

**CONFORMAL EMBEDDINGS OF AFFINE VERTEX
ALGEBRAS IN MINIMAL W -ALGEBRAS II:
DECOMPOSITIONS**

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ABSTRACT. We present methods for computing the explicit decomposition of the minimal simple affine W -algebra $W_k(\mathfrak{g}, \theta)$ as a module for its maximal affine subalgebra $\mathcal{V}_k(\mathfrak{g}^{\natural})$ at a conformal level k , that is, whenever the Virasoro vectors of $W_k(\mathfrak{g}, \theta)$ and $\mathcal{V}_k(\mathfrak{g}^{\natural})$ coincide. A particular emphasis is given on the application of affine fusion rules to the determination of branching rules. In almost all cases when \mathfrak{g}^{\natural} is a semisimple Lie algebra, we show that, for a suitable conformal level k , $W_k(\mathfrak{g}, \theta)$ is isomorphic to an extension of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ by its simple module. We are able to prove that in certain cases $W_k(\mathfrak{g}, \theta)$ is a simple current extension of $\mathcal{V}_k(\mathfrak{g}^{\natural})$. In order to analyze more complicated non simple current extensions at conformal levels, we present an explicit realization of the simple W -algebra $W_k(\mathfrak{sl}(4), \theta)$ at $k = -8/3$. We prove, as conjectured in [3], that $W_k(\mathfrak{sl}(4), \theta)$ is isomorphic to the vertex algebra $\mathcal{R}^{(3)}$, and construct infinitely many singular vectors using screening operators. We also construct a new family of simple current modules for the vertex algebra $V_k(\mathfrak{sl}(n))$ at certain admissible levels and for $V_k(\mathfrak{sl}(m|n))$, $m \neq n$, $m, n \geq 1$ at arbitrary levels.

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1. INTRODUCTION

Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a basic simple Lie superalgebra. Choose a Cartan subalgebra \mathfrak{h} for $\mathfrak{g}_{\overline{0}}$ and let Δ be the set of roots. Choose a subset of positive roots such that the minimal root $-\theta$ is even. Let $W_k(\mathfrak{g}, \theta)$ be the simple minimal affine W -algebra at level k associated to (\mathfrak{g}, θ) [30], [31], [10]. Let $\mathcal{V}_k(\mathfrak{g}^{\natural})$ be its maximal affine subalgebra (see (3.2) and Definition 4.1). In [10] we classified the levels k such that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is conformally embedded into $W_k(\mathfrak{g}, \theta)$, i.e. such that the Virasoro vectors of $W_k(\mathfrak{g}, \theta)$ and $\mathcal{V}_k(\mathfrak{g}^{\natural})$ coincide. We call these levels the *conformal levels*. We proved that, if k is a conformal level, then $W_k(\mathfrak{g}, \theta)$ either collapses to $\mathcal{V}_k(\mathfrak{g}^{\natural})$ or

$$k \in \left\{ -\frac{2}{3}h^{\vee}, -\frac{h^{\vee} - 1}{2} \right\}.$$

(see Section 4 with review of main results from [10]).

In the present paper, we study the decomposition of $W_k(\mathfrak{g}, \theta)$ as a $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module when k is a conformal level. It turns out that we can use methods similar to those we developed for studying conformal embeddings of affine vertex algebras in [9] (see also [7], [33]). Our main technical tool is the representation theory of affine vertex algebras at admissible and negative integer levels, in particular fusion rules for $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules, as we shall explain below.

As in [9], it is natural to discuss the case in which \mathfrak{g}^{\natural} is semisimple separately from the case in which \mathfrak{g}^{\natural} has a nontrivial center. In the latter case,

one has the eigenspace decomposition for the action of the center:

$$W_k(\mathfrak{g}, \theta) \cong \bigoplus_{i \in \mathbb{Z}} W_k(\mathfrak{g}, \theta)^{(i)},$$

and we prove the following result (see Theorems 6.4 and 6.5).

Theorem 1.1. *Consider the conformal embeddings of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ into $W_k(\mathfrak{g}, \theta)$:*

- (1) $\mathfrak{g} = sl(n)$ or $\mathfrak{g} = sl(2|n)$ ($n \geq 4$), $\mathfrak{g} = sl(m|n)$, $m > 2$, $m \neq n + 3, n + 2, n, n - 1$, conformal level $k = -\frac{h^{\vee}-1}{2}$;
- (2) $\mathfrak{g} = sl(n)$ ($n = 5$ or $n \geq 7$), $\mathfrak{g} = sl(2|n)$ ($n \geq 3$), $\mathfrak{g} = sl(m|n)$, $m > 2$, $m \neq n + 6, n + 4, n + 2, n$, conformal level $k = -\frac{2}{3}h^{\vee}$.
- (3) $\mathfrak{g} = sl(4)$, conformal level $k = -\frac{2}{3}h^{\vee} = -\frac{8}{3}$.

Then $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is simple and, in cases (1), (2), each $W_k(\mathfrak{g}, \theta)^{(i)}$ is an irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module, while in case (3) each $W_k(\mathfrak{g}, \theta)^{(i)}$ is an infinite sum of irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules.

It is quite surprising that there is only one case when each $W_k(\mathfrak{g}, \theta)^{(i)}$ is an infinite sum of irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules, namely the conformal embedding of $\mathcal{V}_k(gl(2))$ into $W_k(sl(4), \theta)$ and $k = -\frac{2}{3}h^{\vee} = -8/3$. This is related to the new explicit realization of $W_k(sl(4), \theta)$, conjectured in [3], as the vertex operator algebra $\mathcal{R}^{(3)}$. This conjecture is proven in Section 7 of the present paper. Our realization is based on the logarithmic extension of the Wakimoto modules for $V_{-5/3}(gl(2))$ by singular vectors constructed using screening operators. The simplicity of $\mathcal{R}^{(3)}$ is proved by constructing certain relations in the corresponding Zhu algebra. A generalization of this construction, together with more applications related to vertex algebras appearing in LCFT will be considered in [4].

Theorem 1.1 is proved by studying fusion rules for $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules. Recall that fusion rules (or fusion coefficients) are the dimensions of certain spaces of intertwining operators. The fact that a certain fusion coefficient is zero implies that a given $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module cannot appear in the decomposition of $W_k(\mathfrak{g}, \theta)^{(i)}$. In many cases this is the only information we need to establish both the simplicity of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ and the semisimplicity of $W_k(\mathfrak{g}, \theta)$ as a $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module: see Theorem 6.2. This information, very often, is obtained just from the decomposition of tensor products of certain simple finite dimensional \mathfrak{g}^{\natural} -modules.

In the cases when we are able to compute precisely the fusion rules, we are also able to describe explicitly the decomposition of $W_k(\mathfrak{g}, \theta)$ as a $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module. Indeed, we prove that the modules $W_k(\mathfrak{g}, \theta)^{(i)}$ are simple currents in a suitable category of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules in the following instances: either when $\mathcal{V}_k(\mathfrak{g}^{\natural})$ contains a subalgebra which is an admissible affine vertex algebra of type A (cf. Theorem 8.1), or in the case of the affine W -algebras $W_{-2}(sl(5), \theta)$ (cf. Corollary 8.3), $W_{-2}(sl(n+5|n), \theta)$ (cf. Corollary 9.3, Remark 9.2). We believe that this property of the modules $W_k(\mathfrak{g}, \theta)^{(i)}$ holds in all cases (1) and (2) from Theorem 1.1.

As a byproduct, we construct a new family of simple current modules for the vertex superalgebra $V_k(sl(m|n))$ which belong to the category KL_k (cf. Theorem 9.1).

Next we consider the cases when \mathfrak{g}^{\natural} is a semisimple Lie algebra. Then we have a natural decomposition

$$W_k(\mathfrak{g}, \theta) = W_k(\mathfrak{g}, \theta)^{\bar{0}} \oplus W_k(\mathfrak{g}, \theta)^{\bar{1}},$$

according to the parity of twice the conformal weight.

The subspaces $W_k(\mathfrak{g}, \theta)^{\bar{i}}$ are naturally $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules, so we are reduced to compute the decompositions of the $W_k(\mathfrak{g}, \theta)^{\bar{i}}$. We solve our problem in many cases by showing that the subspaces $W_k(\mathfrak{g}, \theta)^{\bar{i}}$ are actually irreducible as $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules:

Theorem 1.2. *Consider the following conformal embeddings of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ into $W_k(\mathfrak{g}, \theta)$ [10]:*

- (1) $\mathfrak{g} = so(n)$ ($n \geq 8, n \neq 11$), $\mathfrak{g} = osp(4|n)$ ($n \geq 2$), $\mathfrak{g} = osp(m|n)$ ($m \geq 5, m \neq n+r, r \in \{-1, 2, 3, 4, 6, 7, 8, 11\}$) or \mathfrak{g} is of type G_2, F_4, E_6, E_7, E_8 or \mathfrak{g} is a Lie superalgebra $D(2, 1, a)$ ($a = 1, 1/4$), $F(4)$ and $k = -\frac{h^\vee - 1}{2}$.
- (2) $\mathfrak{g} = sp(n)$ ($n \geq 6$), $\mathfrak{g} = spo(2|m)$ ($m \geq 3, m \neq 4$), $\mathfrak{g} = spo(n|m)$ ($n \geq 4, m \neq n+2, n, n-2, n-4$) and $k = -\frac{2}{3}h^\vee$.

Then $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is simple and each $W_k(\mathfrak{g}, \theta)^{\bar{i}}$, $i = 0, 1$, is an irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module.

In most cases, the proof of this theorem follows quite easily combining information on fusion rules coming from tensor product decompositions of \mathfrak{g}^{\natural} -modules with basic principles of vertex algebra theory, such as Galois theory: see Theorem 6.8. In some cases our proof is technically involved and it uses the representation theory of the admissible affine vertex algebra $V_{-1/2}(sl(2))$: see Theorem 6.9.

We also believe that the conformal embeddings listed in Theorem 1.2 provide all cases where $W_k(\mathfrak{g}, \theta)$ is isomorphic to an extension of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ by its simple module. This result is also interesting from the perspective of extensions of affine vertex algebras, since it gives a realization of a broad list of extensions which (so far) cannot be constructed by other methods.

In our forthcoming papers we shall consider embeddings of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ into $W_k(\mathfrak{g}, \theta)$ when critical levels appear.

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Notation. The base field is \mathbb{C} . As usual, tensor product of a family of vector spaces ranging over the empty set is meant to be \mathbb{C} . For

a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ we set $\text{Dim}V = \dim V_{\bar{0}} | \dim V_{\bar{1}}$ and $\text{sdim}V = \dim V_{\bar{0}} - \dim V_{\bar{1}}$.

2. PRELIMINARIES

2.1. Vertex algebras. Here we fix notation about vertex algebras (cf. [28]). Let V be a vertex algebra, with vacuum vector $\mathbf{1}$. The vertex operator corresponding to the state $a \in V$ is denoted by

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad \text{where } a_{(n)} \in \text{End}V.$$

We frequently use the notation of λ -bracket and normally ordered product:

$$[a_\lambda b] = \sum_{i \geq 0} \frac{\lambda^i}{i!} (a_{(i)} b), \quad : ab := a_{(-1)} b.$$

If V admits a Virasoro(=conformal) vector and Δ_a is the conformal weight of a , then we also write the corresponding vertex operator as

$$Y(a, z) = \sum_{m \in \mathbb{Z} - \Delta_a} a_m z^{-m - \Delta_a},$$

so that

$$a_{(n)} = a_{n - \Delta_a + 1}, \quad n \in \mathbb{Z}, \quad a_m = a_{(m + \Delta_a - 1)}, \quad m \in \mathbb{Z} - \Delta_a.$$

The product of two subsets A, B of V is

$$A \cdot B = \text{span}(a_{(n)} b \mid n \in \mathbb{Z}, a \in A, b \in B).$$

This product is associative (cf. [13]).

2.2. Intertwining operators and fusion rules. Let V be a vertex algebra. A V -module is a vector superspace M endowed with a parity preserving map Y^M from V to the superspace of $\text{End}(M)$ -valued fields

$$a \mapsto Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1}$$

such that

- (1) $Y^M(|0\rangle, z) = I_M$,
- (2) for $a, b \in V$, $m, n, k \in \mathbb{Z}$,

$$\begin{aligned} & \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)}^M \\ &= \sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{(m+n-j)}^M b_{(k+j)}^M - p(a, b) (-1)^n b_{(k+n-j)}^M a_{(m+j)}^M), \end{aligned}$$

Given three V -modules M_1, M_2, M_3 , an *intertwining operator of type* $\begin{bmatrix} M_3 \\ M_1 \ M_2 \end{bmatrix}$ (cf. [22], [23]) is a map $I : a \mapsto I(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^I z^{-n-1}$ from M_1 to the space of $End(M_2, M_3)$ -valued fields such that for $a \in V, b \in M_1, m, n \in \mathbb{Z}$,

$$\begin{aligned} & \sum_{j \in \mathbb{N}} \binom{m}{j} (a_{(n+j)}^{M_1} b)_{(m+k-j)}^I \\ &= \sum_{j \in \mathbb{N}} (-1)^j \binom{n}{j} (a_{(m+n-j)}^{M_3} b_{(k+j)}^I - p(a, b) (-1)^n b_{(k+n-j)}^I a_{(m+j)}^{M_2}). \end{aligned}$$

We let $I(\begin{smallmatrix} M_3 \\ M_1 \ M_2 \end{smallmatrix})$ denote the space of intertwining operators of type $\begin{bmatrix} M_3 \\ M_1 \ M_2 \end{bmatrix}$, and set

$$N_{M_1, M_2}^{M_3} = \dim I \left(\begin{smallmatrix} M_3 \\ M_1 \ M_2 \end{smallmatrix} \right).$$

When $N_{M_1, M_2}^{M_3}$ is finite, it is usually called a *fusion coefficient*.

