reduction choices. In particular we set $F_\alpha := x^\alpha + \sum_{\{x^\beta \in \mathcal{N}(J)_t : x^\beta < x^\alpha\}} c_{\alpha\beta} x^\beta$, where t is the degree of x^α , then we obtain the ideal of $\text{St}_h(J, \preceq)$.

For properties of Gröbner strata we refer to [23, 7, 12], but it is noteworthy to point out an unexpected feature of Gröbner strata, i.e. they are homogeneous varieties with respect to some non-standard graduation [7, 12]. Thus, the embedding dimension of $\text{St}_h(J, \preceq)$, denoted by $\text{ed}(\text{St}_h(J, \preceq))$, is the dimension of the Zariski tangent space of the stratum at the origin and can be computed by the same procedure which produces Gröbner strata. In fact, the ideal $\mathcal{L}(J)$ generated in $K[c_{\alpha\beta}]$ by the linear components of the generators of $\mathcal{A}(J)$, as computed above, defines the Zariski tangent space of the stratum at the origin (Theorems 3.6(ii) and 4.3 of [12]). This fact gives a new tool for studying the singularities of Hilbert schemes.

3. SEGMENTS

Definition 3.1. A set $B \subset \mathbb{T}_t$ is a *segment* with respect to (w.r.t., for short) a term order \preceq on \mathbb{T} if, whenever a term τ belongs to B , all the t -degree terms which are greater than τ belong to B . A monomial ideal I is a *segment ideal* w.r.t. \preceq , if $I \cap \mathbb{T}_t$ is a segment w.r.t. \preceq , for every $t \geq 0$.

Lemma 3.2. Let $I \subset S$ be a saturated Borel ideal, \preceq any term order on \mathbb{T} and $p > q$ integers. If I_p is a segment then I_q is a segment too.

Proof. Let x^α be a term of I_q and x^β a term of \mathbb{T}_q such that $x^\alpha \preceq x^\beta$, hence $x_0^{p-q} x^\alpha \preceq x_0^{p-q} x^\beta$ and, since I_p is a segment, $x_0^{p-q} x^\beta$ belongs to I_p . Recalling that I is saturated, x^β belongs to I_q and we are done. \square

Remark 3.3. A segment is a Borel set and a segment ideal is a Borel ideal. Indeed, by recovering the arguments of Remark 1.1(2), $x_i x^\alpha \prec x_h x^\alpha$ if $i < h$, thus $x^\alpha <_B x^\beta$ implies that $x^\alpha \prec x^\beta$, for any term order \preceq . In particular, if \preceq is the lex order and I is a monomial ideal generated in degree $\leq q$ such that I_q is a segment w.r.t. \preceq , then I_p is a segment too, for every $p > q$.

Remark 3.4. (1) To each admissible polynomial $p(z)$ of degree $0 \leq \ell \leq n$ corresponds a unique saturated segment ideal $L(p(z))$ w.r.t. lex order (see [1, 13]). In particular for a constant polynomial $p(z) = d$ we have the following, where $\mathbb{T}(2) := \mathbb{T} \cap K[x_0, x_1]$,

$$L(d) = (x_n, x_{n-1}, \dots, x_2, x_1^d),$$

$$\mathcal{N}(L(d))_j = \begin{cases} \mathbb{T}(2)_j & \text{if } 0 \leq j < d \\ \{x_0^{d+i}, x_0^{d+i-1}x_1, \dots, x_0^{i+1}x_1^{d-1}\} & \text{if } j = d+i, \forall i \geq 0, \end{cases}$$

(2) A segment ideal w.r.t. the revlex order exists if and only if the Hilbert polynomial is constant and the Hilbert function H is non-increasing, i.e. $\Delta H(t) \leq 0$ for every $t > \alpha_H = \min\{t \in \mathbb{N} | H(t) < \binom{t+n}{n}\}$ [6, 15].

(3) The same reasoning of [6, 15] shows that, more in general, a segment ideal J w.r.t. a reverse term order exists if and only if the Hilbert polynomial is constant and the Hilbert function H is non-increasing. Namely, if α_H is the initial degree and $x_1^{\alpha_H} \in \mathcal{N}(J)$ it must be $x_1^{\alpha_H+1} \in J$ otherwise, letting $\tau \succ x_1^{\alpha_H}$ be the smallest degree α_H term in J , it would be $x_1^{\alpha_H+1} \in \mathcal{N}(J)$ with $x_1^{\alpha_H+1} \succ x_0\tau \in J$.

Proposition 3.5. *If an ideal $J \subset S$ of initial degree α_H has the property that there exist an integer $t \geq \alpha_H$ and four terms $x^\alpha, x^\beta \in \mathcal{N}(J)_t$, $x^\gamma, x^\delta \in J_t$ with $x^{\alpha+\beta} = x^{\gamma+\delta}$, then J is not a segment ideal w.r.t. any term order \preceq .*

Proof. If J were a segment ideal w.r.t. some \preceq , by the given assumptions we would have in particular both $\mathcal{N}(J)_t \ni x^\beta \prec x^\delta \in J_t$ and $\mathcal{N}(J)_t \ni x^\alpha \prec x^\gamma \in J_t$. From these it would follow $x^{\alpha+\beta} \prec x^{\alpha+\delta} \prec x^{\gamma+\delta}$ contradicting $x^{\alpha+\beta} = x^{\gamma+\delta}$. \square

Example 3.6. (1) The (saturated) Borel ideal $J = (x_2^3, x_1^3x_2^2, x_1^5x_2, x_1^6) \subset K[x_0, x_1, x_2]$ is not a segment ideal w.r.t. any term order as it satisfies the conditions of Proposition 3.5. Namely, its initial degree is 3 and, for $t = 6 \geq 3$, we have: $J_6 \ni x_0^3x_2^3, x_1^6$ and $x_0^2x_1^2x_2^2, x_0x_1^4x_2 \in \mathcal{N}(J)_6$ with $x_0^3x_2^3 \cdot x_1^6 = x_0^2x_1^2x_2^2 \cdot x_0x_1^4x_2$.

(2) Here we show that Proposition 3.5 cannot be inverted. The Borel ideal $J = (x_2^3, x_1x_2^2, x_1^2x_2, x_0^2x_2^2, x_0^3x_1x_2, x_0^5x_2, x_1^7) \subset K[x_0, x_1, x_2]$ of [4, Example 5.8] has the property that J_3 is a segment w.r.t. revlex order while J_t is a segment w.r.t. lex order, for every $t \geq 4$ so that at each degree it does not satisfy the conditions of Proposition 3.5. Nevertheless J is not a segment w.r.t. any term order \preceq , namely if it were, from $x_1^2x_2 \in J_3$ and $x_0x_2^2 \in \mathcal{N}(J)_3$, it would follow $x_0x_2 \prec x_1^2$ contradicting $(x_0x_2)^2 \in J_4, x_1^4 \in \mathcal{N}(J)_4$. Note also that $J^{sat} = (x_2, x_1^7)$ is the saturated lex segment.

Definition 3.7. Let $I \subset S$ be a non null saturated Borel ideal and \preceq a term order on \mathbb{T} .

- (a) [12] I is a *hilib-segment ideal* if I_r is a segment, where r is the Gotzmann number of the Hilbert polynomial of I ;
- (b) I is a *reg-segment ideal* if I_δ is a segment, where δ is the regularity of I ;
- (c) I is a *gen-segment ideal* if, for every integer s , $G(I)_s$ consists of the greatest terms among the s -degree terms not in $\langle I_{s-1} \rangle$.

Remark 3.8. The criterion given by Proposition 3.5 can be adapted also to hilib-segment ideals and to reg-segment ideals I , by simply verifying it at degree $r = \text{Gotzmann number}$ and at degree $\delta = \text{reg}(I)$, respectively. Computational evidence suggests that the condition of this criterion is also necessary for reg-segment and hilib-segment ideals.

The following results about Gröbner strata motivate the definitions of reg-segment ideal and of hilb-segment ideal, respectively.

Proposition 3.9. (i) Let $I \subset S$ be a Borel saturated ideal generated in degree $\leq r$. If s is the maximum degree of terms in $G(I)$ in which x_1 appears, then $\mathcal{S}t_h(I_{\geq m}) \cong \mathcal{S}t_h(I_{\geq s})$, for every $m \geq s$. In particular, if x_1 does not appear in any term of $G(I)$, then $\mathcal{S}t_h(I_{\geq m}) \cong \mathcal{S}t_h(I_{\geq s})$ for every m (Theorem B and, also, Corollary 4.8(ii),(iii) of [12]).

(ii) An isolated, irreducible component of $\text{Hilb}_{\mathbf{p}(z)}^n$ that contains a smooth point corresponding to a hilb-segment ideal is rational (Theorem D and, also, Corollary 6.8 of [12]).

Proposition 3.10. Let $I \subset S$ be a saturated Borel ideal and \preceq a term order on \mathbb{T} . Then

- (i) I segment ideal $\Rightarrow I$ hilb-segment ideal $\Rightarrow I$ reg-segment ideal $\Rightarrow I$ gen-segment ideal.
- (ii) \preceq is the lex order \Leftrightarrow the implications in (i) are all equivalences, for every ideal I .
- (iii) If the projective scheme defined by I is 0-dimensional, then: I segment ideal $\Leftrightarrow I$ hilb-segment ideal $\Leftrightarrow I$ reg-segment ideal.

Proof. (i) The first implication is obvious. For the second one, it is enough to apply Lemma 3.2 since $r \geq \delta$. For the third implication, recall that I is generated in degrees $\leq \delta$, by definition. Moreover, if I is a reg-segment ideal, by Lemma 3.2 I_t contains the greatest terms of degree t , for every $t \leq \delta$. Thus, in particular, minimal generators of I must be the greatest possible.

(ii) First, suppose that \preceq is the lex order. Then, by (i), it is enough to show that a gen-segment ideal is also a segment ideal. Indeed, by induction on the degree s of terms and with $s = 0$ as base of induction, for $s > 0$ suppose that I_{s-1} is a segment. Thus, by Remark 3.3, we know that $\langle I_{s-1} \rangle_s$ is a segment and, since possible minimal generators are always the greatest possible, we are done.

Vice versa, if \preceq is not the lex order, let s be the minimum degree at which the terms are ordered in a different way from the lex one. Thus, there exist two terms x^α and x^β with maximum variables x_l and x_h , respectively, such that $x^\beta \prec x^\alpha$ but $x_h \succ x_l$. The ideal $I = (x_h, \dots, x_n)$ is a gen-segment ideal but not a segment ideal, since x^β belongs to I and x^α does not.

(iii) It is enough to show that, in the 0-dimensional case, a reg-segment ideal I is also a segment ideal. By induction on the degree s , if $s \leq \delta$, then the thesis follows by the hypothesis and by Lemma 3.2. Suppose that $s > \delta$ and that I_{s-1} is a segment. At degree s there are not minimal generators for I so that a term of I_s is always of type $x^\alpha x_h$ with x^α in I_{s-1} . Let x^β be a term of degree s such that $x^\beta \succ x^\alpha x_h$, thus $x^\beta \succ x^\alpha x_0$. By Proposition 2.3, we have that $(x_1, \dots, x_n)^s \subseteq I$. So, if x^β is not divided by x_0 , then x^β belongs to I_s , otherwise there exists a term x^γ such that $x^\beta = x^\gamma x_0$. Thus $x^\gamma \succ x^\alpha$ and by induction x^γ belongs to I_{s-1} so that $x^\beta = x^\gamma x_0$ belongs to I_s . \square

Example 3.11. Let \preceq be the revlex order.

(1) The ideal $I = (x_2^2, x_1 x_2) \subset K[x_0, x_1, x_2]$ is a hilb-segment ideal, but it is not a segment ideal. In this case, the Hilbert polynomial is $p(z) = z + 2$ with Gotzmann number 2 and $\text{reg}(I) = 2$. We have $x_1^3 \in \mathcal{N}(I)$ and $x_0 x_1 x_2 \in I$ with $x_1^3 \succeq x_0 x_1 x_2$.

(2) $I' = (x_2^3, x_1 x_2^2, x_1^2 x_2) \subset K[x_0, x_1, x_2]$ is a reg-segment ideal, but not a hilb-segment ideal. In this case, the Hilbert polynomial is $p(z) = z + 4$ with Gotzmann number 4 and $\text{reg}(I') = 3$. We get $x_0 x_2^3 \in I'$ with $x_0 x_2^3 \preceq x_1^4 \notin I'$.

(3) $I'' = (x_4^2, x_3 x_4, x_3^3) \subset K[x_0, \dots, x_4]$ is a gen-segment ideal but not a reg-segment ideal. In this case, the Hilbert polynomial is $p(z) = 2z^2 + 2z + 1$ with Gotzmann number 12 and $\text{reg}(I'') = 3$. We get $x_0 x_4^2 \in I''$ with $x_0 x_4^2 \preceq x_3^3 \notin I''$.

Remark 3.12. If I is a saturated Borel ideal and also an almost revlex segment ideal, as defined in [6], then it is a gen-segment ideal w.r.t. revlex order.

Theorem 3.13. To the ideal J generated by a Borel set $B \subset \mathbb{T}_d$, consisting of all but d terms of degree d , corresponds a projective scheme with Hilbert polynomial $p(z) = d$.

Proof. By the Borel condition, we have that x_1^d belongs to J , otherwise $|\mathcal{N}(J)_d| \geq d+1 > d$ by Remark 2.4(1), so that by Remark 2.4(2) we have $|\mathcal{N}(J)_t| = d$, for every $t \geq d$. The ideal $I = J^{\text{sat}}$ is the saturated ideal of a projective subscheme with Hilbert polynomial $p(z) = d$. \square

Remark 3.14. (1) For every positive integer d and any term order \preceq on \mathbb{T} , there exists a unique saturated segment ideal $I \subset S$, with Hilbert polynomial $p(z) = d$. This is a straightforward consequence of Theorem 3.13: it is enough to take the ideal J generated by all but the least d terms of degree d .

(2) In [15] for the revlex order and then in [4] for each term order, it is shown that the generic initial ideal of a set X of d general points in \mathbb{P}^n is a segment ideal with Hilbert polynomial $p(z) = d$. As the Hilbert function of X is the maximum possible, that is $H_X(t) = \min\{\binom{t+n}{t}, d\}$, we deduce that this is the Hilbert function of the saturated segment ideal of (1).

(3) For the revers term orders it is possible to give a direct and constructive proof of (1). If $J \subset S$ is a segment ideal w.r.t. a revers order with Hilbert polynomial $p(z) = d$, its Hilbert function must to be non-increasing by Remark 3.4(2) and strictly increasing until it reaches the value d , after which it is always equal to d , because J is a saturated ideal of Krull dimension 1. Thus, $H_{S/J}(t)$ must to be the maximum possible and we have two items:

- (i) $\alpha_H = \rho_H + 1$, so that $J = (x_1, \dots, x_n)^{\alpha_H}$;
- (ii) $\alpha_H = \rho_H$, so that J is generated only in degrees α_H and $\alpha_H + 1$; more precisely, the minimal generators of degree α_H are the greatest $\binom{\alpha_H+n}{\alpha_H} - d$ terms of \mathbb{T}_{α_H} (so that in $\mathcal{N}(J)_{\alpha_H}$ there are $d - \binom{\alpha_H+n-1}{n-1}$ terms x^β with $\min(x^\beta) \geq 1$) and the minimal generators of degree $\alpha_H + 1$ are the all terms $\tau \succeq x_1^{\alpha_H+1}$ which are not multiples of terms in J_{α_H} (these terms are at least $d - \binom{\alpha_H-1+n}{\alpha_H-1}$, by Remark 2.4).

It follows that in case (i) we have $|G(J)| = \binom{\rho_H+n}{n-1}$ and in case (ii) $|G(J)| \geq \binom{\rho_H+n}{n} - d + d - \binom{\rho_H+n-1}{n} = \binom{\rho_H+n-1}{n-1}$.

3.1. On hilb-segment ideals. Let \preceq be any term order and $p(z)$ an admissible polynomial with Gotzmann number r . We want to see under which conditions there exists a hilb-segment ideal for $p(z)$. In this context, it is immediate to see that, if $r = 1$, then $p(z) = \binom{z+\ell}{\ell}$, where $\ell < n$ is the degree of $p(z)$, so that $I = (x_{\ell+1}, \dots, x_n)$ is the hilb-segment ideal for $p(z)$. Moreover, we have already observed that a hilb-segment ideal exists always for a constant polynomial $p(z) = d$.

Example 3.15. The following saturated Borel ideals are not hilb-segment ideals for any term order:

- 1) $J = (x_2^2, x_1^3 x_2, x_1^4) \subset K[x_0, x_1, x_2]$, (see [4]) as $H = (1, 3, 5, 7, \dots, p(z) = 7, \dots)$ we have $r = 7$ so, if J were a hilb-segment ideal w.r.t. some \preceq , at degree 7 we should have $\mathcal{N}(J)_7 \ni x_0^4 x_1^2 x_2 \prec x_0^5 x_2^2 \in J_7$ and $\mathcal{N}(J)_7 \ni x_0^4 x_1^2 x_2 \prec x_0^3 x_1^4 \in J_7$ contradicting $(x_0^4 x_1^2 x_2)^2 = x_0^5 x_2^2 \cdot x_0^3 x_1^4$.
- 2) $J = (x_2^3, x_1 x_2^2, x_1^2 x_2) \subset K[x_0, x_1, x_2]$, as $H = (1, 3, 6, 7, \dots, p(z) = z + 4, \dots)$ we have $\text{reg}(J) = 3$ and $r = 4$, so we can repeat the same reasoning of 1) with $x_0 x_1^2 x_2 \in J_4$ and $x_0^4, x_0^2 x_2^2 \in \mathcal{N}(J)_4$.