Assume that in a category K of $\mathbb{Z}_{\geq 0}$ -graded V -modules, the irreducible modules $\{M_i \mid i \in I\}$, where I is an index set, have the following properties

- (1) for every $i, j \in I$ $N_{M_i, M_j}^{M_k}$ is finite for any $k \in I$;
- (2) $N_{M_i, M_j}^{M_k} = 0$ for all but finitely many $k \in I$.

Then the algebra with basis $\{e_i \in I\}$ and product

$$e_i \cdot e_j = \sum_{k \in I} N_{M_i, M_j}^{M_k} e_k$$

is called the *fusion algebra* of V, K . (Note that we consider only fusion rules between irreducible modules).

Let K be a category of V -modules. Let M_1, M_2 be irreducible V -modules in K . Given an irreducible V -module M_3 in K , we will say that the fusion rule

$$(2.1) \quad M_1 \times M_2 = M_3$$

holds in K if $N_{M_1, M_2}^{M_3} = 1$ and $N_{M_1, M_2}^R = 0$ for any other irreducible V -module R in K which is not isomorphic to M_3 .

We say that an irreducible V -module M_1 is a *simple current* in K if M_1 is in K and, for every irreducible V -module M_2 in K , there is an irreducible V -module M_3 in K such that the fusion rule (2.1) holds in K (see [19]).

2.3. Commutants. Let U be a vertex subalgebra of V . Then the commutant of U in V (cf. [22]) is the following vertex subalgebra of V :

$$\text{Com}(U, V) = \{v \in V \mid [Y(a, z), Y(v, w)] = 0, \forall a \in U\}.$$

In the case when U is an affine vertex algebra, say $U = V_k(\mathfrak{g})$ (see below), it is easy to see that

$$\text{Com}(U, V) = \{v \in V \mid x_{(n)}v = 0 \ \forall x \in \mathfrak{g}, n \geq 0\}.$$

2.4. Zhu algebras. Assume that V is a vertex algebra, endowed with a conformal vector ω such that the conformal weights Δ_v of $v \in V$ are in $\frac{1}{2}\mathbb{Z}$. Then

$$V = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} V(r), \quad V(r) = \{v \in V \mid \Delta_v = r\}.$$

Set

$$V^{\bar{0}} = \bigoplus_{r \in \mathbb{Z}} V(r), \quad V^{\bar{1}} = \bigoplus_{r \in \frac{1}{2} + \mathbb{Z}} V(r).$$

We define two bilinear maps $*$: $V \times V \rightarrow V$, \circ : $V \times V \rightarrow V$ as follows: for homogeneous $a, b \in V$, let

$$(2.2) \quad a * b = \begin{cases} \text{Res}_x Y(a, x) \frac{(1+x)^{\Delta_a}}{x} b & \text{if } a, b \in V^{\bar{0}}, \\ 0 & \text{if } a \text{ or } b \in V^{\bar{1}} \end{cases}$$

$$(2.3) \quad a \circ b = \begin{cases} \text{Res}_x Y(a, x) \frac{(1+x)^{\Delta_a}}{x^2} b & \text{if } a \in V^{\bar{0}}, \\ \text{Res}_x Y(a, x) \frac{(1+x)^{\Delta_a - \frac{1}{2}}}{x} b & \text{if } a \in V^{\bar{1}}. \end{cases}$$

Next, we extend $*$ and \circ to $V \otimes V$ linearly, and denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. The space $A(V)$ has a unital associative algebra structure, with multiplication induced by $*$. The algebra $A(V)$ is called the Zhu algebra of V . The image of $v \in V$, under the natural map $V \mapsto A(V)$ will be denoted by $[v]$. In the case when $V^{\bar{0}} = V$ we get the usual definition of Zhu algebra for vertex operator algebras.

An important result, proven by Zhu [38] for \mathbb{Z} -graded V and later by Kac-Wang [35] for $\frac{1}{2}\mathbb{Z}$ -graded V , is the following theorem.

Theorem 2.1. *There is a one-to-one correspondence between irreducible $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded V -modules and irreducible $A(V)$ -modules.*

2.5. Affine vertex algebras. Let \mathfrak{a} be a Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form B . The universal affine vertex algebra $V^B(\mathfrak{a})$ is the universal enveloping vertex algebra of the Lie conformal superalgebra $R = (\mathbb{C}[T] \otimes \mathfrak{a}) \oplus \mathbb{C}$ with λ -bracket given by

$$[a_\lambda b] = [a, b] + \lambda B(a, b), \quad a, b \in \mathfrak{a}.$$

In the following, we shall say that a vertex algebra V is an *affine vertex algebra* if it is a quotient of some $V^B(\mathfrak{a})$.

If \mathfrak{a} is simple or is a one dimensional abelian Lie algebra, then one usually fixes a nondegenerate invariant supersymmetric bilinear form $(\cdot | \cdot)$ on \mathfrak{a} . Any invariant supersymmetric bilinear form is therefore a constant multiple of $(\cdot | \cdot)$. In particular, if $B = k(\cdot | \cdot)$ ($k \in \mathbb{C}$), then we denote $V^B(\mathfrak{a})$ by $V^k(\mathfrak{a})$.

Let $h_{\mathfrak{a}}^{\vee}$ be half of the eigenvalue of the Casimir element corresponding to $(\cdot|\cdot)$; if $h_{\mathfrak{a}}^{\vee} \neq -k$, then $V^k(\mathfrak{a})$ admits a unique irreducible quotient which we denote by $V_k(\mathfrak{a})$.

In the same hypothesis, $V^k(\mathfrak{a})$ is equipped with a Virasoro vector

$$(2.4) \quad \omega_{sug}^{\mathfrak{a}} = \frac{1}{2(k + h_{\mathfrak{a}}^{\vee})} \sum_{i=1} : b_i a_i :,$$

where $\{a_i\}$ is a basis of \mathfrak{a} and $\{b_i\}$ is its dual basis with respect to $(\cdot|\cdot)$. If a vertex algebra V is some quotient of $V^k(\mathfrak{a})$, we will say that k is the level of V .

A module M for $\widehat{\mathfrak{a}}$ is said to be of level k if K acts on M by kI_M . Finally recall that an irreducible highest weight module $L(\lambda)$, over an affine algebra $\widehat{\mathfrak{a}}$ is said to be *admissible* [26] if

- (1) $(\lambda + \rho)(\alpha) \notin \{0, -1, -2, \dots\}$, for each positive coroot α ;
- (2) the rational linear span of positive simple coroots equals the rational linear span of the coroots which are integral valued on $\lambda + \rho$.

2.6. Heisenberg vertex algebras. In the special case when \mathfrak{a} is an abelian Lie algebra, $V^B(\mathfrak{a})$ is of course a Heisenberg vertex algebra. If this is the case, we will denote $V^B(\mathfrak{a})$ by $M_{\mathfrak{a}}(B)$. In the special case when \mathfrak{a} is one dimensional, then we can choose a basis $\{\alpha\}$ of \mathfrak{a} and the form $(\cdot|\cdot)$ so that $(\alpha|\alpha) = 1$. With these choices we denote $V^k(\mathfrak{a})$ by $M_{\alpha}(k)$ or simply by $M(k)$ when the reference to the generator α need not to be explicit. The vertex algebra $M(k)$ is called the universal Heisenberg vertex algebra of level k generated by α . Recall that, if $k \neq 0$, $M_{\alpha}(k)$ is simple and that $M(k) \cong M(1)$. The irreducible $M_{\alpha}(k)$ -modules are the modules $M_{\alpha}(k, s)$ (or simply $M(k, s)$) generated by a vector v_s with action, for $n \in \mathbb{Z}_+$, given by $\alpha_{(n)}v_s = \delta_{n,0}sv_s$.

The irreducible modules of the Heisenberg vertex algebra $M(k)$ are all simple currents in the category of $M(k)$ -modules. Indeed we have the following fusion rules (cf.[22]):

$$(2.5) \quad M(k, s_1) \times M(k, s_2) = M(k, s_1 + s_2) \quad (s_1, s_2 \in \mathbb{C}).$$

If \mathfrak{a} is simple or one-dimensional even abelian with fixed nondegenerate invariant supersymmetric bilinear form $(\cdot|\cdot)$, the affinization of \mathfrak{a} is the Lie superalgebra $\widehat{\mathfrak{a}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{a}) \oplus \mathbb{C}d \oplus \mathbb{C}K$ where K is a central element, and d acts as td/dt . We choose the central element K so that

$$[t^s \otimes x, t^r \otimes y] = t^{s+r} \otimes [x, y] + \delta_{r,-s}rK(x|y).$$

Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{a} and $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ a Cartan subalgebra of $\widehat{\mathfrak{a}}$. Let $\Lambda_0 \in \widehat{\mathfrak{h}}^*$ be defined by $\Lambda_0(K) = 1$, $\Lambda_0(\mathfrak{h}) = \Lambda_0(d) = 0$. We fix a set of simple roots for $\widehat{\mathfrak{a}}$ and denote by $\rho \in \widehat{\mathfrak{h}}^*$ a corresponding Weyl vector. We shall denote by $L_{\mathfrak{a}}(\lambda)$ the irreducible highest weight $V^k(\mathfrak{a})$ -module of highest weight $\lambda \in \widehat{\mathfrak{h}}^*$. Sometimes, if no confusion may arise, we simply write $L(\lambda)$. Similarly, we shall denote by $V_{\mathfrak{a}}(\lambda)$ or simply by $V(\lambda)$ the irreducible highest

weight \mathfrak{a} -module of highest weight $\lambda \in \mathfrak{h}^*$. Note that in the case when \mathfrak{a} is a one dimensional abelian Lie algebra $\mathbb{C}\alpha$, then $M_\alpha(k, s) = L(k\Lambda_0 + s\lambda_1)$ where $\lambda_1 \in (\mathbb{C}\alpha)^*$ is defined by setting $\lambda_1(\alpha) = 1$.

2.7. Rank one lattice vertex algebra $V_{\mathbb{Z}\alpha}$. Assume that L is an integral, positive definite lattice; let L° be its dual lattice. Set V_L to be the lattice vertex algebra (cf. [28]) associated to L .

The set of isomorphism classes of irreducible V_L -modules is parametrized by L°/L (cf. [16]). Let $V_{\bar{\lambda}}$ denote the irreducible V_L -module corresponding to $\bar{\lambda} = \lambda + L \in L^\circ/L$. Every irreducible V_L -module is a simple current.

We shall now consider rank one lattice vertex algebras. For $n \in \mathbb{Z}_{>0}$, let $M(n)$ be the universal Heisenberg vertex algebra generated by α and let F_n denote the lattice vertex algebra $V_{\mathbb{Z}\alpha} = M(n) \otimes \mathbb{C}[\mathbb{Z}\alpha]$ associated to the lattice $L = \mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = n$. The dual lattice of L is $L^\circ = \frac{1}{n}L$. For $i \in \{0, \dots, n-1\}$, set $F_n^i = V_{\frac{i}{n}\alpha + \mathbb{Z}\alpha}$. Then the set

$$\{F_n^i \mid i = 0, \dots, n-1\}$$

provides a complete list of non-isomorphic irreducible F_n -module. We choose the following Virasoro vector in F_n :

$$\omega_{F_n} = \frac{1}{2n} : \alpha \alpha : .$$

As a $M(n)$ -module, F_n decomposes as

$$(2.6) \quad F_n = \bigoplus_{j \in \mathbb{Z}} M(n) e^{j\alpha} = \bigoplus_{j \in \mathbb{Z}} M(n, jn)$$

The following result is a consequence of the result of H. Li and X. Xu [37] on characterization of lattice vertex algebras.

Proposition 2.2. *Assume that $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a \mathbb{Z} -graded vertex algebra satisfying the following properties*

- (1) V is a subalgebra of a simple vertex algebra W ;
- (2) there exists a Heisenberg vector $\alpha \in V_0$ such that $V_0 = M_\alpha(n)$, and $V_i \cong M_\alpha(n, in)$ as a V_0 -module.

Then V is a simple vertex algebra and $V \cong F_n$.

Proof. The Main Theorem of [37] implies that a simple vertex algebra satisfying condition (2) is isomorphic to F_n . To prove simplicity, we first observe that

$$(2.7) \quad Y(v, z)w \neq 0 \quad \forall v, w \in V,$$

which in our setting holds since W is simple. Now (2.7) and the fusion rules (2.5) imply that

$$V_i \cdot V_j = V_{i+j} \quad (i, j \in \mathbb{Z}).$$

This implies that V is simple, and the claim follows.

3. MINIMAL QUANTUM AFFINE W -ALGEBRAS

In this section we briefly recall some results of [30] and [10]. We include an example which contains explicit λ -bracket formulas for $W^k(\mathfrak{sl}(4), \theta)$ which we shall need in Section 7.

We first recall the definition of minimal affine W -algebras.

Let \mathfrak{g} be a basic simple Lie superalgebra. Choose a Cartan subalgebra \mathfrak{h} for $\mathfrak{g}_{\bar{0}}$ and let Δ be the set of roots. Fix a minimal root $-\theta$ of \mathfrak{g} . (A root $-\theta$ is called *minimal* if it is even and there exists an additive function $\varphi : \Delta \rightarrow \mathbb{R}$ such that $\varphi|_{\Delta} \neq 0$ and $\varphi(\theta) > \varphi(\eta)$, $\forall \eta \in \Delta \setminus \{\theta\}$). We choose root vectors e_{θ} and $e_{-\theta}$ such that

$$[e_{\theta}, e_{-\theta}] = x \in \mathfrak{h}, \quad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of adx defines a *minimal* $\frac{1}{2}\mathbb{Z}$ -gradation ([30, (5.1)]):

$$(3.1) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$. One has

$$(3.2) \quad \mathfrak{g}_0 = \mathfrak{g}^{\natural} \oplus \mathbb{C}x, \quad \mathfrak{g}^{\natural} = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}.$$

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on \mathfrak{g} by the condition

$$(3.3) \quad (\theta|\theta) = 2.$$

The dual Coxeter number h^{\vee} of the pair (\mathfrak{g}, θ) is defined to be half the eigenvalue of the Casimir operator of \mathfrak{g} corresponding to $(\cdot|\cdot)$.