Proposition 3.16. *In $K[x_0, x_1, x_2]$ every saturated Borel ideal with Hilbert polynomial $p(z) = d \leq 6$ is a hilb-segment ideal. Whereas for every $p(z) = d \geq 7$, a saturated Borel ideal, which is not a hilb-segment for any term order, always exists.*

Proof. We give a direct constructive proof of the result, based in part on the characterization of the Borel subsets in three variables of [14].

- $d \leq 2$ there exists a unique saturated Borel ideal (x_2, x_1^d) , which is the hilb-segment ideal w.r.t. lex order;
- $d = 3$ there are only two saturated Borel ideals: the hilb-segment ideals (x_2, x_1^3) (w.r.t. lex) and $(x_2^2, x_1 x_2, x_1^2)$ (w.r.t. revlex);

- $d = 4$ there are only two saturated Borel ideals: the hilb-segment ideals (x_2, x_1^4) (w.r.t. lex) and (x_2^2, x_1x_2, x_1^3) (w.r.t. revlex);
- $d = 5$ there are three saturated Borel ideals: the hilb-segment ideals (x_2, x_1^5) (w.r.t lex), (x_2^2, x_1x_2, x_1^4) (w.r.t. $(4, 2, 1)$ -term order) and $(x_2^2, x_1^2x_2, x_1^3)$ (w.r.t revlex);
- $d = 6$ there are four saturated Borel ideals: the hilb-segment ideals (x_2, x_1^6) (w.r.t. lex), (x_2^2, x_1x_2, x_1^5) (w.r.t. $(5, 2, 1)$ -term order), $(x_2^2, x_1^2x_2, x_1^4)$ (w.r.t. $(3, 2, 1)$ -term order) and $(x_2^3, x_1x_2^2, x_1^2x_2, x_1^3)$ (w.r.t revlex);
- $d \geq 7$ (i) Let us firstly consider the case $d = 2a + 1, a \geq 3$ and the ideal $J = (x_2^2, x_1^ax_2, x_1^{a+1})$. It has Hilbert polynomial $p(z) = 2a + 1$, in fact in degree $2a + 1$, the $2a + 1$ monomials $\{x_0^{a+i}x_1^{a-i}x_2, x_0^{a+j+1}x_1^{a-j}, i = 1, \dots, a, j = 0, \dots, a\}$, belong to the quotient. Moreover $x_0^{2a-1}x_2^2, x_0^ax_1^{a+1} \in J$ and $x_0^{2a-2}x_1^2x_2, x_0^{a+1}x_1^{a-1}x_2 \notin J$, but $x_0^{2a-1}x_2^2 \cdot x_0^ax_1^{a+1} = x_0^{2a-2}x_1^2x_2 \cdot x_0^{a+1}x_1^{a-1}x_2$ (if $a = 3$ this is exactly the ideal of Example 3.15 1)).
- (ii) In the case $d = 2a, a \geq 4$, let us consider the ideal $J = (x_2^3, x_1x_2^2, x_1^2x_2, x_1^{2a-3})$. It has Hilbert polynomial $p(z) = 2a$, namely $\mathcal{N}(J)_{2a} = \{x_0^{2a-2}x_2^2, x_0^{2a-2}x_1x_2, x_0^{2a-1}x_2, x_0^{2a-i}x_1^i, i = 0, \dots, 2a-4\}$. Moreover $x_0^{2a-3}x_1^2x_2 \in J_{2a}$, $x_0^{2a-2}x_2^2, x_0^{2a-4}x_1^4 \in \mathcal{N}(J)_{2a}$, and $(x_0^{2a-3}x_1^2x_2)^2 = x_0^{2a-2}x_2^2 \cdot x_0^{2a-4}x_1^4$. \square

Proposition 3.17. *Let \preceq be any reverse term order and $p(z)$ an admissible polynomial of positive degree with Gotzmann number r .*

- (1) *If $p(r) \leq \binom{r-1+n}{n}$, then there is not the hilb-segment ideal for $p(z)$.*
- (2) *If $p(z) = dz + 1 - g$, there exists the hilb-segment ideal J for $p(z)$ if and only if*
 - (i) *$r = d$ or $r = d + 1$, when $n = 2$;*
 - (ii) *$r = d = 1$, when $n > 2$.*

Proof. (1) By the hypothesis we have that x_1^r belongs to the ideal, hence the Krull dimension must to be 1 by Proposition 2.3, and we are done.

(2) In this case, the hilb-segment ideal J exists if and only if $p(r) = \binom{n+r-1}{n} + d$. Infact, the sous-escalier of $(J, x_0)_r$ contains the least d terms not divided by x_0 and, since the term order is reverse and $r \geq d$, also the sous-escalier of J_r must contain the same least d terms not divided by x_0 . Hence, by the Borel property, all the terms divided by x_0 must to belong to the sous-escalier of J_r , too. Thus, since $r = \binom{d}{2} + 1 - g$ by Example 1.2, we get:

$$dr + r - \binom{d}{2} = \binom{n+r-1}{n} + d \Leftrightarrow d = \frac{1}{2} \left(2r - 1 \pm \sqrt{8 \binom{r+1}{2} - 8 \binom{r+n-1}{n} + 1} \right)$$

so that J exists if and only if the argument Δ under the square root is not negative. By an easy calculation we obtain the thesis. \square

3.2. On gen-segment ideals for revlex order. We describe some procedure to construct gen-segment ideals w.r.t revlex order with a given admissible polynomial $p(z)$. We have already observed that a hilb-segment ideal exists always and, thus, also a gen-segment ideal for a constant polynomial $p(z) = d$.

Lemma 3.18. *If $p(z) = dz + 1 - g$ is an admissible polynomial with Gotzmann number r , there exist two integers $n \geq 2$ and $j(n) > 0$ such that $\binom{j(n)-1+n}{n} \leq p(j(n) - 1)$ and $p(j(n) + h) < \binom{j(n)+h+n}{n}$ for every $h \geq 0$.*

Proof. Any projective scheme of dimension 1 with Hilbert polynomial $p(z)$ has regularity $\leq r$, so that $p(r) < \binom{r+n}{n}$, for any $n \geq 2$. Now, it is enough to show that there exist integers $n \geq 2$ and $t < r$ such that $p(t) \geq \binom{t+n}{n}$. Since in the plane, i.e. for $n = 2$, it holds $g \leq \frac{1}{2}(d-1)(d-2)$, then $p(t) = dt + 1 - g \geq dt + 1 - \frac{1}{2}(d-1)(d-2)$ and, for $t = d-1$, we have $d(d-1) + 1 - \frac{1}{2}(d-1)(d-2) = \binom{d-1+2}{2}$. Thus, $n = 2, d \leq j(n) \leq r$. \square

Proposition 3.19. *Let $p(z) = dz + 1 - g$ be an admissible polynomial. For any $n \geq 2$ there exists a gen-segment ideal $I(n) \subset S$ w.r.t. revlex order with Hilbert polynomial $p(z)$.*

Proof. By Lemma 3.18 we can take an integer $n \geq 2$ for which there exists $j(n) > 0$ such that $\binom{j(n)-1+n}{n} \leq p(j(n)-1)$ and $p(j(n)+h) < \binom{j(n)+h+n}{n}$ for every $h \geq 0$. First, we prove the thesis in this case.

Under the given assumptions we have $p(j(n)) - \binom{j(n)-1+n}{n} \geq d = p(j(n)) - p(j(n)-1)$, thus, by Remark 2.4 (3), in $\Lambda_{p(j(n)),j(n)}$ there are at least d terms x^α such that $\min(x^\alpha) \geq 1$ and we let $\tau_1 < \dots < \tau_d$ be the least w.r.t. revlex order among them. We also set $N(t) := \mathbb{T}_t$, for every $0 \leq t < j(n)$, $N(j(n)) := \Lambda_{p(j(n)),j(n)}$, $N(t) := x_0 \cdot N_{t-1} \sqcup x_1^h \cdot \{\tau_1, \dots, \tau_d\}$, for every $t = j(n) + h$, $h \geq 1$ and $N := \sqcup_{t \geq 0} N(t)$. By construction $N \subset \mathbb{T}$ is such that $N_t = N(t)$, $\forall t \geq 0$ and $|N_t| = p(t)$, $\forall t \geq j(n)$. Thus, the monomial ideal $I(n) \subset S$ such that $\mathcal{N}(I(n)) = N$ is, by construction, a gen-segment ideal with Hilbert polynomial $p(z)$. Moreover $G(I(n))_t = \emptyset$, for $t < j(n)$ and $t > j(n) + 1$, so that $\text{reg}(I(n)) \leq j(n) + 1 \leq r$.