The complete list of the Lie superalgebras \mathfrak{g}^{\natural} , the \mathfrak{g}^{\natural} -modules $\mathfrak{g}_{\pm 1/2}$ (they are isomorphic and self-dual), and h^{\vee} for all possible choices of \mathfrak{g} and of θ (up to isomorphism) is given in Tables 1,2,3 of [30], and it is as follows

Table 1

\mathfrak{g} is a simple Lie algebra.

\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}	\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}
$sl(n), n \geq 3$	$gl(n-2)$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$	n	F_4	$sp(6)$	$\bigwedge^3 \mathbb{C}^6$	9
$so(n), n \geq 5$	$sl(2) \oplus so(n-4)$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-2$	E_6	$sl(6)$	$\bigwedge^3 \mathbb{C}^6$	12
$sp(n), n \geq 2$	$sp(n-2)$	\mathbb{C}^{n-2}	$n/2+1$	E_7	$so(12)$	$spin_{12}$	18
G_2	$sl(2)$	$S^3 \mathbb{C}^2$	4	E_8	E_7	$\dim = 56$	30

Table 2

\mathfrak{g} is not a Lie algebra but \mathfrak{g}^{\natural} is and $\mathfrak{g}_{\pm 1/2}$ is purely odd ($m \geq 1$).

\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}	\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}
$sl(2 m), m \neq 2$	$gl(m)$	$\mathbb{C}^m \oplus (\mathbb{C}^m)^*$	$2-m$	$D(2, 1; a)$	$sl(2) \oplus sl(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	0
$psl(2 2)$	$sl(2)$	$\mathbb{C}^2 \oplus \mathbb{C}^2$	0	$F(4)$	$so(7)$	$spin_7$	-2
$spo(2 m)$	$so(m)$	\mathbb{C}^m	$2-m/2$	$G(3)$	G_2	$\text{Dim} = 0 7$	-3/2
$osp(4 m)$	$sl(2) \oplus sp(m)$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$2-m$				

Table 3

Both \mathfrak{g} and \mathfrak{g}^{\natural} are not Lie algebras ($m, n \geq 1$).

\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}
$sl(m n), m \neq n, m > 2$	$gl(m-2 n)$	$\mathbb{C}^{m-2 n} \oplus (\mathbb{C}^{m-2 n})^*$	$m-n$
$psl(m m), m > 2$	$sl(m-2 m)$	$\mathbb{C}^{m-2 m} \oplus (\mathbb{C}^{m-2 m})^*$	0
$spo(n m), n \geq 4$	$spo(n-2 m)$	$\mathbb{C}^{n-2 m}$	$1/2(n-m)+1$
$osp(m n), m \geq 5$	$osp(m-4 n) \oplus sl(2)$	$\mathbb{C}^{m-4 n} \otimes \mathbb{C}^2$	$m-n-2$
$F(4)$	$D(2,1;2)$	Dim = 6 4	3
$G(3)$	$osp(3 2)$	Dim = 4 4	2

In this paper we shall exclude the case of $\mathfrak{g} = sl(n+2|n)$, $n > 0$. In all other cases the Lie superalgebra \mathfrak{g}^{\natural} decomposes in a direct sum of ideals, called components of \mathfrak{g}^{\natural} :

$$(3.4) \quad \mathfrak{g}^{\natural} = \bigoplus_{i \in I} \mathfrak{g}_i^{\natural},$$

where each summand is either the (at most 1-dimensional) center of \mathfrak{g}^{\natural} or is a basic classical simple Lie superalgebra different from $psl(n|n)$. We will also exclude $\mathfrak{g} = sl(2)$.

It follows from the tables that the index set I has cardinality $r = 0, 1, 2$, or 3. The case $r = 0$, i.e. $\mathfrak{g}^{\natural} = \{0\}$, happens if and only if $\mathfrak{g} = spo(2|1)$. In the case when the center is non-zero (resp. $\{0\}$) we use $I = \{0, 1, \dots, r-1\}$ (resp. $I = \{1, \dots, r\}$) as the index set, and denote the center of \mathfrak{g}^{\natural} by $\mathfrak{g}_0^{\natural}$.

Let $C_{\mathfrak{g}_i^{\natural}}$ be the Casimir operator of $\mathfrak{g}_i^{\natural}$ corresponding to $(\cdot|\cdot)_{|\mathfrak{g}_i^{\natural} \times \mathfrak{g}_i^{\natural}}$. We define the dual Coxeter number $h_{0,i}^{\vee}$ of $\mathfrak{g}_i^{\natural}$ as half of the eigenvalue of $C_{\mathfrak{g}_i^{\natural}}$ acting on $\mathfrak{g}_i^{\natural}$ (which is 0 if $\mathfrak{g}_i^{\natural}$ is abelian). Their values are given in Table 4 of [30].

Let $W^k(\mathfrak{g}, e_{-\theta})$ be the minimal W-algebras of level k studied in [30]. It is known that, for k non-critical, i.e., $k \neq -h^{\vee}$, the vertex algebra $W^k(\mathfrak{g}, e_{-\theta})$ has a unique simple quotient, denoted by $W_k(\mathfrak{g}, e_{-\theta})$.

To simplify notation, we set

$$W^k(\mathfrak{g}, \theta) = W^k(\mathfrak{g}, e_{-\theta}), \quad W_k(\mathfrak{g}, \theta) = W_k(\mathfrak{g}, e_{-\theta}).$$

Throughout the paper we shall assume that $k \neq -h^{\vee}$. In such a case, it is known that $W^k(\mathfrak{g}, f)$ has a Virasoro vector ω , [30, (2.2)] that has central charge [30, (5.7)]

$$(3.5) \quad c(\mathfrak{g}, k) = \frac{k \operatorname{sdim} \mathfrak{g}}{k + h^{\vee}} - 6k + h^{\vee} - 4.$$

It is proven in [30] that the universal minimal W-algebra $W^k(\mathfrak{g}, \theta)$ is freely and strongly generated by the elements $J^{\{a\}}$ (a runs over a basis of \mathfrak{g}^{\natural}), $G^{\{u\}}$ (u runs over a basis of $\mathfrak{g}_{-1/2}$), and the Virasoro vector ω . Furthermore the elements $J^{\{a\}}$ (resp. $G^{\{u\}}$) are primary of conformal weight 1 (resp. 3/2), with respect to ω . The λ -brackets satisfied by these generators have been

given in [30] and, in a simplified form, in [10]. This simplified form reads:

$$(3.6) \quad [J^{\{a\}} \lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda \left((k + h^\vee/2)(a|b) - \frac{1}{4}\kappa_0(a, b) \right), \quad a, b \in \mathfrak{g}^\natural,$$

$$(3.7) \quad [J^{\{a\}} \lambda G^{\{u\}}] = G^{\{[a,u]\}}, \quad a \in \mathfrak{g}^\natural, \quad u \in \mathfrak{g}_{-1/2}.$$

(3.8)

$$\begin{aligned} [G^{\{u\}} \lambda G^{\{v\}}] &= -2(k + h^\vee)(e_\theta|[u, v])\omega + (e_\theta|[u, v]) \sum_{\alpha=1}^{\dim \mathfrak{g}^\natural} : J^{\{u^\alpha\}} J^{\{v^\alpha\}} : \\ &+ \sum_{\gamma=1}^{\dim \mathfrak{g}_{1/2}} : J^{\{[u, u^\gamma]^\natural\}} J^{\{[u, v]^\natural\}} : + 2(k+1)\partial J^{\{[[e_\theta, u], v]^\natural\}} \\ &+ 4\lambda \sum_{i \in I} \frac{p(k)}{k_i} J^{\{[[e_\theta, u], v]_i^\natural\}} + 2\lambda^2 (e_\theta|[u, v])p(k)\mathbf{1}. \end{aligned}$$

Here κ_0 is the Killing form of \mathfrak{g}_0 ; $\{u_\alpha\}$ (resp. $\{v_\gamma\}$) is a basis of \mathfrak{g}^\natural (resp. $\mathfrak{g}_{1/2}$) and $\{u^\alpha\}$ (resp. $\{u^\gamma\}$) is the corresponding dual basis w.r.t. $(\cdot|\cdot)$ (resp. w.r.t. $\langle \cdot, \cdot \rangle_{\text{ne}} = (e_{-\theta}|\cdot, \cdot)$), a^\natural is the image of $a \in \mathfrak{g}_0$ under the orthogonal projection of \mathfrak{g}_0 on \mathfrak{g}^\natural , a_i^\natural is the projection of a^\natural on the i th minimal ideal \mathfrak{g}_i^\natural of \mathfrak{g}^\natural , $k_i = k + \frac{1}{2}(h^\vee - h_{0,i}^\vee)$, and $p(k)$ is the monic polynomial given in the following table [10]:

Table 4

\mathfrak{g}	$p(k)$	\mathfrak{g}	$p(k)$
$sl(m n), n \neq m$	$(k+1)(k+(m-n)/2)$	E_6	$(k+3)(k+4)$
$psl(m m)$	$k(k+1)$	E_7	$(k+4)(k+6)$
$osp(m n)$	$(k+2)(k+(m-n-4)/2)$	E_8	$(k+6)(k+10)$
$spo(n m)$	$(k+1/2)(k+(n-m+4)/4)$	F_4	$(k+5/2)(k+3)$
$D(2, 1; a)$	$(k-a)(k+1+a)$	G_2	$(k+4/3)(k+5/3)$
$F(4), \mathfrak{g}^\natural = so(7)$	$(k+2/3)(k-2/3)$	$G(3), \mathfrak{g}^\natural = G_2$	$(k-1/2)(k+3/4)$
$F(4), \mathfrak{g}^\natural = D(2, 1; 2)$	$(k+3/2)(k+1)$	$G(3), \mathfrak{g}^\natural = osp(3 2)$	$(k+2/3)(k+4/3)$

Note that the linear polynomials k_i always divide $p(k)$ so the coefficients in (3.8) depend polynomially on k .

Example 3.1. Consider $\mathfrak{g} = sl(4)$. Set

$$c = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case $\mathfrak{g}^\natural = \mathfrak{g}_0^\natural \oplus \mathfrak{g}_1^\natural$ with

$$\mathfrak{g}_0^\natural = \mathbb{C}c, \quad \mathfrak{g}_1^\natural = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid A \in sl(2) \right\} \simeq sl(2),$$

so $\mathfrak{g}^\natural \simeq gl(2)$, while $\mathfrak{g}_{-1/2} = \text{span}(e_{2,1}, e_{3,1}, e_{4,2}, e_{4,3})$.

The λ -brackets $[G^{\{u\}}{}_{\lambda}G^{\{v\}}]$ are as follows:

$$\begin{aligned}
[G^{\{e_{2,1}\}}{}_{\lambda}G^{\{e_{2,1}\}}] &= [G^{\{e_{3,1}\}}{}_{\lambda}G^{\{e_{3,1}\}}] = 0 \\
[G^{\{e_{4,2}\}}{}_{\lambda}G^{\{e_{4,2}\}}] &= [G^{\{e_{4,3}\}}{}_{\lambda}G^{\{e_{4,3}\}}] = 0 \\
[G^{\{e_{2,1}\}}{}_{\lambda}G^{\{e_{3,1}\}}] &= [G^{\{e_{4,3}\}}{}_{\lambda}G^{\{e_{4,2}\}}] = 0 \\
[G^{\{e_{2,1}\}}{}_{\lambda}G^{\{e_{4,3}\}}] &= 2 : J^{\{c\}} J^{\{e_{2,3}\}} : - (k+2) \partial J^{\{e_{2,3}\}} - \lambda 2(k+2) J^{\{e_{2,3}\}} \\
[G^{\{e_{3,1}\}}{}_{\lambda}G^{\{e_{4,2}\}}] &= 2 : J^{\{c\}} J^{\{e_{3,2}\}} : - (k+2) \partial J^{\{e_{3,2}\}} - \lambda 2(k+2) J^{\{e_{3,2}\}} \\
[G^{\{e_{2,1}\}}{}_{\lambda}G^{\{e_{4,2}\}}] &= \\
(k+4)\omega - 2 : J^{\{e_{2,3}\}} J^{\{e_{3,2}\}} : &- \frac{1}{2} : J^{\{e_{2,2-e_{3,3}\}} J^{\{e_{2,2-e_{3,3}\}}} : \\
- \frac{3}{2} : J^{\{c\}} J^{\{c\}} : &+ : J^{\{c\}} J^{\{e_{2,2-e_{3,3}\}} : + (k+1) \partial J^{\{c\}} - \frac{k}{2} \partial J^{\{e_{2,2-e_{3,3}\}} \\
+ \lambda 2(k+1) J^{\{c\}} - \lambda(k+2) J^{\{e_{2,2-e_{3,3}\}} &- \lambda^2(k+1)(k+2) \mathbf{1} \\
[G^{\{e_{4,3}\}}{}_{\lambda}G^{\{e_{3,1}\}}] &= \\
- (k+4)\omega + 2 : J^{\{e_{2,3}\}} J^{\{e_{3,2}\}} : &+ \frac{1}{2} : J^{\{e_{2,2-e_{3,3}\}} J^{\{e_{2,2-e_{3,3}\}}} : \\
+ \frac{3}{2} : J^{\{c\}} J^{\{c\}} : &+ : J^{\{c\}} J^{\{e_{2,2-e_{3,3}\}} : + (k+1) \partial J^{\{c\}} + \frac{k}{2} \partial J^{\{e_{2,2-e_{3,3}\}} \\
+ \lambda 2(k+1) J^{\{c\}} + \lambda(k+2) J^{\{e_{2,2-e_{3,3}\}} &+ \lambda^2(k+1)(k+2) \mathbf{1}.
\end{aligned}$$

4. A CLASSIFICATION OF CONFORMAL LEVELS FROM [10]

In this section we recall the definition of conformal embeddings of affine vertex subalgebras into minimal affine W -algebras and review results from [10] on the classification of conformal levels.

Let $\mathcal{V}^k(\mathfrak{g}^{\natural})$ be the subalgebra of the vertex algebra $W^k(\mathfrak{g}, \theta)$, generated by $\{J^{\{a\}} \mid a \in \mathfrak{g}^{\natural}\}$. By (3.6), $\mathcal{V}^k(\mathfrak{g}^{\natural})$ is isomorphic to a universal affine vertex algebra. More precisely, the map $a \mapsto J^{\{a\}}$ extends to an isomorphism

$$(4.1) \quad \mathcal{V}^k(\mathfrak{g}^{\natural}) \simeq \bigotimes_{i \in I} V^{k_i}(\mathfrak{g}_i^{\natural}).$$

Definition 4.1. We set $\mathcal{V}_k(\mathfrak{g}^{\natural})$ to be the image of $\mathcal{V}^k(\mathfrak{g}^{\natural})$ in $W_k(\mathfrak{g}, \theta)$.

Clearly we can write

$$\mathcal{V}_k(\mathfrak{g}^{\natural}) \simeq \bigotimes_{i \in I} \mathcal{V}_{k_i}(\mathfrak{g}_i^{\natural}),$$

where $\mathcal{V}_{k_i}(\mathfrak{g}_i^{\natural})$ is some quotient (not necessarily simple) of $V^{k_i}(\mathfrak{g}_i^{\natural})$.

If $k_i + h_{0,i}^\vee \neq 0$, then $V^{k_i}(\mathfrak{g}_i^{\natural})$ is equipped with the Virasoro vector $\omega_{sug}^{\mathfrak{g}_i^{\natural}}$ (cf. (2.4)). If $k_i + h_{0,i}^\vee \neq 0$ for all i , we set

$$\omega_{sug} = \sum_{i \in I} \omega_{sug}^{\mathfrak{g}_i^{\natural}}.$$

Define

$$\mathcal{K} = \{k \in \mathbb{C} \mid k + h^\vee \neq 0, k_i + h_{0,i}^\vee \neq 0 \text{ whenever } k_i \neq 0\}.$$

If $k \in \mathcal{K}$ we also set

$$\omega'_{sug} = \sum_{i \in I: k_i \neq 0} \omega_{sug}^{\mathfrak{g}_i^{\natural}}.$$

We define

$$c_{sug} = \text{central charge of } \omega'_{sug}.$$

Definition 4.2. Assume $k \in \mathcal{K}$. We say that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is conformally embedded in $W_k(\mathfrak{g}, \theta)$ if $\omega'_{sug} = \omega$. The level k is called a *conformal level*.

If $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^{\natural})$, we say that k is a *collapsing level*.

Remark 4.3. The above definition of conformal level is slightly more general than the one given in the Introduction. Indeed it makes sense also when $k_i = h_{0,i}^\vee = 0$.

Next we recall the classification of collapsing levels from [10].

Proposition 4.1. [10, Theorem 3.3]

The level k is collapsing if and only if $p(k) = 0$ where p is the polynomial listed in the Table 4.

The classification of non-collapsing conformal levels is given in Section 4 of [10]. It may be summarized as follows.

Proposition 4.2.

(I). Assume that \mathfrak{g}^{\natural} is either zero or simple or 1-dimensional.

If $\mathfrak{g} = sl(3)$, or $\mathfrak{g} = spo(n|n+2)$ with $n \geq 2$, $\mathfrak{g} = spo(n|n-1)$ with $n \geq 2$, $\mathfrak{g} = spo(n|n-4)$ with $n \geq 4$, then there are no non-collapsing conformal levels. In all other cases the non-collapsing conformal levels are

- (1) $k = -\frac{h^\vee - 1}{2}$ if \mathfrak{g} is of type $G_2, F_4, E_6, E_7, E_8, F(4)(\mathfrak{g}^{\natural} = so(7)), G(3)$ ($\mathfrak{g}^{\natural} = G_2, osp(3|2)$), or $\mathfrak{g} = psl(m|m)$ ($m \geq 2$);
- (2) $k = -\frac{2}{3}h^\vee$ if $\mathfrak{g} = sp(n)$ ($n \geq 6$), or $\mathfrak{g} = spo(2|m)$ ($m \geq 2$), or $\mathfrak{g} = spo(n|m)$ ($n \geq 4$).

(II). Assume that $\mathfrak{g}^{\natural} = \mathfrak{g}_0^{\natural} \oplus \mathfrak{g}_1^{\natural}$ with $\mathfrak{g}_0^{\natural} \simeq \mathbb{C}$ and $\mathfrak{g}_1^{\natural}$ simple.

If $\mathfrak{g} = sl(m|m-3)$ with $m \geq 4$, then there are no non-collapsing conformal levels. In other cases the non-collapsing conformal levels are

- (1) $k = -\frac{2}{3}h^\vee$ if $\mathfrak{g} = sl(m|m+1)$ ($m \geq 2$), and $\mathfrak{g} = sl(m|m-1)$ ($m \geq 3$);

(2) $k = -\frac{2}{3}h^\vee$ and $k = -\frac{h^\vee-1}{2}$ in all other cases.

(III). Assume that $\mathfrak{g}^\natural = \sum_{i=1}^r \mathfrak{g}_i^\natural$ with $\mathfrak{g}_1^\natural \simeq sl(2)$ and $r \geq 2$. If $\mathfrak{g} = osp(n+5|n)$ with $n \geq 2$ or $\mathfrak{g} = D(2, 1; a)$ with $a = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, then there are no non-collapsing conformal levels. In the other cases the non-collapsing conformal levels are

- (1) $k = -\frac{h^\vee-1}{2}$ if $\mathfrak{g} = D(2, 1; a)$ ($a \notin \{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$), $\mathfrak{g} = osp(n+8|n)$ ($n \geq 0$), $\mathfrak{g} = osp(n+2|n)$ ($n \geq 2$), $\mathfrak{g} = osp(n-4|n)$ ($n \geq 8$);
- (2) $k = -\frac{2}{3}h^\vee$ if $\mathfrak{g} = osp(n+7|n)$ ($n \geq 0$), $\mathfrak{g} = osp(n+1|n)$ ($n \geq 4$);
- (3) $k = -\frac{2}{3}h^\vee$ and $k = -\frac{h^\vee-1}{2}$ in all other cases.

It is important to observe that, if k is a conformal level, we have the following identification of the Zhu algebra of $W_k(\mathfrak{g}, \theta)$.

Proposition 4.3. *Assume that k is a conformal non-collapsing level. Let \mathcal{J} be any proper ideal in $W^k(\mathfrak{g}, \theta)$ which contains $\omega - \omega_{sug}$. Then there is a surjective homomorphism of associative algebras*

$$A(\mathcal{V}^k(\mathfrak{g}^\natural)) \rightarrow A(W^k(\mathfrak{g}, \theta)/\mathcal{J}).$$

In particular, $A(W_k(\mathfrak{g}, \theta))$ is isomorphic to a certain quotient of $U(\mathfrak{g}^\natural)$.

Proof. Recall first that if a vertex algebra V is strongly generated by the set $S \subset V$, then Zhu's algebra $A(V)$ is generated by the set $\{[a], a \in S\}$ (cf. [1, Proposition 2.5], [17]). Since $W^k(\mathfrak{g}, \theta)/\mathcal{J}$ is strongly generated by the set

$$\{G^{\{u\}}, u \in \mathfrak{g}_{-1/2}\} \cup \{J^{\{x\}}, x \in \mathfrak{g}^\natural\},$$

we have that $A(W^k(\mathfrak{g}, \theta)/\mathcal{J})$ is generated by the set

$$\{[G^{\{u\}}], u \in \mathfrak{g}_{-1/2}\} \cup \{[J^{\{x\}}], x \in \mathfrak{g}^\natural\}.$$

On the other hand, since $G^{\{u\}} = G^{\{u\}} \circ \mathbf{1} \in O(W^k(\mathfrak{g}, \theta)/\mathcal{J})$, we have $[G^{\{u\}}] = 0$ in $A(W^k(\mathfrak{g}, \theta)/\mathcal{J})$ for every $u \in \mathfrak{g}_{-1/2}$. Therefore, $A(W^k(\mathfrak{g}, \theta)/\mathcal{J})$ is only generated by the set $\{[J^{\{x\}}], x \in \mathfrak{g}^\natural\}$. This gives a surjective homomorphism $A(\mathcal{V}^k(\mathfrak{g}^\natural)) = U(\mathfrak{g}^\natural) \rightarrow A(W^k(\mathfrak{g}, \theta)/\mathcal{J})$. \square

We should also mention that a conjectural generalization of our results to conformal embeddings of affine vertex algebras into more general W -algebras have been recently proposed by T. Creutzig in [15].

5. SOME RESULTS ON ADMISSIBLE AFFINE VERTEX ALGEBRAS

Assume \mathfrak{g} is a simple Lie superalgebra. Let \mathcal{O}^k be the category of $\widehat{\mathfrak{g}}$ -modules from the category \mathcal{O} of level k . Let KL^k be the subcategory of \mathcal{O}^k consisting of modules on which \mathfrak{g} acts locally finitely. Note that modules from KL^k are $V^k(\mathfrak{g})$ -modules. Moreover, every irreducible module M from

KL^k has finite-dimensional weight spaces with respect to $(\omega_{\mathfrak{su}\mathfrak{g}}^{\mathfrak{g}})_0$ and admits the following $\mathbb{Z}_{\geq 0}$ -gradation:

$$M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n), \quad (\omega_{\mathfrak{su}\mathfrak{g}}^{\mathfrak{g}})_0|_{M(n)} \equiv (n+h)\text{Id} \quad (h \in \mathbb{C}),$$

(cf. [34]; such modules are usually called ordinary modules in the terminology of vertex operator algebra theory [20]). The graded component $M(0)$ is usually called the lowest graded component.

5.1. Fusion rules for certain affine vertex algebras. The classification of irreducible modules in the category \mathcal{O}^k for affine vertex algebras $V_k(\mathfrak{g})$ at admissible levels was conjectured in [5] and proved by Arakawa in [12]. We need the classification result in the subcategory KL^k of the category \mathcal{O}^k .

Definition 5.1. We define KL_k to be the category of all modules M in KL^k which are $V_k(\mathfrak{g})$ -modules.

The classification of irreducible modules in the category KL_k coincides with the classification of irreducible $V_k(\mathfrak{g})$ -modules having finite-dimensional weight spaces with respect to $(\omega_{\mathfrak{su}\mathfrak{g}}^{\mathfrak{g}})_0$ [5], [12].

We restrict our attention to $\mathfrak{g} = \mathfrak{sl}(n)$ with $(\cdot|\cdot)$ the trace form. We choose a set of positive roots for \mathfrak{g} and let $\omega_i \in \mathfrak{h}^*$ ($i = 1, \dots, n-1$) denote the corresponding fundamental weights. Set $\Lambda_i = \Lambda_0 + \omega_i$. Recall from 2.2 the definition of fusion rules.

Proposition 5.1. *Let $k = \frac{1}{2} - n$, $n \geq 2$.*

(1) *The set*

$$(5.1) \quad \{L_{\mathfrak{sl}(2n-2)}(k\Lambda_0 + \Lambda_i) \mid i = 0, \dots, 2n-3\}$$

provides a complete list of irreducible $V_{k+1}(\mathfrak{sl}(2n-2))$ -modules in the category KL_{k+1} .

(2) *The following fusion rule holds in KL_{k+1} :*

$$L_{\mathfrak{sl}(2n-2)}(k\Lambda_0 + \Lambda_{i_1}) \times L_{\mathfrak{sl}(2n-2)}(k\Lambda_0 + \Lambda_{i_2}) = L_{\mathfrak{sl}(2n-2)}(k\Lambda_0 + \Lambda_{i_3})$$

where $0 \leq i_1, i_2, i_3 \leq 2n-3$ are such that $i_1 + i_2 \equiv i_3 \pmod{2n-2}$.

In particular, the modules in (5.1) are simple currents in the category KL_{k+1} .

Proof. First we notice that the set of admissible weights of level $k+1$ which are dominant with respect to $\mathfrak{sl}(2n-2)$ is $\{k\Lambda_0 + \Lambda_i \mid i = 0, \dots, 2n-3\}$. Now assertion (1) follows from the main result from [12].

Assertion (2) follows from (1) and the fact that the tensor product

$$V_{\mathfrak{sl}(2n-2)}(\omega_{i_1}) \otimes V_{\mathfrak{sl}(2n-2)}(\omega_{i_2})$$

contains a component $V_{\mathfrak{sl}(2n-2)}(\omega_{i_3})$ if and only if $i_1 + i_2 \equiv i_3 \pmod{2n-2}$. \square

The proof of the following result is completely analogous to the proof of Proposition 5.1.

Proposition 5.2. *Let $k = \frac{2}{3}(n-2) \notin \mathbb{Z}$. Then*

(1) *The set*

$$(5.2) \quad \{L_{sl(n)}(-(k+2)\Lambda_0 + \Lambda_i) \mid i = 0, \dots, n-1\}$$

provides a complete list of irreducible $V_{-k-1}(sl(n))$ -modules in the category KL_{-k-1} .

(2) *The following fusion rules hold in KL_{-k-1} :*

$$L_{sl(n)}(-(k+2)\Lambda_0 + \Lambda_{i_1}) \times L_{sl(n)}(-(k+2)\Lambda_0 + \Lambda_{i_2}) = L_{sl(n)}(-(k+2)\Lambda_0 + \Lambda_{i_3})$$

where $0 \leq i_1, i_2, i_3 \leq n-1$ are such that $i_1 + i_2 \equiv i_3 \pmod{n}$.

In particular, the modules in (5.2) are simple currents in the category KL_{-k-1} .

Remark 5.2. It is also interesting to notice that the fusion algebra generated by irreducible modules for $V_{3/2-n}(sl(2n-2))$ in the category $KL_{3/2-n}$ (resp. for $V_{-\frac{2n-1}{3}}(sl(n))$ in the category $KL_{-\frac{2n-1}{3}}$) is isomorphic to the fusion algebra for the rational affine vertex algebra $V_1(sl(2n-2))$ (resp. $V_1(sl(n))$). Moreover, all irreducible modules in the KL_k category for these vertex algebras are simple currents.