Now, suppose that n is such that $p(t) < \binom{t+n}{n}$ for every $t \geq 0$ and let $n_0 := \max\{n' \mid \exists j(n') : \binom{j(n')-1+n'}{n'} \leq p(j(n')-1) \text{ and } p(j(n')) < \binom{j(n')+n'}{n'}\}$. Above we proved that for such an n_0 there exists a gen-segment ideal $I(n_0) \subset K[x_0, \dots, x_{n_0}]$ w.r.t. revlex order with Hilbert polynomial $p(z)$. Now, it is enough to observe that $I(n) := (I(n_0), x_{n_0+1}, \dots, x_n) \subset S$ is a gen-segment ideal w.r.t. revlex order as claimed. \square

Remark 3.20. Given an admissible polynomial $p(z) = dz + 1 - g$, if $n > 2$ is such that $p(t) < \binom{t+n}{n}$ for every $t \geq 0$ and “there exists $l(n) := \min\{l \in \mathbb{N} : \sum_{i=1}^{n-1} \lambda_{i,l}(N_l) \geq d\}$ ”, by a similar procedure we can construct a gen-segment ideal $J(n) \subset S$ w.r.t. revlex order with Hilbert polynomial $p(z)$ different from those coming from the smaller n' 's as in the proof of Proposition 3.19. Indeed, under the given assumptions, $\Lambda_{p(l(n)),l(n)} \subset \mathbb{T}_{l(n)}$ does no longer contain at least d terms x^α with $\min(x^\alpha) \geq 1$, but surely its expansion in degree $l(n) + 1$ does it and we let $\bar{\tau}_1 < \dots < \bar{\tau}_d$ be the least w.r.t. revlex order of them. Similarly as before, we take $M(t) := \mathbb{T}_t$, for every $0 \leq t < l(n)$, $M(l(n)) := \Lambda_{p(l(n)),l(n)}$, $M(l(n) + 1) := x_0 \cdot M(l(n)) \sqcup \{\bar{\tau}_1, \dots, \bar{\tau}_d\}$, $M(t) := x_0 \cdot M(t-1) \sqcup x_1^{t-l(n)-1} \{\bar{\tau}_1, \dots, \bar{\tau}_d\}$ for every $t > l(n) + 1$. We finally let $J(n)$ be the gen-segment ideal such that $\mathcal{N}(J(n)) = M := \sqcup_{t \geq 0} M(t)$ and note that it has $p(z)$ as Hilbert polynomial and regularity $\leq l(n) + 2$.

Example 3.21. (1) The Gotzmann number of the admissible polynomial $p(z) = 6z - 3$ is 12 and we obtain the following gen-segment ideals:

- (i) if $n = 2$, we can apply the procedure described in the proof of Proposition 3.19 with $j(2) = 9$ and construct the ideal $I(2) = (x_2^9, x_1x_2^8, x_1^2x_2^7, x_1^3x_2^6)$;
- (ii) if $n = 3$, there is not a $j(3)$, yet we can apply the procedure described in Remark 3.20 with $l(3) = 2$, since $p(t) < \binom{3+t}{t}$, for every $t > 0$, obtaining $J(3) = (x_3^2, x_2^2x_3, x_2^4)$ besides $(I(2), x_3)$;
- (iii) if $n \geq 4$, neither $j(n)$ nor $l(n)$ exist and we have only $(I(2), x_3, \dots, x_n)$ and $(J(3), x_4, \dots, x_n)$.

(2) The Gotzmann number of the admissible polynomial $p(z) = 7z + 1$ is 22 and we have $j(2) = 12$, $j(3) = 4$, $j(4) = 3$, $j(5) = j(6) = j(7) = 2$, while for $n \geq 8$, there is not a $j(n)$. Moreover, by definition of $j(n)$, we have $\alpha(n) := p(j(n)) - \binom{j(n)-1+n}{n} \geq 7$, thus we can apply the procedure described in the proof of Proposition 3.19 and

- (i) for $n = 2$, as $\alpha(2) = 85 - 78 = 7$, we obtain $I(2) = (x_2^{12}, x_1x_2^{11}, \dots, x_1^5x_2^7)$ ($\text{reg}(I(2)) = j(2)$);
- (ii) for $n = 3$, as $\alpha(3) = 29 - 20 = 9 > 7$, we obtain $I(3) = (x_3^4, x_2x_3^3, x_2^2x_3^2, x_2^3x_3, x_1^4x_3^3, x_1^2x_2x_3^2, x_1^2x_2^2x_3)$ ($\text{reg}(I(3)) = j(3) + 1$), besides $(I(2), x_3)$;
- (iii) for $n = 4$, as $\alpha(4) = 22 - 15 = 7$, we obtain $I(4) = ((x_4, x_3, x_2)^3, x_1(x_4, x_3)^2)$ ($\text{reg}(I(4)) = j(4)$), besides $(I(2), x_3, x_4)$ and $(I(3), x_4)$;

- (iv) for $n = 5$, as $\alpha(5) = 15 - 6 = 9 > 7$, we obtain $I(5) = ((x_5, x_4, x_3)^2, x_2^2 x_5, x_1 x_2 x_5, x_2^2 x_4, x_1 x_2 x_4, x_2^2 x_3, x_2^3)$ ($\text{reg}(I(5)) = j(5) + 1$), besides $(I(2), x_3, x_4, x_5)$, $(I(3), x_4, x_5)$, $(I(4), x_5)$;
- (v) for $n = 6$, as $\alpha(6) = 15 - 7 = 8 > 7$, we obtain $I(6) = ((x_6, x_5, x_4, x_3)^2, x_2 x_6, x_2 x_5, x_2 x_4, x_2^2 x_3, x_1 x_2 x_3, x_2^3)$, ($\text{reg}(I(6)) = j(6) + 1$), besides $(I(2), x_3, x_4, x_5, x_6)$, $(I(3), x_4, x_5, x_6)$, $(I(4), x_5, x_6)$ and $(I(5), x_6)$;
- (vi) for $n = 7$, as $\alpha(7) = 15 - 8 = 7$, we obtain $I(7) = (x_7, x_6, x_5, x_4, x_3, x_2)^2$, ($\text{reg}(I(7)) = j(7)$), besides $(I(2), x_3, x_4, x_5, x_6, x_7)$, $(I(3), x_4, x_5, x_6, x_7)$, $(I(4), x_5, x_6, x_7)$, $(I(5), x_6, x_7)$ and $(I(6), x_7)$;
- (vii) for $n \geq 8$, $j(n)$ does not exist yet, as $\lambda_{1,1}(\Lambda_{p(1),1}) \geq 7$ we have $J(n) = (J(8), x_9, \dots, x_n)$ besides the ideals coming from $I(m), m \leq 7$.

Proposition 3.22. *The saturated segment ideal $L(p(z)) \subset S$ w.r.t. lex order with Hilbert polynomial $p(z)$ is a gen-segment ideal w.r.t. the revlex order if and only if $\deg(p(Z)) \leq 1$ or there are only two generators of degree > 1 .*

Proof. In section 1 we have already recalled that, given an admissible polynomial $p(z)$ of degree ℓ , there exist unique integers $m_0 \geq m_1 \geq \dots \geq m_\ell \geq 0$ such that $p(z) = \sum_{i=0}^{\ell} \binom{z+i}{i+1} - \binom{z+i-m_i}{i+1}$ [13, 10, 1]. Let $a_\ell := m_\ell, a_{\ell-1} := m_{\ell-1} - m_\ell, \dots, a_0 := m_0 - m_1$. Note that $L(p(z)) \subset S$ has the $n + 1 - \ell - 2$ greatest variables as generators of degree 1, i.e. $\mathcal{N}(L(p(z)))_1 = \{x_0, \dots, x_{\ell+1}\}$. Thus, for every $j \leq a_\ell$, the greatest term of $\mathcal{N}(L(p(z)))_j$ is $x_{\ell+1}^j$, w.r.t. both lex and revlex orders (namely $\mathcal{N}(L(p(z)))_j = \mathbb{T}_j \cap K[x_0, \dots, x_{\ell+1}]$). In degree $a_\ell + 1$ the ideal $L(p(z))$ has a new generator $x_{\ell+1}^{a_\ell+1}$, so that $\mathcal{N}(L(p(z)))_{a_\ell+1} = (\mathbb{T}_{a_\ell+1} \cap K[x_0, \dots, x_{\ell+1}]) \setminus \{x_{\ell+1}^{a_\ell+1}\}$, therefore its greatest term, w.r.t. both lex and revlex orders, is $x_\ell x_{\ell+1}^{a_\ell}$, and so on until there is a new generator in degree $a_\ell + a_{\ell-1} + 1$ if $a_{\ell-2} \neq 0$, which is $x_\ell^{a_{\ell-1}+1} x_{\ell+1}^{a_\ell}$ (or, if $a_{\ell-2} = 0$, the new generator is $x_\ell^{a_{\ell-1}} x_{\ell+1}^{a_\ell}$). At this point, the greatest term in $\mathcal{N}(L(p(z)))_{a_\ell+a_{\ell-1}+1}$ is $x_\ell^{a_{\ell-1}+2} x_{\ell+1}^{a_\ell-1}$ w.r.t. revlex order and $x_{\ell-1} x_\ell^{a_{\ell-1}} x_{\ell+1}^{a_\ell}$ w.r.t. lex order (similarly for the case in parenthesis). Moreover, since the new generator of $L(p(z))$ at degree $a_\ell + a_{\ell-1} + a_{\ell-2} + 1$ is $x_{\ell-1}^{a_{\ell-2}+1} x_\ell^{a_{\ell-1}} x_{\ell+1}^{a_\ell}$ (if $\ell = 2$ the third generator of degree > 1 is $x_1^{a_0} x_2^{a_1} x_3^{a_2}$ or, if $a_{\ell-3} = 0$, is $x_{\ell-1}^{a_{\ell-2}} x_\ell^{a_{\ell-1}} x_{\ell+1}^{a_\ell}$), it is not the greatest term w.r.t. revlex order. \square

Example 3.23. (i) The ideal $L(p(z)) = (x_4, x_3^5, x_2^3 x_3^4, x_1^6 x_2^2 x_3^4)$ is the saturated segment ideal w.r.t. lex in $K[x_0, \dots, x_4]$, with Hilbert polynomial $p(z) = 2z^2 + 2z + 1$ and Gotzmann number 12, but is not a gen-segment ideal w.r.t. revlex order.