5.2. The vertex algebra $V_k(sl(2))$. Recall that a level k is called admissible if $k\Lambda_0$ is admissible. If $\mathfrak{g} = sl(2)$ then k is admissible if and only if $k+2 = \frac{p}{q}$, $p, q \in \mathbb{N}$, $(p, q) = 1$, $p \geq 2$ [32]. Let e, h, f be the usual Chevalley generators for $sl(2)$.

Theorem 5.3. *Assume that $k = \frac{p}{q} - 2$ is an admissible level for \widehat{sl}_2 . Then we have:*

- (1) [32, Corollary 1]. *The maximal ideal in J^k in $V^k(sl(2))$ is generated by a singular vector v_λ of weight $\lambda = (k-2(p-1))\Lambda_0 + 2(p-1)\Lambda_1$.*
- (2). *The ideal J^k is simple.*

Proof. We provide here a proof of (2) which uses Virasoro vertex algebras and Hamiltonian reduction. This result can be also proved by using embedding diagrams for submodules of the Verma modules for \widehat{sl}_2 .

Assume first that $k \notin \mathbb{Z}_{\geq 0}$. Let $V^{Vir}(c_{p,q})$ be the universal Virasoro vertex algebra of central charge $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$. Then the maximal ideal in $V^{Vir}(c_{p,q})$ is irreducible and it is generated by a singular vector of conformal weight $(p-1)(q-1)$ (cf. [24], Theorem 4.2.1). So $V^{Vir}(c_{p,q})$ contains a unique ideal which we shall denote by $I_{p,q}$. Then $L^{Vir}(c_{p,q}) = V^{Vir}(c_{p,q})/I_{p,q}$ is a simple vertex algebra.

Recall that by quantum Hamiltonian reduction

$$W^k(sl(2), \theta) = V^{Vir}(c_{p,q}).$$

Let H_{Vir} be the corresponding functor (denoted in [11] by $H_f^{\infty+0}$), which maps $V^k(sl(2))$ -modules to $V^{Vir}(c_{p,q})$ -modules. Assume that I is a non-trivial, proper ideal in $V^k(sl(2))$. By using the main result of [11], we get

that $H_{Vir}(I) \neq 0$, $H_{Vir}(I) \neq V^{Vir}(c_{p,q})$. So $H_{Vir}(I) = I_{p,q}$. Since the functor H_{Vir} is exact, we get that

$$H_{Vir}(V^k(sl(2))/I) = V^{Vir}(c_{p,q}, 0)/I_{p,q} = L^{Vir}(c_{p,q}).$$

By using again the exactness and non-triviality result of the functor H_{Vir} we conclude that $V^k(sl(2))/I$ is simple. So I is the maximal ideal.

If $k \in \mathbb{Z}_{\geq 0}$, then the maximal ideal is $J^k = V^k(sl(2)) \cdot (e_{(-1)})^{k+1} \mathbf{1}$ and we have

$$H_{Vir}(J^k) = W^k(sl(2), \theta) = V^{Vir}(c_{k+2,1}) = L^{Vir}(c_{k+2,1}).$$

Since $H_{Vir}(J^k)$ is irreducible, the properties of the functor H_{Vir} imply that J^k is a simple ideal. \square

It follows from [24], Theorem 9.1.2, that, if \mathfrak{g} is a simple Lie algebra different from $sl(2)$, then the maximal ideal in $V^k(\mathfrak{g})$ is either zero or it is not simple.

5.2.1. *Representation theory of $V_{-1/2}(sl(2))$.* We now recall some known facts on the representation theory of the vertex algebra $V_{-1/2}(sl(2))$ (cf. [5] and Theorem 5.3).

We first fix notation. Let $\mathcal{L}_{sl(2)}(\lambda)$ be a highest weight $V^{-1/2}(sl(2))$ -module with highest weight λ , and let v_λ be the corresponding highest weight vector. Writing $\lambda = -1/2\Lambda_0 + \mu$ with $\mu \in \mathfrak{h}^*$, we let $N_{sl(2)}(\lambda)$ denote the generalized Verma module induced from the simple $sl(2)$ -module $V_{sl(2)}(\mu)$.

Let $\omega_{sug}^{sl(2)}$ be the Sugawara Virasoro vector for $V^{-1/2}(sl(2))$. For $i \in \mathbb{Z}_{\geq 0}$ we define the following weights:

$$\lambda_i = -(i + 1/2)\Lambda_0 + i\Lambda_1 = -1/2\Lambda_0 + i\omega_1.$$

Then one has:

(1). The maximal ideal of $V^{-1/2}(sl(2))$ is generated by the singular vector $v_{\lambda_4} \in V^{-1/2}(sl(2))$ of weight λ_4 . In particular,

$$V_{-1/2}(sl(2)) = V^{-1/2}(sl(2))/V^{-1/2}(sl(2)) \cdot v_{\lambda_4}.$$

Moreover $V^{-1/2}(sl(2)) \cdot v_{\lambda_4}$ is simple.

(2). There is a singular vector $v_{\lambda_3} \in N_{sl(2)}(\lambda_1)$ of weight λ_3 such that

$$L(\lambda_1) = N_{sl(2)}(\lambda_1)/V^{-1/2}(sl(2)) \cdot v_{\lambda_3}.$$

Moreover $V^{-1/2}(sl(2)) \cdot v_{\lambda_3}$ is simple.

(3). $L_{sl(2)}(\lambda_i)$, $i = 0, 1$, are irreducible $V_{-1/2}(sl(2))$ -modules.

Every $V_{-1/2}(sl(2))$ -module from the category $KL_{-\frac{1}{2}}$ is completely reducible and isomorphic to a direct sum of certain copies of $L_{sl(2)}(\lambda_i)$, $i = 0, 1$.

(4). The following fusion rule holds in $KL_{-\frac{1}{2}}$:

$$(5.3) \quad L_{sl(2)}(\lambda_1) \times L_{sl(2)}(\lambda_1) = V_{-1/2}(sl(2)).$$

This fusion rule follows from the tensor product decomposition

$$V_{sl(2)}(\omega_1) \otimes V_{sl(2)}(\omega_1) = V_{sl(2)}(2\omega_1) + V_{sl(2)}(0)$$

and the classification of irreducible modules for $V_{-1/2}(sl(2))$ -modules from [5]). In particular, we only need to note that $L_{sl(2)}(\lambda_2)$ is not a $V_{-1/2}(sl(2))$ -module.

6. SEMISIMPLICITY OF CONFORMAL EMBEDDINGS

The main goal of this section is to give criteria for establishing the simplicity of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ together with the semisimplicity of $W_k(\mathfrak{g}, \theta)$ as a $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module when k is a non-collapsing conformal level. We will give two separate criteria: one for the cases when \mathfrak{g}^{\natural} has a nontrivial center and another for the cases when \mathfrak{g}^{\natural} is centerless.

6.1. Semisimplicity with nontrivial center of \mathfrak{g}^{\natural} . The next result collects some structural facts proven in [10, Proposition 4.6] describing the structure of $\mathfrak{g}_{-1/2}$ as a \mathfrak{g}^{\natural} -module.

Lemma 6.1. *Assume that \mathfrak{g}^{\natural} is a Lie algebra and $\mathfrak{g}_0^{\natural} \neq \{0\}$ (which happens only for $\mathfrak{g} = sl(n)$ or $\mathfrak{g} = sl(2|n)$, $n \neq 2$). Then*

- (1) $\dim \mathfrak{g}_0^{\natural} = 1|0$.
- (2) A basis $\{c\}$ of $\mathfrak{g}_0^{\natural}$ can be chosen so that the eigenvalues of $ad(c)$ acting on $\mathfrak{g}_{-1/2}$ are ± 1 .
- (3) Let U^+ (resp. U^-) be the eigenspace for $ad(c)|_{\mathfrak{g}_{-1/2}}$ corresponding to the eigenvalue 1 (resp. -1). Then $\mathfrak{g}_{-1/2} = U^+ \oplus U^-$ with U^{\pm} irreducible finite dimensional mutually contragredient \mathfrak{g}^{\natural} -modules.

By (3.7) and the above Lemma, $J_{(0)}^{\{c\}}$ defines a \mathbb{Z} -gradation on $W_k(\mathfrak{g}, \theta)$:

$$W_k(\mathfrak{g}, \theta) = \bigoplus W_k(\mathfrak{g}, \theta)^{(i)}, \quad W_k(\mathfrak{g}, \theta)^{(i)} = \{v \in W_k(\mathfrak{g}, \theta) \mid J_{(0)}^{\{c\}} v = iv\}.$$

Recall that a primitive vector in a module M for an affine vertex algebra is a vector that is singular in some subquotient of M .

In light of Lemma 6.1, we have that, in the Grothendieck group of finite dimensional representations of \mathfrak{g}^{\natural} , we can write

$$U^+ \otimes U^- = V(0) + \sum_{\nu_i \neq 0} V(\nu_i).$$

Theorem 6.2. *Assume that the embedding of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ in $W_k(\mathfrak{g}, \theta)$ is conformal and that $W_k(\mathfrak{g})^{(0)}$ does not contain $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -primitive vectors of weight ν_r .*

Then $W_k(\mathfrak{g})^{(0)} = \mathcal{V}_k(\mathfrak{g}^{\natural})$, $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra and $W_k(\mathfrak{g}, \theta)^{(i)}$ are simple $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules.

Proof. Let $\mathcal{U}^\pm = \text{span}(G^{\{u\}} \mid u \in U^\pm)$. Let $A^\pm = \mathcal{V}_k(\mathfrak{g}^{\natural}) \cdot \mathcal{U}^\pm$. We claim that

$$(6.1) \quad A^- \cdot A^+ \subset \mathcal{V}_k(\mathfrak{g}^{\natural}).$$

To check this, it is enough to check for all $n \in \mathbb{Z}$, $u \in U^+$, and $v \in U^-$, that $G^{\{u\}}_{(n)} G^{\{v\}} \in \mathcal{V}_k(\mathfrak{g}^{\natural})$. Assume that this is not the case. Then we can choose n maximal such that there are $u \in U^+$, $v \in U^-$ such that $G^{\{u\}}_{(n)} G^{\{v\}} \notin \mathcal{V}_k(\mathfrak{g}^{\natural})$. Since the map

$$\phi : U^+ \otimes U^- \rightarrow W_k(\mathfrak{g})/\mathcal{V}_k(\mathfrak{g}^{\natural}), \quad \phi : u \otimes v \mapsto G^{\{u\}}_{(n)} G^{\{v\}} + \mathcal{V}_k(\mathfrak{g}^{\natural})$$

is \mathfrak{g}^{\natural} -equivariant, we can choose a weight vector $w = \sum u_i \otimes v_i \in U^+ \otimes U^-$ of weight ν such that $\phi(w)$ is a highest weight vector in $\phi(U^+ \otimes U^-)$. Since, by maximality of n , $\phi(w)$ is singular for $\mathcal{V}_k(\mathfrak{g}^{\natural})$, we have that $y = \sum G^{\{u_i\}}_{(n)} G^{\{v_i\}}$ is primitive in $W_k(\mathfrak{g}, \theta)$ so, by our hypothesis, $\nu = 0$. Since the embedding is conformal, y is an eigenvector for $\omega'_{\text{ Sug}}$ and since $\phi(w)$ is singular for $\mathcal{V}_k(\mathfrak{g}^{\natural})$ of weight 0, we see that its eigenvalue is zero. Since the embedding is conformal we have that y has conformal weight zero in $W_k(\mathfrak{g}, \theta)$ so $y \in \mathbf{C}\mathbf{1} \subset \mathcal{V}_k(\mathfrak{g}^{\natural})$, a contradiction.

Since the embedding is conformal, $W_k(\mathfrak{g}, \theta)$ is strongly generated by

$$\text{span} \left\{ J^{\{a\}} \mid a \in \mathfrak{g}^{\natural} \right\} + \mathcal{U}^+ + \mathcal{U}^-.$$

It follows that $W_k(\mathfrak{g}, \theta)^{(0)}$ is contained in the sum of all fusion products of type $A_1 \cdot A_2 \cdot \dots \cdot A_r$ with $A_i \in \{A^+, A^-, \mathcal{V}_k(\mathfrak{g}^{\natural})\}$ such that

$$\#\{i \mid A_i = A^+\} = \#\{i \mid A_i = A^-\}.$$

By the associativity of fusion products, we see that (6.1) implies that $A_1 \cdot A_2 \cdot \dots \cdot A_r \subset \mathcal{V}_k(\mathfrak{g}^{\natural})$, so $W_k(\mathfrak{g}, \theta)^{(0)} = \mathcal{V}_k(\mathfrak{g}^{\natural})$. It follows that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra and $W_k(\mathfrak{g}, \theta)^{(i)}$ is a simple $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module for all i . \square

If $V(\mu)$, $\mu \in (\mathfrak{h}^{\natural})^*$, is an irreducible \mathfrak{g}^{\natural} -module, then we can write

$$(6.2) \quad V(\mu) = \bigotimes_{j \in I} V_{\mathfrak{g}_j^{\natural}}(\mu^j),$$

where $V_{\mathfrak{g}_j^{\natural}}(\mu^j)$ is an irreducible $\mathfrak{g}_j^{\natural}$ -module. Let ρ_0^j be the Weyl vector in $\mathfrak{g}_j^{\natural}$ (with respect to the positive system induced by the choice of positive roots for \mathfrak{g}).

Corollary 6.3. *If the embedding of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ in $W_k(\mathfrak{g}, \theta)$ is conformal and, for each irreducible subquotient $V(\mu)$ with $\mu \neq 0$ of the \mathfrak{g}^{\natural} -module $U^+ \otimes U^-$, we have*

$$(6.3) \quad \sum_{i \in I, k_i \neq 0} \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})} \notin \mathbb{Z}_+,$$

then $W_k(\mathfrak{g})^{(0)} = \mathcal{V}_k(\mathfrak{g}^{\natural})$, $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra and the $W_k(\mathfrak{g}, \theta)^{(i)}$ are simple $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules.