(ii) The ideal $L(p(z)) = (x_5, x_4^5, x_3^2 x_4^4)$ is the saturated segment ideal w.r.t. lex in $K[x_0, \dots, x_5]$, with Hilbert polynomial $p(z) = 2/3 z^3 + 2z^2 - 11/3 z + 10$ and Gotzmann number 6, and is also a gen-segment ideal w.r.t. revlex order.

4. SATURATIONS OF BOREL IDEALS AND HILBERT POLYNOMIAL

Let $J \subset S$ be a Borel ideal. Recall that in our notation the (Borel) ideal J^{sat} is obtained by setting $x_0 = 1$ in each minimal generator of J (Proposition 2.3(i)). In this section we let $J_{x_0} := J^{\text{sat}}$ and denote $J_{x_0 x_1}$ the Borel ideal obtained by setting $x_0 = x_1 = 1$ in the minimal generators of J . We call $J_{x_0 x_1}$ the x_1 -saturation of J and say that J is x_1 -saturated if $J = J_{x_0 x_1}$, so that an ideal J x_1 -saturated is also saturated.

Remark 4.1. An ideal $J \subset S$, which is x_1 -saturated and has Hilbert polynomial $p(z) := p_{S/J}(z)$, has the same minimal generators of the saturated Borel ideal $J \cap K[x_1, \dots, x_n] \subset K[x_1, \dots, x_n]$, for which the Hilbert polynomial is $\Delta p(z)$.

The following result is analogous to Theorem 3 of [21], where the notion of “fan” is used. Here we apply the combinatorial properties of Borel ideals only.

Proposition 4.2. *Let $J \subset S$ be a saturated Borel ideal with Hilbert polynomial $p(z)$ and Gotzmann number r . Let $I = J_{x_0 x_1}$ be the x_1 -saturation of J and let $q := \dim_K I_r - \dim_K J_r$. Then*

- (i) $p_{S/I}(z) = p(z) - q$;
- (ii) q is equal to the sum of the exponents of x_1 in the minimal generators of J .

Proof. (i) We show that if $q = \dim_K I_s - \dim_K J_s$ then $q = \dim_K I_{s+1} - \dim_K J_{s+1}$, for every $s \geq r$. Let $x^{\beta_1}, \dots, x^{\beta_q}$ be the terms of $I_s \setminus J_s$. Thus, $x_0 x^{\beta_1}, \dots, x_0 x^{\beta_q}$ are terms of $I_{s+1} \setminus J_{s+1}$ and so $\dim_K I_{s+1} - \dim_K J_{s+1} \geq q$, since $x_0 x^{\beta_i}$ belongs to J_{s+1} if and only if x^{β_i} belongs to J_s being J saturated. Now, for obtaining the opposite inequality it is enough to show that every term of $I_{s+1} \setminus J_{s+1}$ is divided by x_0 . Let $x^\gamma \in I_{s+1}$ be such that $\min(x^\gamma) \geq 1$ and let x^α be a minimal generator of I such that $x^\gamma = x^\alpha x^\delta$. Since J is saturated and I is the x_1 -saturation of J , $x^\alpha x_1^a$ is a minimal generator of J for some non negative integer a . Hence, for every $x^{\delta'}$ of degree $s+1 - |\alpha|$ and with $\min(x^{\delta'}) \geq 1$, by the Borel property $x^\alpha x^{\delta'}$ belongs to J_s . In particular, $x^\gamma \in J_{s+1}$.

(ii) Let $x^{\alpha_1} x_1^{s_1}, \dots, x^{\alpha_h} x_1^{s_h}$ be the minimal generators of J , with x_1 not dividing $x^{\alpha_i}, \forall 1 \leq i \leq h$. As the $\sum s_i$ terms $x^{\alpha_i} x_1^{s_i-t} x_0^{r-|\alpha_i|-s_i+t}, 1 \leq t \leq s_i$, are in $I_r \setminus J_r$, one has $q \geq \sum s_i$. Vice versa, we show that each term x^δ in $I_r \setminus J_r$ is of the previous type. We can write $x^\delta = x^\beta x_0^{r-|\beta|-u} x_1^u$, with $\min(x^\beta) \geq 2$ and $u < s_i$. Let s be the minimum non negative integer such that $x^\beta x_1^s$ is in J . Then there exists i such that $x^{\alpha_i} x_1^{s_i} | x^\beta x_1^s$, i.e. $x^{\alpha_i} | x^\beta$ and $s_i \leq s$. By the definition of s , we get $s_i = s$ and there exists x^γ with $\min(x^\gamma) \geq 2$ such that $x^\beta = x^{\alpha_i} x^\gamma$. Since x^β does not belong to J we have $|\gamma| < s_i = s$, otherwise $x^{\alpha_i} x_1^{|\gamma|}$ and hence, by the Borel property, $x^\beta = x^{\alpha_i} x^\gamma$ should belong to J . Now we can take $x^\beta x_1^{s-|\gamma|}$ and observe that this term belongs to J because it follows $x^{\alpha_i} x_1^s$ in the Borel relation. Thus $s \leq s - |\gamma|$, so that $\gamma = 0$, i.e. $x^\beta = x^{\alpha_i}$ as claimed. \square

Proposition 4.3. *Let $J \subset S$ be a saturated Borel ideal with Hilbert polynomial $p(z)$ and Gotzmann number r . Let $x^\beta x_0$ be a term of J of degree $s \geq r$ which is minimal in J w.r.t. $<_B$. Then the ideal $I := (G((J_s)) \setminus \{x^\beta x_0\})$ is Borel and $p_{S/I}(z) = p(z) + 1$.*

Proof. First, note that I_s is closed w.r.t. $<_B$ by Remark 2.6. We show that, for every $t \geq 0$, $x^\beta x_0^{1+t}$ is the unique term in $J_{s+t} \setminus I_{s+t}$. For $t = 0$ we have the hypothesis. For $t > 0$, note that $x^\beta x_0^{1+t}$ cannot belong to I . On the contrary, there would be a term $x^\gamma \in I_s$ such that $x^\gamma \neq x^\beta x_0$ and $x^\gamma | x^\beta x_0^{1+t}$. But every degree s factor of $x^\beta x_0^{1+t}$ different from $x^\beta x_0$ is lower w.r.t. $<_B$ and so it cannot belong to I_s . Then, $x^\beta x_0^{1+t} \notin I_{s+t}$. If x^α is a term of $J_{s+t} \setminus I_{s+t}$, there exists a term of $J_{s+t-1} \setminus I_{s+t-1}$ which divides x^α . By induction, this term is $x^\beta x_0^t$ and the thesis follows from the fact that every multiple of degree $s+t$ of $x^\beta x_0^t$, different from $x^\beta x_0^{1+t}$, belongs to I_{s+t} . \square

Proposition 4.4. *Let I and J be Borel ideals of S . If for every $s \gg 0$ we have $I_s \subset J_s$ and $p_{S/I}(z) = p_{S/J}(z) + a$, with $a \in \mathbb{N}$, then I and J have the same x_1 -saturation.*

Proof. Let $s \geq \max\{\text{reg}(I), \text{reg}(J)\}$. In case $a = 1$, there exists a unique term in $J_{s+t} \setminus I_{s+t}$, for every $t \geq 0$. Let x^α be the unique term in $J_s \setminus I_s$. Then, both $x^\alpha x_0$ and $x^\alpha x_1$ belong to J_{s+1} . By the Borel property, $x^\alpha x_1$ must be in I_{s+1} and so the unique term in $J_{s+t} \setminus I_{s+t}$ is $x^\alpha x_1^t$. This is enough to say that I and J have the same x_1 -saturation. If $a > 1$, the thesis follows by induction applying Proposition 4.3. \square

Corollary 4.5. *Let $p(z)$ be an admissible polynomial of degree $h \leq n$ and $P := \{q(z) = p(z) + u \mid u \in \mathbb{Z} \text{ and } q(z) \text{ admissible}\}$ the set of all admissible polynomials of degree h which differ from $p(z)$ only for an integer. Then*

- (i) *there is a polynomial $\hat{p}(z)$ in P such that, for every $q(z)$ in P , $q(z) = \hat{p}(z) + c$ with $c \geq 0$;*
- (ii) *every saturated Borel ideal I with Hilbert polynomial $p_{S/I} = \hat{p}(z)$ is x_1 -saturated.*

Proof. (i) Every admissible polynomial $p(z)$ has a unique saturated lex segment ideal $L(p(z))$. If H is the saturated lex segment ideal of $p(z) + u$, then we have $H \subset L(p(z))$ if $u > 0$ and $L(p(z)) \subset H$ if $u < 0$. Thus, we can apply Proposition 4.4, obtaining that $L(p(z))$ and H have the same x_1 -saturation I . We claim that $\hat{p}(z)$ is the Hilbert polynomial of I . Indeed, by Proposition 4.2 the Hilbert polynomial of I is of type $p(z) - q$. If $\hat{p}(z) = p(z) - q - t$ with $t \geq 0$, then the saturated lex segment ideal of $\hat{p}(z)$ should have I as x_1 -saturation and should be contained in I , that is possible only if $t = 0$.

(ii) Let J be a Borel ideal with $\hat{p}(z)$ as Hilbert polynomial. If J were not x_1 -saturated, by Proposition 4.2 the x_1 -saturation of J should have a Hilbert polynomial of type $\hat{p}(z) - q$, with $q > 0$, against the definition of $\hat{p}(z)$, \square

Definition 4.6. The polynomial $\hat{p}(z)$ of Corollary 4.5 is called *minimal polynomial*.

Remark 4.7. An alternative proof of the previous statement can be obtained by following the construction of the Gotzmann number.

Example 4.8. By Proposition 4.2, a Borel ideal with a minimal Hilbert polynomial is x_1 -saturated. The vice versa is not true. For example, the ideal $I = (x_3^2, x_2x_3, x_2^2) \subset K[x_0, x_1, x_2, x_3]$ is x_1 -saturated and is a reg-segment ideal w.r.t. the revlex order. The corresponding Hilbert polynomial is $p_{S/I}(z) = 3z + 1$ which is not minimal because the Borel ideal (x_3, x_2^3) has Hilbert polynomial $3z$.

Remark 4.9. From the proof of Corollary 4.5 we deduce the following fact. Let $I \subset K[x_0, \dots, x_n]$ be a Borel ideal with Hilbert polynomial $p(z)$. If $I = I_{x_1} \cdot K[x_0, \dots, x_n]$, where $I_{x_1} \subset K[x_1, \dots, x_n]$ is the segment ideal w.r.t. lex order with Hilbert polynomial $\Delta p(z)$, then $p(z) = \hat{p}(z)$.

5. AN ALGORITHM TO COMPUTE SATURATED BOREL IDEALS

In this section, by exploiting the arguments of section 4, we describe an algorithm for computing all the saturated Borel ideals with a given Hilbert polynomial $p(z)$. We first give an efficient strategy to find the minimal elements in a Borel set B , that consists in representing B by a connected planar graph, in which the nodes are the terms of B and the edges are the elementary moves which connect the terms. In Figure 1 we give some examples showing that it is easy to single out the minimal terms looking at these graphs.

Let $0 \leq k < n$ be an integer. Recall that, if $I \subset K[x_k, \dots, x_n]$ is a saturated Borel ideal which has a non null Hilbert polynomial $p(z)$ with Gotzmann number r , then $J := \frac{(I, x_k)}{(x_k)} \subset K[x_{k+1}, \dots, x_n]$ has Hilbert polynomial $\Delta p(z)$, being x_k a non zero-divisor on $\frac{K[x_k, \dots, x_n]}{I}$.

This fact tells that every saturated Borel ideal $I \subset K[x_k, \dots, x_n]$ with Hilbert polynomial $p(z)$ “comes from” a Borel ideal $J \subset K[x_{k+1}, \dots, x_n]$ with Hilbert polynomial $\Delta p(z)$ and generated in degrees $\leq r$. So, our idea to construct all saturated Borel ideal with given Hilbert polynomial $p(z)$ consists in applying a recursion on the number of variables: in the hypothesis of knowing all Borel ideals J in $n - k$ variables generated in degrees $\leq r$ with Hilbert polynomial $\Delta p(z)$, we construct the saturated Borel ideals I in $n - k + 1$ variables such that $J := \frac{(I, x_k)}{(x_k)}$ for some of the ideals J .

Let $J \subset K[x_{k+1}, \dots, x_n]$ be a Borel ideal with Hilbert polynomial $\Delta p(z)$ and $\bar{I} := (J^{sat} \cdot K[x_k, \dots, x_n])_r$, where r is the Gotzmann number of $p(z)$. Let \bar{N} be the set of terms x^α of $K[x_k, \dots, x_n]_r$ such that there exists a composition F of elementary moves of type e_j^- and a term τ of $\mathcal{N}(J)_r$ such that $F(\tau) = x^\alpha$. Hence, by construction, the terms of $\bar{N} \setminus \mathcal{N}(J)$ are not maximal and \bar{N} is contained in the *sous-escalier* of any ideal of $K[x_k, \dots, x_n]$ having J as hyperplane section. Note that the Gotzmann number of $\Delta^{k+1}p(z)$ is not higher than the Gotzmann number of $\Delta^k p(z)$.

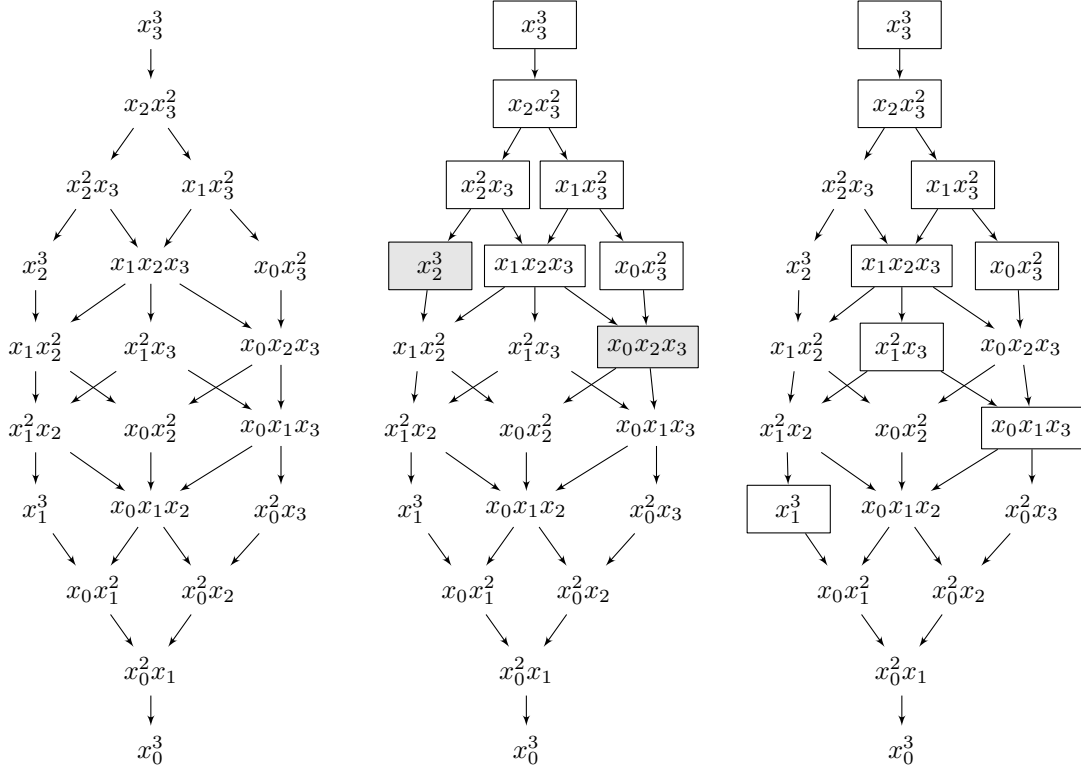


FIGURE 1. Here are: on the left the graph of $K[x_0, \dots, x_3]_3$, in the center the graph of $(x_3^2, x_2x_3, x_2^3)_3$, where we coloured the minimal elements, on the right the graph of $(x_3^2, x_1x_3, x_1^3)_3$ which is not Borel (the terms in the ideal are the boxed ones).

Lemma 5.1. $\mathcal{N}(\bar{I})_r = \bar{N}$.

Proof. It is enough to show that $K[x_k, \dots, x_n]_r = (\bar{I}, \bar{N})$ (Figure 2). Indeed, let $x^\gamma = x_k^{\gamma_k} \dots x_n^{\gamma_n}$ be in $K[x_k, \dots, x_n]_r$. The term $x^\beta = (e_k^+)^{\gamma_k}(x^\gamma)$ belongs to $K[x_{k+1}, \dots, x_n]_r$, hence is in J_r or in $\mathcal{N}(J)_r$. If x^β is in J_r , then $x_{k+2}^{\gamma_{k+2}} \dots x_n^{\gamma_n}$ belongs to J_r^{sat} , hence to \bar{I} , otherwise $x^\gamma = (e_{k+1}^-)^{\gamma_k}(x^\beta)$ is in \bar{N} . \square

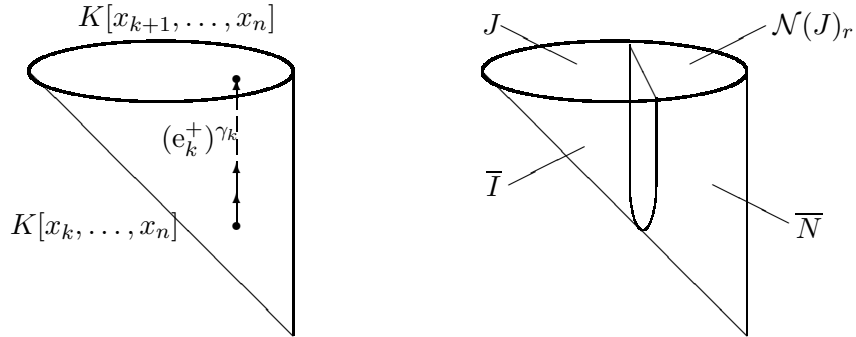


FIGURE 2. Partition of $K[x_k, \dots, x_n]$.