Proof. In order to apply Theorem 6.2, we need to check that if $\mu \neq 0$ and $V(\mu)$ is an irreducible subquotient of $U^+ \otimes U^-$, then there is no primitive vector v in $W_k(\mathfrak{g}, \theta)^{(0)}$ with weight μ . Since the embedding is conformal, ω'_{sug} acts diagonally on $W_k(\mathfrak{g}, \theta)$. In particular, we can assume that v is an eigenvector for ω'_{sug} . Let $N \subset M \subset W_k(\mathfrak{g}, \theta)$ be submodules such that $v + N$ is a singular vector in M/N . Then $v + N$ is an eigenvector for the action of ω'_{sug} on M/N and the corresponding eigenvalue is $\sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})}$. It follows that the eigenvalue for ω'_{sug} acting on v is $\sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})}$. It is easy to check that the conformal weights of elements in $W_k(\mathfrak{g}, \theta)^{(0)}$ are positive integers hence, since $\omega'_{sug} = \omega$, $\sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})}$ must be in \mathbb{Z}_+ , a contradiction. \square

We now apply Corollary 6.3 to the cases where \mathfrak{g}^{\natural} is a basic Lie superalgebra with nontrivial center. These can be read off from Tables 1–3 and correspond to taking $\mathfrak{g} = sl(n)$ ($n \geq 3$), $\mathfrak{g} = sl(2|n)$ ($n \geq 1$, $n \neq 2$) or $\mathfrak{g} = sl(m|n)$ ($n \neq m > 2$).

Theorem 6.4. *Assume that we are in the following cases of conformal embedding of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ into $W_k(\mathfrak{g}, \theta)$.*

- (1) $\mathfrak{g} = sl(n)$, $n \geq 4$, conformal level $k = -\frac{n-1}{2} = -\frac{h^{\vee}-1}{2}$;
- (2) $\mathfrak{g} = sl(n)$, $n \geq 5$, $n \neq 6$, conformal level $k = -\frac{2n}{3} = -\frac{2h^{\vee}}{3}$;
- (3) $\mathfrak{g} = sl(2|n)$, $n \geq 4$, conformal level $k = \frac{n-1}{2} = -\frac{h^{\vee}-1}{2}$;
- (4) $\mathfrak{g} = sl(2|n)$, $n \geq 3$, conformal level $k = \frac{2(n-2)}{3} = -\frac{2h^{\vee}}{3}$.
- (5) $\mathfrak{g} = sl(m|n)$, $m > 2$, $m \neq n+3, n+2, n, n-1$, conformal level $k = \frac{n-m+1}{2} = -\frac{h^{\vee}-1}{2}$;
- (6) $\mathfrak{g} = sl(m|n)$, $m > 2$, $m \neq n+6, n+4, n+2, n$, conformal level $k = \frac{2(n-m)}{3} = -\frac{2h^{\vee}}{3}$.

Then $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra and $W_k(\mathfrak{g}, \theta)^{(i)}$ is an irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module for every $i \in \mathbb{Z}$. In particular, $W_k(\mathfrak{g}, \theta)$ is a semisimple $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module.

Proof. We verify that the assumptions of Corollary 6.3 hold. In cases (1) and (2), $\mathfrak{g}^{\natural} \simeq gl(n-2) = \mathbb{C}Id \oplus sl(n-2)$, hence $\mathfrak{g}_0^{\natural} \simeq \mathbb{C}$ and $\mathfrak{g}_1^{\natural} \simeq sl(n-2)$. Moreover $U^+ = \mathbb{C}^n$ and $U^- = (\mathbb{C}^n)^*$. The tensor product $U^+ \otimes U^-$ decomposes as $V(0) \oplus V(\mu)$ with

$$\mu^0 = 0, \quad \mu^1 = \omega_1 + \omega_{n-3}.$$

Moreover $k_0 = k + h^{\vee}/2 \neq 0$ and $k_1 = k + 1 \neq 0$. Since

$$\sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})} = \sum_{i=0}^1 \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})} = \frac{n-2}{n+k-1},$$

we see that (6.3) holds in cases (1) and (2).

In cases (3) and (4), $\mathfrak{g}^{\natural} \simeq \mathfrak{gl}(n) = \mathbb{C}Id \oplus \mathfrak{sl}(n)$, hence $\mathfrak{g}_0^{\natural} \simeq \mathbb{C}$ and $\mathfrak{g}_1^{\natural} = \mathfrak{sl}(n)$. Moreover $U^+ = \mathbb{C}^n$ and $U^- = (\mathbb{C}^n)^*$. The tensor product $U^+ \otimes U^-$ decomposes as $V(0) \oplus V(\mu)$ with

$$\mu^0 = 0, \quad \mu^1 = \omega_1 + \omega_{n-1}.$$

Moreover $k_0 = k + h^\vee/2 \neq 0$ and $k_1 = k + 1 \neq 0$. Since

$$\sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^\vee)} = \sum_{i=0}^1 \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^\vee)} = \frac{n}{n+k},$$

we see that (6.3) holds in cases (3) and (4).

In cases (5) and (6) we have $\mathfrak{g}^{\natural} \simeq \mathfrak{gl}(m-2|n) = \mathbb{C}Id \oplus \mathfrak{sl}(m-2|n)$ (recall that we are assuming $m \neq n+2$), hence $\mathfrak{g}_0^{\natural} \simeq \mathbb{C}$ and $\mathfrak{g}_1^{\natural} = \mathfrak{sl}(m-2|n)$. Moreover $U^+ = \mathbb{C}^{m-2|n}$ and $U^- = (\mathbb{C}^{m-n|n})^*$. Then, as \mathfrak{g}^{\natural} -modules

$$U^+ \otimes U^- \simeq \mathfrak{sl}(m-2|n) \oplus \mathbb{C}.$$

With notation as in [25], choose $\{\epsilon_1 - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m\}$ as simple roots for \mathfrak{g} , so that the highest root is even. The set of positive roots induced on \mathfrak{g}^{\natural} has as simple roots $\{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_2, \dots, \epsilon_{m-2} - \epsilon_{m-1}\}$. Then we have $\mu_0 = 0, \mu_1 = \delta_1 - \epsilon_{m-1}$. Since $2(\rho_0^1)_1 = -n(\epsilon_2 + \dots + \epsilon_{m-1}) + (m-2)(\delta_1 + \dots + \delta_n)$, $2(\rho_0^1)_0 = (m-3)\epsilon_2 + (m-5)\epsilon_3 + \dots + (3-m)\epsilon_{m-1} + (n-1)\delta_1 + (n-3)\delta_2 + \dots + (1-n)\delta_n$, we have that

$$(\delta_1 - \epsilon_{m-1}, 2\rho_0^1) = -n + 1 + m + 2 - (3-m) - n = 2(m-n-2).$$

In case (5), we have $k = -\frac{h^\vee-1}{2}$. Then $k_1 + h_{0,1}^\vee = \frac{m-n-1}{2}$ and $(\mu, \mu + 2\rho_0^1) = (\delta_1 - \epsilon_{m-1}, 2\rho_0^1) = 2(m-n-2)$. Therefore

$$\frac{(\mu_1, \mu_1 + 2\rho_0^1)}{2(k_1 + h_{0,1}^\vee)} = 2 \frac{m-n-2}{m-n-1} = 2 \left(1 - \frac{1}{m-n-1}\right)$$

which is not an integer unless $m = n+3, n+2, n, n-1$.

In case (6), we have $k = -\frac{2}{3}h^\vee$. Then $k_1 + h_{0,1}^\vee = \frac{m-n-3}{3}$ and

$$\frac{(\mu_1, \mu_1 + 2\rho_0^1)}{2(k_1 + h_{0,1}^\vee)} = 3 \frac{m-n-2}{m-n-3} = 3 \left(1 + \frac{1}{m-n-3}\right)$$

which not an integer unless $m = n+6, n+4, n+2, n$.

In Section 8 we will discuss explicit decompositions for some occurrences of cases (1) and (4) of Theorem 6.4, exploiting the fact that some of the levels k_i may be admissible for $V_{k_i}(\mathfrak{g}_i^{\natural})$. We shall determine explicitly the decomposition of $W_k(\mathfrak{g}, \theta)$ as a module for this admissible vertex algebra.

We now list the cases which are not covered by Theorem 6.4. Recall that, if $\mathfrak{g} = \mathfrak{sl}(m|n)$, then we excluded the case $m = n+2$ from the beginning

while the case $m = n$ had to be excluded because $\mathfrak{g} = sl(n|n)$ is not simple. The remaining cases are

- (1) $sl(n-1|n)$, $k = 1$;
- (2) $sl(n+3|n)$ $n \geq 0$, $k = -1$;
- (3) $sl(n+4|n)$, $k = -\frac{8}{3}$;
- (4) $sl(n+6|n)$, $k = -4$;
- (5) $sl(2|1) = spo(2|2)$, $k = -\frac{2}{3}$.

If $\mathfrak{g} = sl(n-1|n)$ and $k = 1$ then $k + h^\vee = 0$, so we have to exclude this case.

By Theorem 3.3 of [10] (stated in this paper as Proposition 4.1), $k = -1$ is a collapsing level for $\mathfrak{g} = sl(n+3|n)$. It follows that $W_{-1}(sl(n+3|n)) = \mathcal{V}_{-1}(gl(n+1|n))$. If $n = 0$, we obtain $W_{-1}(sl(3)) = \mathcal{V}_{-1}(gl(1))$, which is the Heisenberg vertex algebra $V_{\frac{1}{2}}(\mathbb{C}c)$.

In the case $\mathfrak{g} = sl(2|1)$, $k = -\frac{2}{3}$, $W_k(\mathfrak{g}, \theta)$ is the simple $N = 2$ superconformal vertex algebra $V_{\mathbf{c}}^{N=2}$ (cf. [28], [30], [2]) with central charge $\mathbf{c} = 1$. In this case $\mathcal{V}_k(gl(1))$ is the Heisenberg vertex algebra $M(-\frac{2}{3})$, so we have conformal embedding of $M(-\frac{2}{3})$ into $W_k(\mathfrak{g}, \theta)$. It is well-known that $V_{\mathbf{c}}^{N=2}$ admits the free-field realization as the lattice vertex algebra F_3 . Using this realization we see that each $W_k(\mathfrak{g}, \theta)^{(i)}$ is an irreducible $M(-\frac{2}{3})$ -module.

Case (3) with $n = 0$ is of special interest: it turns out that $W_k(\mathfrak{g}, \theta)$ is isomorphic to the vertex algebra $\mathcal{R}^{(3)}$ introduced in [3]. This will require a thoughtful discussion which will be performed in Section 7. There we will prove the following

Theorem 6.5. *Let \mathfrak{g} be of type A_3 with $k = -\frac{8}{3}$. Then, for all $i \in \mathbb{Z}$, $W_k(\mathfrak{g}, \theta)^{(i)}$ is an infinite sum of irreducible $\mathcal{V}_k(gl(2))$ -modules.*

Remark 6.1. The only remaining open case is \mathfrak{g} of type A_5 with $k = -4$. It is surprising that this case (possibly) and the case discussed in Theorem 6.5 are the only instances of conformal embeddings into $W_k(\mathfrak{g}, \theta)$ where $W_k(\mathfrak{g}, \theta)^{(i)}$ is not a finite sum of irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules.

6.2. Finite decomposition when \mathfrak{g}^{\natural} has trivial center. Recall that $W_k(\mathfrak{g}, \theta)$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded vertex algebra by conformal weight. It admits the following natural \mathbb{Z}_2 -gradation

$$W_k(\mathfrak{g}, \theta) = W_k(\mathfrak{g}, \theta)^{\bar{0}} \oplus W_k(\mathfrak{g}, \theta)^{\bar{1}},$$

where

$$W_k(\mathfrak{g}, \theta)^{\bar{i}} = \{v \in W_k(\mathfrak{g}, \theta) \mid \Delta_v \in i/2 + \mathbb{Z}\}.$$

Similarly to what we have done in Section 6.1, we start by developing a criterion for checking when $\mathcal{V}_k(\mathfrak{g}^{\natural}) = W_k(\mathfrak{g}, \theta)^{\bar{0}}$, so that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra, and $W_k(\mathfrak{g}, \theta)^{\bar{1}}$ is an irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module. In particular, in these cases, we have a finite decomposition of $W_k(\mathfrak{g}, \theta)$ as a $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module.

One checks, browsing Tables 1–3, that when \mathfrak{g}^{\natural} has trivial center, then $\mathfrak{g}_{-1/2}$ is an irreducible \mathfrak{g}^{\natural} -module. Then, in the Grothendieck group of finite dimensional representations of \mathfrak{g}^{\natural} , we can write

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V(0) + \sum_{\nu_i \neq 0} V(\nu_i).$$

Theorem 6.6. *Assume that the embedding of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ in $W_k(\mathfrak{g}, \theta)$ is conformal and that $W_k(\mathfrak{g})^{\bar{0}}$ does not contain $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -primitive vectors of weight ν_i .*

Then $W_k(\mathfrak{g})^{\bar{0}} = \mathcal{V}_k(\mathfrak{g}^{\natural})$, $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra, and $W_k(\mathfrak{g}, \theta)^{\bar{1}}$ is a simple $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module.

Proof. Let $\mathcal{U} = \text{span}\{G^{\{u\}} \mid u \in \mathfrak{g}_{-1/2}\}$. Let $A = \mathcal{V}_k(\mathfrak{g}^{\natural}) \cdot \mathcal{U}$. Arguing as in the proof of Theorem 6.2, we have

$$(6.4) \quad A \cdot A \subset \mathcal{V}_k(\mathfrak{g}^{\natural}).$$

From (6.4) we obtain that $\mathcal{V}_k(\mathfrak{g}^{\natural}) + A$ is a vertex subalgebra of $W_k(\mathfrak{g}, \theta)$. Since the embedding is conformal, $W_k(\mathfrak{g}, \theta)$ is strongly generated by

$$\text{span}(J^{\{a\}} \mid a \in \mathfrak{g}^{\natural}) + \mathcal{U},$$

hence $\mathcal{V}_k(\mathfrak{g}^{\natural}) + A$ contains a set of generators for $W_k(\mathfrak{g}, \theta)$. It follows that

$$W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^{\natural}) + A, \quad W_k(\mathfrak{g}, \theta)^{\bar{0}} = \mathcal{V}_k(\mathfrak{g}^{\natural}), \quad W_k(\mathfrak{g}, \theta)^{\bar{1}} = \mathcal{V}_k(\mathfrak{g}^{\natural}) \cdot \mathcal{U}.$$

The statement now follows by a simple application of quantum Galois theory, for the cyclic group of order 2. \square

The same argument of Corollary 6.3 provides a numerical criterion for a finite decomposition, actually, for a decomposition in a sum of two irreducible submodules:

Corollary 6.7. *If the embedding of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ in $W_k(\mathfrak{g}, \theta)$ is conformal and, for any irreducible subquotient $V(\mu)$ of $\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2}$ with $\mu \neq 0$, we have (see (6.2))*

$$(6.5) \quad \sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})} \notin \mathbb{Z}_+,$$

then $W_k(\mathfrak{g})^{\bar{0}} = \mathcal{V}_k(\mathfrak{g}^{\natural})$, $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra and $W_k(\mathfrak{g}, \theta)^{\bar{1}}$ is an irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module.

We now apply Corollary 6.7 to the cases where \mathfrak{g}^{\natural} is a basic Lie super-algebra. These can be read off from Tables 1–2 and correspond to taking $\mathfrak{g} = \mathfrak{so}(n)$ ($n \geq 7$), $\mathfrak{g} = \mathfrak{sp}(n)$ ($n \geq 4$), $\mathfrak{g} = \mathfrak{psl}(2|2)$, $\mathfrak{g} = \mathfrak{spo}(2|m)$ ($m \geq 3$), $\mathfrak{g} = \mathfrak{osp}(4|m)$ ($m \geq 1$), $\mathfrak{g} = \mathfrak{psl}(m|m)$ ($m > 2$), $\mathfrak{g} = \mathfrak{spo}(m|n)$ ($n \geq 4$), $\mathfrak{g} = \mathfrak{osp}(m|n)$ ($m \geq 5$) or \mathfrak{g} of the following exceptional types: G_2 , F_4 , E_6 , E_7 , E_8 , $F(4)$, $G(3)$, $D(2, 1; a)$.

Theorem 6.8. *Assume that we are in the following cases of conformal embedding of $\mathcal{V}_k(\mathfrak{g}^\natural)$ into $W_k(\mathfrak{g}, \theta)$:*

- (1) $\mathfrak{g} = so(n)$ ($n \geq 8, n \neq 11$), $\mathfrak{g} = osp(4|n)$ ($n \geq 2$), $\mathfrak{g} = osp(m|n)$ ($m \geq 5, m \neq n+r, r \in \{-1, 2, 3, 4, 6, 7, 8, 11\}$) or \mathfrak{g} is of type $G_2, F_4, E_6, E_7, E_8, F(4)$ and $k = -\frac{h^\vee-1}{2}$.
- (2) $\mathfrak{g} = sp(n)$ ($n \geq 6$), $\mathfrak{g} = spo(2|m)$ ($m \geq 3, m \neq 4$), $\mathfrak{g} = spo(n|m)$ ($n \geq 4, m \neq n+2, n, n-2, n-4$) and $k = -\frac{2}{3}h^\vee$.

Then $W_k(\mathfrak{g}, \theta)^i, i = 0, 1$, are irreducible $\mathcal{V}_k(\mathfrak{g}^\natural)$ -modules

Proof. We shall show case-by-case that the numerical criterion of Corollary 6.7 holds. We start by listing all cases explicitly.

- (1) \mathfrak{g} is of type $D_n, n \geq 5, k = -\frac{h^\vee-1}{2} = \frac{3}{2} - n$;
- (2) \mathfrak{g} is of type $B_n, n \geq 4, n \neq 5, k = -\frac{h^\vee-1}{2} = 1 - n$;
- (3) \mathfrak{g} is of type $G_2, k = -\frac{h^\vee-1}{2} = -3/2$;
- (4) \mathfrak{g} is of type $F_4, k = -\frac{h^\vee-1}{2} = -4$;
- (5) \mathfrak{g} is of type $E_6, k = -\frac{h^\vee-1}{2} = -11/2$;
- (6) \mathfrak{g} is of type $E_7, k = -\frac{h^\vee-1}{2} = -17/2$;
- (7) \mathfrak{g} is of type $E_8, k = -\frac{h^\vee-1}{2} = -29/2$;
- (8) $\mathfrak{g} = F(4), k = -\frac{h^\vee-1}{2} = 3/2$;
- (9) $\mathfrak{g} = osp(4|2n), n \geq 2, k = -\frac{h^\vee-1}{2} = n - 1/2$;
- (10) \mathfrak{g} is of type $C_{n+1}, n \geq 2, k = -\frac{2}{3}h^\vee = -\frac{2}{3}(n+2)$;
- (11) $\mathfrak{g} = spo(2|2n), n \geq 3, k = -\frac{2}{3}h^\vee = \frac{2}{3}(n-2)$;
- (12) $\mathfrak{g} = spo(2|2n+1), n \geq 1, k = -\frac{2}{3}h^\vee = \frac{2}{3}(n-3/2)$;
- (13) $\mathfrak{g} = spo(n|m), n \geq 4, k = -\frac{2}{3}h^\vee = \frac{m-n-2}{3}$;
- (14) $\mathfrak{g} = osp(m|n), m \geq 5, k = -\frac{h^\vee-1}{2} = \frac{n-m+3}{2}$.

If $V(\mu)$ is an irreducible representation of \mathfrak{g}^\natural , we set

$$h_\mu = \sum_{i=0, k_i \neq 0}^r \frac{(\mu^i, \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^\vee)}.$$

For each case listed above we give $\mathfrak{g}^\natural, \mathfrak{g}_{-1/2}$, and the decomposition of $\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2}$ in irreducible modules for \mathfrak{g}^\natural . Then we list all values h_μ for all irreducible components $V(\mu)$ of $\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2}$ with $\mu \neq 0$, showing that they are not positive integers. We also exhibit the decomposition of $W_k(\mathfrak{g}, \theta)$ as a $\mathcal{V}_k(\mathfrak{g}^\natural)$ -module. In cases (13)–(14) we will use the usual $\epsilon - \delta$ notation for roots in Lie superalgebras explained e.g. in [25].

Case (1): \mathfrak{g}^\natural of Type $A_1 \times D_{n-2}, \mathfrak{g}_{-1/2} = V_{A_1}(\omega_1) \otimes V_{D_{n-2}}(\omega_1)$

$$\begin{aligned} & \mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} \\ &= (V_{A_1}(2\omega_1) + V_{A_1}(0)) \otimes (V_{D_{n-2}}(0) + V_{D_{n-2}}(\omega_2) + V_{D_{n-2}}(2\omega_1)). \end{aligned}$$

Values of h_μ 's:

$$\begin{aligned} h_{2\omega_1,0} &= \frac{4}{3}, & h_{0,\omega_2} &= \frac{4n-12}{2n-5}, & h_{0,2\omega_1} &= \frac{4n-8}{2n-5}, \\ h_{2\omega_1,\omega_2} &= \frac{4}{3} + \frac{4n-12}{2n-5}, & h_{2\omega_1,2\omega_1} &= \frac{4}{3} + \frac{4n-8}{2n-5}. \end{aligned}$$

These values are non-integral for $n \geq 5$.

Decomposition:

$$\begin{aligned} W_k(D_n) &= L_{A_1}(-\frac{1}{2}\Lambda_0) \otimes L_{D_{n-2}}((\frac{7}{2}-n)\Lambda_0) \\ &\quad \oplus L_{A_1}(-\frac{3}{2}\Lambda_0 + \Lambda_1) \otimes L_{D_{n-2}}((\frac{5}{2}-n)\Lambda_0 + \Lambda_1). \end{aligned}$$

Case (2): \mathfrak{g}^\natural of Type $A_1 \times B_{n-2}$, $\mathfrak{g}_{-1/2} = V_{A_1}(\omega_1) \otimes V_{B_{n-2}}(\omega_1)$

$$\begin{aligned} &\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} \\ &= (V_{A_1}(2\omega_1) + V_{A_1}(0)) \otimes (V_{B_{n-2}}(0) + V_{B_{n-2}}(\omega_2) + V_{B_{n-2}}(2\omega_1)). \end{aligned}$$

Values of h_μ 's:

$$\begin{aligned} h_{2\omega_1,0} &= \frac{4}{3}, & h_{0,\omega_2} &= \frac{2n-5}{n-2}, & h_{0,2\omega_1} &= \frac{2n-3}{n-2}, \\ h_{2\omega_1,\omega_2} &= \frac{4}{3} + \frac{2n-5}{n-2}, & h_{2\omega_1,2\omega_1} &= \frac{4}{3} + \frac{2n-3}{n-2}. \end{aligned}$$

These values are non-integral for $n \geq 4$, $n \neq 5$.

Decomposition:

$$\begin{aligned} W_k(B_n) &= L_{A_1}(-\frac{1}{2}\Lambda_0) \otimes L_{B_{n-2}}((3-n)\Lambda_0) \\ &\quad \oplus L_{A_1}(-\frac{3}{2}\Lambda_0 + \Lambda_1) \otimes L_{B_{n-2}}((2-n)\Lambda_0 + \Lambda_1). \end{aligned}$$

Case (3): \mathfrak{g}^\natural of Type A_1 , $\mathfrak{g}_{-1/2} = V_{A_1}(3\omega_1)$,

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{A_1}(6\omega_1) + V_{A_1}(4\omega_1) + V_{A_1}(2\omega_1) + V_{A_1}(0).$$

Values of h_μ 's:

$$h_{2i\omega_1} = \frac{2}{5}i(i+1) \notin \mathbb{Z} \quad (i = 1, 2, 3).$$

Decomposition:

$$W_k(G_2) = L_{A_1}(\frac{1}{2}\Lambda_0) \oplus L_{A_1}(-\frac{5}{2}\Lambda_0 + 3\Lambda_1).$$

Case (4): \mathfrak{g}^\natural of Type C_3 , $\mathfrak{g}_{-1/2} = V_{C_3}(\omega_3)$

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{C_3}(0) + V_{C_3}(2\omega_1) + V_{C_3}(2\omega_3).$$

Values of h_μ 's:

$$h_{2\omega_1} = \frac{8}{5}, \quad h_{2\omega_3} = \frac{18}{5}.$$

Decomposition:

$$W_k(F_4) = L_{C_3}(-\frac{3}{2}\Lambda_0) \oplus L_{C_3}(-\frac{5}{2}\Lambda_0 + \Lambda_3).$$

Case (5): \mathfrak{g}^\natural of Type A_5 , $\mathfrak{g}_{-1/2} = V_{A_5}(\omega_3)$

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{A_5}(0) + V_{A_5}(\omega_1 + \omega_5) + V_{A_5}(\omega_2 + \omega_4) + V_{A_5}(2\omega_3).$$

Values of h_μ 's:

$$h_{\omega_1 + \omega_5} = \frac{12}{7}, \quad h_{\omega_2 + \omega_4} = \frac{20}{7}, \quad h_{2\omega_3} = \frac{24}{7}.$$

Decomposition:

$$W_k(E_6) = L_{A_5}(-\frac{5}{2}\Lambda_0) \oplus L_{A_5}(-\frac{7}{2}\Lambda_0 + \Lambda_3).$$

Case (6): \mathfrak{g}^\natural of Type D_6 , $\mathfrak{g}_{-1/2} = V_{D_6}(\omega_6)$

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{D_6}(0) + V_{D_6}(\omega_2) + V_{D_6}(\omega_4) + V_{D_5}(2\omega_6).$$

Values of h_μ 's:

$$h_{2\omega_6} = \frac{36}{11}, \quad h_{\omega_4} = \frac{32}{11}, \quad h_{\omega_2} = \frac{20}{11}.$$

Decomposition:

$$W_k(E_7) = L_{D_6}(-\frac{9}{2}\Lambda_0) \oplus L_{D_6}(-\frac{11}{2}\Lambda_0 + \Lambda_6).$$

Case (7): \mathfrak{g}^\natural of Type E_7 , $\mathfrak{g}_{-1/2} = V_{E_7}(\omega_7)$

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{E_7}(0) + V_{E_7}(\omega_1) + V_{E_7}(\omega_6) + V_{E_7}(2\omega_7).$$

Values of h_μ 's:

$$h_{2\omega_7} = \frac{60}{19}, \quad h_{\omega_6} = \frac{56}{19}, \quad h_{\omega_1} = \frac{36}{19}.$$

Decomposition:

$$W_k(E_8) = L_{E_7}(-\frac{17}{2}\Lambda_0) \oplus L_{E_7}(-\frac{19}{2}\Lambda_0 + \Lambda_7).$$

Case (8): \mathfrak{g}^\natural of Type B_3 , $\mathfrak{g}_{-1/2} = V_{B_3}(\omega_3)$

$$\begin{aligned} &\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} \\ &= V_{B_3}(\omega_3) \otimes V_{B_3}(\omega_3) = V_{B_3}(2\omega_3) + V_{B_3}(\omega_2) + V_{B_3}(\omega_1) + V_{B_3}(0). \end{aligned}$$

Values of h_μ 's:

$$h_{2\omega_3} = \frac{24}{7}, \quad h_{\omega_2} = \frac{20}{7}, \quad h_{\omega_1} = \frac{12}{7},$$

which are not integers. Decomposition

$$W_k(F(4)) = L_{B_3}(-\frac{13}{4}\Lambda_0) \oplus L_{B_3}(-\frac{17}{4}\Lambda_0 + \Lambda_3).$$

Case (9): \mathfrak{g}^\natural of Type $A_1 \times C_n$, $\mathfrak{g}_{-1/2} = V_{A_1}(\omega_1) \otimes V_{C_n}(\omega_1)$

$$\begin{aligned} &\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} \\ &= (V_{A_1}(2\omega_1) + V_{A_1}(0)) \otimes (V_{C_n}(0) + V_{C_n}(\omega_2) + V_{C_n}(2\omega_1)). \end{aligned}$$

Values of h_μ 's:

$$h_{2\omega_1,0} = \frac{4}{3}, \quad h_{0,\omega_2} = \frac{4n}{2n+1}, \quad h_{0,2\omega_1} = \frac{4n+4}{2n+1},$$

$$h_{2\omega_1,\omega_2} = \frac{4}{3} + \frac{4n}{2n+1}, \quad h_{2\omega_1,2\omega_1} = \frac{4}{3} + \frac{4n+4}{2n+1}.$$

which are not integers if $n \geq 2$.