Proposition 5.2. *With the above notation, the Hilbert polynomial $\bar{p}(z)$ for \bar{I} differs from $p(z)$ only for a constant. If $\bar{q} = p(r) - \bar{p}(r) > 0$, execute the following instruction \bar{q} times: select a minimal term τ in \bar{I}_r and set $\bar{I} := (G((\bar{I}_r)) \setminus \{\tau\})$. After these \bar{q} steps, the new ideal obtained has Hilbert polynomial $p(z)$.*

Proof. The theses follow from the results presented in section 4. \square

Proposition 5.2 suggests the design of the following two routines BORELGENERATOR and REMOVE, that have been implemented by the second author in a software with an applet available at <http://www.dm.unito.it/dottorato/dottorandi/lella/borelEN.html>.

```

procedure BORELGENERATOR( $n, p(z), r, k$ )  $\rightarrow \mathcal{F}$ 
  if  $p(z) = 0$  then
    return  $\{(1)\}$ ;
  else
     $\mathcal{E} \leftarrow \text{BORELGENERATOR}(n, \Delta p(z), r, k + 1)$ ;
     $\mathcal{F} \leftarrow \emptyset$ ;
    for all  $J \in \mathcal{E}$  do
       $\bar{I} \leftarrow J \cdot k[x_k, \dots, x_n]$ ;
       $q \leftarrow p(r) - \dim_k k[x_k, \dots, x_n]_r + \dim_k \bar{I}_r$ ;
      if  $q \geq 0$  then
         $\mathcal{F} \leftarrow \mathcal{F} \cup \text{REMOVE}(n, k, r, \bar{I}, q)$ ;
      end if
    end for
    return  $\mathcal{F}$ ;
  end if
end procedure

procedure REMOVE( $n, k, r, \bar{I}, q$ )  $\rightarrow \mathcal{E}$ 
   $\mathcal{E} \leftarrow \emptyset$ ;
  if  $q = 0$  then
    return  $\mathcal{E} \cup \bar{I}^{\text{sat}}$ ;
  else
     $\mathcal{F} \leftarrow \text{MINIMALELEMENTS}(\bar{I}, r)$ 
    for all  $x^\alpha \in \mathcal{F}$  do
       $\mathcal{E} \leftarrow \mathcal{E} \cup \text{REMOVE}(n, k, r, (G((\bar{I}_r)) \setminus x^\alpha), q - 1)$ ;
    end for
    return  $\mathcal{E}$ ;
  end if
end procedure

```

Remark 5.3. The terms removed by our strategy are minimal in \bar{I} . An alternative strategy could consists in adding to $J_r K[x_k, \dots, x_n]$ maximal terms of $\bar{I}_r \setminus J$. In this case, since we want that $\dim_K I_r = \binom{n-k+r}{r} - p(r)$ and we have already $\binom{n-(k+1)+r}{r} - \Delta p(r)$ terms of J , we should add

$$q' = \binom{n-k+r}{r} - p(r) - \binom{n-(k+1)+r}{r} + \Delta p(r) = \binom{n-k-1+r}{r-1} - p(r-1)$$

terms for any J , where q' depends only on r , $n-k$ and $p(z)$; hence, we will write $q'(r, n-k, p(z))$ instead of q' . On the other hand, the value of $\bar{q} = p(r) - |\bar{N}_r| = p(0) - \bar{p}(0)$ depends on J . Note that $q' + \bar{q} = \dim_K \bar{I}_r - \dim_K J_r$. Anyway, we observe that if $n-k > \deg(p(z)) + 1$ then $q' \geq \bar{q}$.

The minimal polynomial $\hat{p}(z)$ of Definition 4.6 can be recovered from $\Delta p(z)$ by the decomposition of Gotzmann in the following way. If

$$\Delta p(z) = \binom{z+b_1}{b_1} + \binom{z+b_2-1}{b_2} + \dots + \binom{z+b_t-(t-1)}{b_t}$$

with $b_1 \geq b_2 \geq \dots \geq b_t \geq 0$, then

$$\hat{p}(z) = \binom{z+a_1}{a_1} + \binom{z+a_2-1}{a_2} + \dots + \binom{z+a_t-(t-1)}{a_t}$$

where $a_i = b_i + 1$. The Gotzmann number of $\Delta p(z)$ is also the Gotzmann number \hat{r} of $\hat{p}(z)$. If r is the Gotzmann number of $p(z)$, then $r - \hat{r} = p(0) - \hat{p}(0) \geq p(0) - \bar{p}(0) = \bar{q}$. We prove that $q' \geq \bar{q}$ by induction on $c = r - \hat{r}$. If $c = 0$ then we get $\bar{q} = 0$. If $c > 0$, by induction we have that $q'(r-1, n-k, p(z)-1) \geq \bar{q}-1$, hence

$$\begin{aligned} q'(r, n-k, p(z)) &= \binom{r-1+n-k}{n-k} - p(r-1) = \\ &= \binom{r-2+n-k}{n-k} + \binom{r-1+n-k-1}{n-k-1} - p(r-1) + \\ &\quad + p(r-2) - p(r-2) = \\ &= q'(r-1, n-k, p(z)-1) + \binom{r-1+n-k-1}{n-k-1} - \Delta p(r-1) - 1 \geq \\ &\geq \bar{q} + \binom{r-1+n-k-1}{n-k-1} - \Delta p(r-1) - 2 \end{aligned}$$

and $\binom{r-1+n-k-1}{n-k-1} - \Delta p(r-1) \geq 2$ since $r-1$ is a bound from above of the Gotzmann number of $\Delta p(z)$ and J is not a hypersurface because $n-k-1 > \deg(\Delta p(z)) + 1$.

Example 5.4. (a) If $p(z) = d$, then $r = d$ and $\hat{r} = 0$, so $\bar{q} = d$ and $q' = \binom{d-1+n}{n} - d$. If moreover $n = \deg(p(z)) + 1$, then $q' = 0$.

(b) The Gotzmann number of $p(z) = 3z + 1$ is $r = 4$ and, if $n = 3$ and $k = 0$, then $q'(r, n, p(z)) = \binom{r-1+n}{n} - p(r-1) = 20 - 10 = 10$ and $r - \hat{r} = 1$. If $J_4 = (x_3, x_2^3)_4$ we get $|\bar{N}_r| = 12$ and $\bar{q} = 1$, meanwhile if $J_4 = (x_3^2, x_2x_3, x_2^2)_4$ we obtain $|\bar{N}_r| = 13$ and $\bar{q} = 0$.

6. REVERSE LEXICOGRAPHIC POINT

In this section exploiting results of [12] we study the points corresponding to hilb-segment ideals in the Hilbert scheme Hilb_d^n of subschemes of \mathbb{P}^n with Hilbert polynomial $p(z) = d$, where d is a fixed positive integer. Recall that for $p(z) = d$ the Gotzmann number is d itself.

From now, $J \subset S$ is a hilb-segment ideal with respect to some term order \preceq and with Hilbert polynomial $p(z) = d$ and let $\mathcal{B} := \{x^\beta \in \mathcal{N}(J)_d : x_1x^\beta \in J\}$. Recall that $G(I)$ denotes the set of minimal generators of I and $\text{ed}(St_h(J, \preceq))$ is the embedding dimension of the Gröbner stratum $St_h(J, \preceq)$.

Lemma 6.1. *With the notation above, we obtain that $\text{ed}(St_h(J, \preceq)) \geq |G(J)| \cdot |\mathcal{B}|$.*

Proof. With the same notation introduced in section 2, by Corollary 4.8(i) of [12] it is enough to look at the variables $c_{\alpha\beta}$ appearing in the polynomials F_α such that $x^\alpha = x^\gamma x_0^{d-|\gamma|}$, where x^γ belongs to $G(J)$. More precisely, we need to count the number of such variables which do not correspond to a pivot in a Gauss reduction of the generators of $L(J)$ (see also Theorem 4.3 of [12]).

First we note that in every S -polynomial which involves such an F_α , the polynomial F_α itself is multiplied by a term in which at least a variable x_h appears, with $h > 0$ (otherwise the

other polynomial involved in the S -polynomial should have x_0^d as initial term). It is enough to investigate the terms $x^\beta x_1$, where x^β belongs to \mathcal{B} , because, if $x^\beta x_1$ belongs to J_{d+1} , then $x^\beta x_h$ belongs to J_{d+1} for any $h > 0$. Since J is a hilb-segment ideal, every term x^β of \mathcal{B} is less than x^α . By the definition of \mathcal{B} , every term x^β of \mathcal{B} is always involved in a reduction step so that it does not appear in any generator of $L(J)$ (see Criterion 4.6 of [12]). The number of such terms is at least $|G(J)| \cdot |\mathcal{B}|$ and we are done. \square

Theorem 6.2. *If for the hilb-segment ideal J we have $|G(J)| \cdot |\mathcal{B}| > nd$, then J corresponds to a singular point in Hilb_d^n .*

Proof. Let H_{RS} be the unique irreducible component of Hilb_d^n containing the lexicographic point [20]. Recall that H_{RS} has dimension equal to nd and that every Borel ideal belongs to H_{RS} [21]. Since J is a hilb-segment ideal w.r.t. \preceq , the Groebner stratum $St_h(J, \preceq)$ is an open subset of H_{RS} (Corollary 6.7 of [12]) and hence $\dim St_h(J, \preceq) = nd$. The point J is smooth for Hilb_d^n if and only if it is smooth for the Groebner stratum $St_h(J, \preceq)$ (see Corollary 4.5 of [12]). Thus, J is smooth if and only if $ed(St_h(J, \preceq)) = nd$ (Corollary 4.5 of [12]). By Lemma 6.1 the thesis is proved. \square

Example 6.3. Let I be the generic initial ideal of 7 general points in \mathbb{P}^3 w.r.t. revlex order, i.e. the (saturated) hilb-segment ideal with Hilbert polynomial $p(z) = 7$. We obtain $G(I) = \{x_3^2, x_2x_3, x_2^2, x_1^2x_3, x_1^2x_2, x_1^3\}$ and $\mathcal{B} = \{x_0^5x_1^2, x_0^5x_1x_2, x_0^5x_2^2\}$. Thus $|G(I)| \cdot |\mathcal{B}| = 6 \cdot 3 = 18 < nd = 21$. But, as it is shown in [12], we can compute directly the Gröbner stratum of $I_{\geq 7}$ showing that its embedding dimension is $27 > nd = 21$. Actually, in [12] the authors construct the stratum of $I_{\geq 3}$ which is isomorphic to the stratum of $I_{\geq 7}$, obtaining a big improvement of the computation.

For 8 points in \mathbb{P}^3 , we have $\mathcal{N}(I)_8 = \{x_0^8, x_0^7x_1, x_0^7x_2, x_0^7x_3, x_0^6x_1^2, x_0^6x_1x_2, x_0^6x_1x_3, x_0^6x_2^2\}$, so that $\mathcal{B} = \{x_0^6x_1^2, x_0^6x_1x_2, x_0^6x_1x_3, x_0^6x_2^2\}$ and $|\mathcal{B}| = 4$. Since $|G(I)| = 7$, we get $|G(I)| \cdot |\mathcal{B}| = 7 \cdot 4 = 28 > 3 \cdot 8 = 24$.

Theorem 6.4. *For every $d > n \geq 3$, the hilb-segment ideal J w.r.t the revlex order corresponds to a singular point in Hilb_d^n .*

Proof. In Remark 3.14 we observed that J must have maximal Hilbert function, so that the regularity ρ_H of its Hilbert function is the integer such that $\binom{\rho_H - 1 + n}{n} < d \leq \binom{\rho_H + n}{n}$. Moreover, if $d = \binom{\rho_H + n}{n}$ then $|G(J)| = \binom{\rho_H + n}{n-1}$, otherwise $|G(J)| \geq \binom{\rho_H + n - 1}{n-1}$.

If $d = n + 1$ then $\rho_H = 1$ and $J = (x_1, \dots, x_n)^2$, so that $|G(J)| = \binom{2+n-1}{n-1} = \binom{n+1}{2}$. Moreover, \mathcal{B} consists of the terms of type $x_0^{d-1}x_i$ with $i > 0$, thus $|\mathcal{B}| = n$ and the statement is true because $\binom{n+1}{2} \cdot n > n(n+1)$ for every $n \geq 3$.

If $d \geq n + 2$ then $\rho_H \geq 2$.

If $d = \binom{\rho_H + n}{n}$, we show that $|\mathcal{B}| > \rho_H + 1$. If we multiply every term of degree ρ_H in the variables x_1, \dots, x_n by $x_0^{d-\rho_H}$, we obtain terms of degree d that multiplied by x_1 give $\binom{\rho_H + n - 1}{n-1}$ terms which belong to \mathcal{B} . Thus $|\mathcal{B}| \geq \binom{\rho_H + n - 1}{n-1} > \rho_H + 1$ and $|G(J)| \cdot |\mathcal{B}| > \frac{dn}{\rho_H + 1} \cdot (\rho_H + 1) = dn$.

If $d < \binom{\rho_H + n}{n}$ and $\rho_H \geq 3$, we show that $|\mathcal{B}| \geq \rho_H + n$. Let x^β any of the $\binom{\rho_H + n - 2}{n-1}$ terms of degree $\rho_H - 1$ in the variables x_1, \dots, x_n . Thus, if $x^\beta x_1$ belongs to J , then $x^\beta x_0^{d-\rho_H+1}$ belongs to \mathcal{B} ; otherwise, if $x^\beta x_1$ does not belong to I , then $x^\beta x_0^{d-\rho_H}x_1$ belongs to \mathcal{B} . Anyway, the term $x^\beta x_1^2$ belongs to J because it is not divided by x_0 and has degree $\rho_H + 1$ and the terms of $\mathcal{N}(I)_{\rho_H+1}$ are all divided by x_0 . Such terms are all distinct, so that $|\mathcal{B}| \geq \binom{\rho_H + n - 2}{n-1}$. Now it is easy to check that $\binom{\rho_H + n - 2}{n-1} \geq \rho_H + n$, for every $\rho_H \geq 3$ and $n \geq 3$. Thus, $|G(J)| \cdot |\mathcal{B}| \geq \binom{\rho_H + n - 1}{n-1} \cdot (\rho_H + n) > nd$, by Remark 3.14(3).

It remains to study the case $\rho_H = 2$ in which $|G(I)| \geq \binom{n+1}{2}$ and $|\mathcal{B}| \geq n$, because of the arguments above, with $n = \binom{n+1}{n} < d < \binom{2+n}{n}$. If $d < \binom{n+1}{2}$, then we get immediately $|G(J)| \cdot |\mathcal{B}| > nd$. If $\binom{n+1}{2} < d < \binom{n+1}{2}$, all the d terms of $\mathcal{N}(J)_d$ are in \mathcal{B} except at most the

$n + 1$ terms divided by x_0^{d-1} . Thus, in this case $|\mathcal{B}| \geq d - (n + 1)$, which is $\geq n + 2$ except for $n = 3$ and $d = 7, 8$. These last two cases have been directly studied in Example 6.3. \square

Observing that:

- (i) a segment ideal w.r.t. revlex order gives rise to a singular point in \mathcal{Hilb}_d^n and defines a scheme not contained in any hyperplane;
- (ii) a segment ideal w.r.t. lex order gives rise to a smooth point in \mathcal{Hilb}_d^n and defines a scheme contained in some hyperplane;

one might guess that there is a relationship between the smoothness of a point in \mathcal{Hilb}_d^n corresponding to a (saturated) monomial ideal and the presence of linear forms in the ideal. But, the next example (for which we are indebted to G. Floystad) shows that this is not the case.

Example 6.5. (i) Let $I = (x_1^{a_1}, \dots, x_i^{a_i}, \dots, x_n^{a_n})$ be a (saturated monomial) complete intersection ideal defining a 0-dimensional scheme \mathbb{X} of degree $d = \prod_i a_i$ in \mathbb{P}^n and let \mathbf{z}_I denote the corresponding point of \mathcal{Hilb}_d^n . Being I a monomial ideal, \mathbf{z}_I lies in the closure of the lexicographic point component of \mathcal{Hilb}_d^n (see for example Corollary 18.30 of [17]). Using the normal sheaf to \mathbb{X} , we get that the dimension of the tangent space to \mathcal{Hilb}_d^n at \mathbf{z}_I is nd , coinciding with that of the lexicographic point component. Thus I gives an example of a monomial ideal which does not contain linear forms and corresponds to a smooth point in \mathcal{Hilb}_d^n .

(ii) Let $J \subset K[x_1, \dots, x_n]$ be a saturated monomial ideal giving a singular point \mathbf{z}_J of \mathcal{Hilb}_d^{n-1} , so that the dimension of the tangent space to \mathcal{Hilb}_d^{n-1} in \mathbf{z}_J is $\alpha > (n - 1)d$. Taking $\tilde{J} = ((x_0) + J) \subset K[x_0, \dots, x_n]$, the dimension of the tangent space to \mathcal{Hilb}_d^n in $\mathbf{z}_{\tilde{J}}$ is $\alpha + d > (n - 1)d + d = nd$, hence $\mathbf{z}_{\tilde{J}}$ is singular too.

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