Decomposition:

$$W_k(\mathfrak{osp}(4|2n)) = L_{A_1}\left(-\frac{1}{2}\Lambda_0\right) \otimes L_{C_n}\left(-\frac{2n+3}{4}\Lambda_0\right)$$

$$\oplus L_{A_1}\left(-\frac{3}{2}\Lambda_0 + \Lambda_1\right) \otimes L_{C_n}\left(-\frac{2n+7}{4}\Lambda_0 + \Lambda_1\right).$$

Case (10): \mathfrak{g}^\natural of Type C_n , $\mathfrak{g}_{-1/2} = V_{C_n}(\omega_1)$,

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{C_n}(2\omega_1) + V_{C_n}(\omega_2) + V_{C_n}(0).$$

Values of h_μ 's:

$$h_{2\omega_1} = \frac{6(n+1)}{2n+1}, \quad h_{\omega_2} = \frac{6n}{2n+1}.$$

For $n \geq 2$ we have that $h_{2\omega_1}$ and h_{ω_2} are non-integral.

Decomposition:

$$W_k(C_{n+1}) = L_{C_n}\left(-\frac{4n+5}{6}\Lambda_0\right) \oplus L_{C_n}\left(-\frac{4n+11}{6}\Lambda_0 + \Lambda_1\right).$$

Case (11): \mathfrak{g}^\natural of Type D_n , $\mathfrak{g}_{-1/2} = V_{D_n}(\omega_1)$,

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = V_{D_n}(0) \oplus V_{D_n}(2\omega_1) \oplus V_{D_n}(\omega_2).$$

Values of h_μ 's:

$$h_{2\omega_1} = 3 + \frac{3}{2n-1}, \quad h_{\omega_2} = 3 - \frac{3}{2n-1}.$$

These values are non-integral for $n \geq 3$.

Decomposition:

$$W_k(\mathfrak{spo}(2|2n)) = L_{D_n}\left(-\frac{4n-5}{3}\Lambda_0\right) \oplus L_{D_n}\left(-\frac{4n-2}{3}\Lambda_0 + \Lambda_1\right).$$

Case (12): \mathfrak{g}^\natural of Type B_n , $\mathfrak{g}_{-1/2} = \begin{cases} V_{B_n}(\omega_1) & \text{if } n \geq 2 \\ V_{A_1}(2\omega_1) & \text{if } n = 1 \end{cases}$,

$$\mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = \begin{cases} V_{B_n}(0) \oplus V_{B_n}(2\omega_1) \oplus V_{B_n}(\omega_2) & \text{if } n \geq 2 \\ V_{A_1}(0) \oplus V_{A_1}(2\omega_1) \oplus V_{A_1}(4\omega_1) & \text{if } n = 1 \end{cases}.$$

Values of h_μ 's:

$$h_{2\omega_1} = 3 + \frac{3}{2n}, \quad h_{\omega_2} = 3 - \frac{3}{2n}.$$

These values are non-integral for $n \geq 1$.

Decomposition:

$$W_k(\mathfrak{spo}(2|2n+1)) = L_{B_n}\left(-\frac{4n-3}{3}\Lambda_0\right) \oplus L_{B_n}\left(-\frac{4n}{3}\Lambda_0 + \Lambda_1\right),$$

and for $n = 1$

$$W_k(\mathfrak{spo}(2|3)) = L_{A_1}\left(-\frac{2}{3}\Lambda_0\right) \oplus L_{A_1}\left(-\frac{8}{3}\Lambda_0 + 2\Lambda_1\right)$$

Case (13) $\mathfrak{g}^\natural = \mathfrak{spo}(n-2|m)$, $\mathfrak{g}_{-1/2} = \mathbb{C}^{n-2|m}$. We have

$$(6.6) \quad \mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = S^2\mathbb{C}^{n-2|m} \oplus \wedge^2\mathbb{C}^{n-2|m}.$$

As \mathfrak{g}^\natural -module, the first summand in the r.h.s. of (6.6) is the adjoint representation of \mathfrak{g}^\natural (which is irreducible, since \mathfrak{g}^\natural is simple), and the second summand in the sum of a trivial representation and an irreducible summand. Fix in \mathfrak{g} the distinguished set of positive roots Π_B if m is odd and Π_{D_1} if m is even (notation as in [25, 4.4]). This choice induces on \mathfrak{g}^\natural a distinguished set of positive roots and, with respect to it, the nonzero highest weights of the irreducible \mathfrak{g}^\natural -modules appearing in (6.6) are $2\delta_2$, $\delta_2 + \delta_3$. Values of h_μ 's:

$$h_{2\delta_2} = 3\left(1 + \frac{1}{n-m-1}\right), \quad h_{\delta_2+\delta_3} = 3\left(1 - \frac{1}{n-m-1}\right).$$

This values are integers if and only if $m = n+2, n, n-2, n-4$.

Case (14): $\mathfrak{g}^\natural = \mathfrak{osp}(m-4|n) \oplus \mathfrak{sl}(2)$, $\mathfrak{g}_{-1/2} = \mathbb{C}^{m-4|n} \otimes \mathbb{C}^2$. We have

$$(6.7) \quad \mathfrak{g}_{-1/2} \otimes \mathfrak{g}_{-1/2} = (\wedge^2\mathbb{C}^{m-4|n} \oplus S^2\mathbb{C}^{m-4|n}) \otimes (\mathfrak{sl}(2) \oplus \mathbb{C}).$$

As $\mathfrak{osp}(m-4|n)$ -modules, $\wedge^2\mathbb{C}^{m-4|n}$ is the adjoint representation (which is irreducible, since $\mathfrak{osp}(m-4|n)$ is simple), and $S^2\mathbb{C}^{m-4|n}$ is the sum of a trivial representation and an irreducible summand. If $m = 2t+1$ is odd, we fix in \mathfrak{g} the set of positive roots corresponding to the diagram [25, (4.20)] with α_t odd isotropic, the short root odd non-isotropic, and the other roots even. If m is even we choose the set of positive roots corresponding to the diagram Π_{D_2} of [25]. With respect to the induced set of positive roots for $\mathfrak{osp}(m-4|n)$ the highest weight of $\wedge^2\mathbb{C}^{m-4|n}$ is $\epsilon_3 + \epsilon_4$ and the highest weight of the nontrivial irreducible component of $S^2\mathbb{C}^{m-4|n}$ is $2\epsilon_3$. The highest weight of $\mathfrak{sl}(2)$ is $\epsilon_1 - \epsilon_2$. Values of h_μ 's:

$$h_{2\epsilon_3, \epsilon_1 - \epsilon_2} = \frac{10}{3} + \frac{2}{m-n-5}, \quad h_{\epsilon_3 + \epsilon_4, \epsilon_1 - \epsilon_2} = \frac{10}{3} - \frac{2}{m-n-5},$$

$$h_{2\epsilon_3, 0} = 2\left(1 + \frac{1}{m-n-5}\right), \quad h_{\epsilon_3 + \epsilon_4, 0} = 2\left(1 - \frac{1}{m-n-5}\right), \quad h_{0, \epsilon_1 - \epsilon_2} = \frac{4}{3}.$$

This values are not in \mathbb{Z}_+ for m, n in the range showed in the statement. \square

We now list the cases that are not covered by Corollary 6.7. We list here only the cases where there is a non-collapsing conformal level as described in Proposition 4.2.

- (1) \mathfrak{g} of type $G(3)$, $k = \frac{5}{4}$;
- (2) $\mathfrak{g} = D(2, 1; a)$ ($a \notin \{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$), $k = \frac{1}{2}$;

- (3) $\mathfrak{g} = \mathfrak{psl}(2|2), k = \frac{1}{2}$;
- (4) $\mathfrak{g} = \mathfrak{spo}(m+r|m), (m \geq 4, r \in \{0, 2\}), k = -\frac{r+2}{3}$;
- (5) $\mathfrak{g} = \mathfrak{osp}(n+r|n), (r \in \{-1, 3, 4, 6, 8, 11\}), k = \frac{3-r}{2}$;
- (6) $\mathfrak{g} = \mathfrak{osp}(m|n), k = \frac{2}{3}(n-m+2)$;
- (7) $\mathfrak{g} = F(4), k = -1$;
- (8) $\mathfrak{g} = G(3), k = -\frac{1}{2}$.

Sometimes $W_k(\mathfrak{g}, \theta)$ still decomposes finitely as a $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module. More explicitly, we have the following result:

Theorem 6.9. $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is a simple affine vertex algebra and $W_k(\mathfrak{g}, \theta)^{\bar{i}}, i = 0, 1$, are irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules in the following cases:

- (1) $\mathfrak{g} = \mathfrak{so}(8), k = -\frac{5}{2}$;
- (2) $\mathfrak{g} = D(2, 1; 1) = \mathfrak{osp}(4|2), k = \frac{1}{2}$;
- (3) $\mathfrak{g} = D(2, 1; 1/4), k = \frac{1}{2}$.

The proof of Theorem 6.9 requires some representation theory of the vertex algebra $V_{-1/2}(\mathfrak{sl}(2))$.

Remark 6.2. Let $k = \frac{1}{2}$. In the case when $\frac{k-a}{a}$, where $a \in \mathbb{Q}$, is an admissible level for $\widehat{\mathfrak{sl}}_2$ we also expect that $W_k(\mathfrak{g}, \theta)$ is a finite sum of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules, but the decomposition is more complicated. We think that the methods developed in [21] can be applied for this conformal embedding. Here we shall only consider the cases $a = 1$ and $a = 1/4$, where we can apply fusion rules for affine vertex algebras.

We will prove cases (1), (2), (3) of Theorem 6.9 in Sections 6.3.1, 6.3.2, and 6.3.3, respectively.

The next result shows that in case (3) above we have an infinite decomposition:

Theorem 6.10. If $\mathfrak{g} = \mathfrak{psl}(2|2)$ and $k = \frac{1}{2}$ then $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is simple and $W_k(\mathfrak{g})$ decomposes into an infinite direct sum of irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -modules.

Proof. In this case $W_k(\mathfrak{g}, \theta)$ is isomorphic to the $N = 4$ superconformal vertex algebra $V_c^{N=4}$ with $c = -9$ (cf. [30]). The explicit realization of this vertex algebra from [3] gives the result.

Remark 6.3. The remaining open cases are

- \mathfrak{g} of type $G(3), k = \frac{5}{4}$;
- $\mathfrak{g} = D(2, 1; a), (a \notin \{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, 1, \frac{1}{4}\}), k = \frac{1}{2}$;
- $\mathfrak{g} = \mathfrak{so}(11), k = -4$;
- \mathfrak{g} of type $B_n (n \geq 3), k = -\frac{4n-2}{3}$;
- \mathfrak{g} of type $D_n (n \geq 5), k = -\frac{4n-4}{3}$;
- $\mathfrak{g} = \mathfrak{osp}(4|n) (n > 2, n \neq 8), k = -\frac{4-2n}{3}$.

6.3. Proof of Theorem 6.9. As in Theorem 6.6, we set

$$\mathcal{U} = \text{span}\{G^{\{u\}} \mid u \in \mathfrak{g}_{-1/2}\}, \quad A = \mathcal{V}_k(\mathfrak{g}^{\natural}) \cdot \mathcal{U}.$$

In order to apply Theorem 6.6, we have to prove that

$$A \cdot A \subset \mathcal{V}_k(\mathfrak{g}^{\natural}).$$

We first prove that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is simple and that A is a simple $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module. Since $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is admissible, we have that $A \cdot A$ is completely reducible [32]. Let $A \cdot A = \sum_i M_i$ be its decomposition into simple modules. We have to show that the only summands appearing are vacuum modules. This is guaranteed by the fusion rules (5.3) presented in Subsection 5.2.1. Next we provide details in each of the three cases.

6.3.1. Proof of Theorem 6.9, case (1). We claim that, as $\widehat{sl(2)}$ -modules,

$$W_{-5/2}(so(8), \theta) = L_{sl(2)}(-\frac{1}{2}\Lambda_0)^{\otimes 3} \oplus L_{sl(2)}(-\frac{3}{2}\Lambda_0 + \Lambda_1)^{\otimes 3}.$$

Recall that in this case $\mathfrak{g}^{\natural} \simeq sl(2) \oplus sl(2) \oplus sl(2)$. Then A is a highest weight $V^{-1/2}(sl(2))^{\otimes 3}$ -module with highest weight vector $v_{\lambda_1} \otimes v_{\lambda_1} \otimes v_{\lambda_1}$.

Assume now that A is not irreducible. Then, as observed in Subsection 5.2.1 (2), a quotient of $N_{sl(2)}(\lambda_1)$ is either irreducible or it is $N_{sl(2)}(\lambda_1)$ itself. It follows that

$$A \cong N_{sl(2)}(\lambda_1) \otimes \tilde{L}_{sl(2)}(\lambda_1) \otimes \tilde{L}_{sl(2)}(\lambda_1)$$

where $\tilde{L}_{sl(2)}(\lambda_1)$ and $\tilde{\tilde{L}}_{sl(2)}(\lambda_1)$ are certain quotients of $N_{sl(2)}(\lambda_1)$.

Let

$$w_3 = v_{\lambda_3} \otimes v_{\lambda_1} \otimes v_{\lambda_1}, \quad W_3 = U(\mathfrak{g}^{\natural})w_3 \subset A.$$

By using tensor product decompositions

$$\begin{aligned} V_{sl(2)}(3\omega_1) \otimes V_{sl(2)}(\omega_1) &= V_{sl(2)}(4\omega_1) \oplus V_{sl(2)}(2\omega_1), \\ V_{sl(2)}(\omega_1) \otimes V_{sl(2)}(\omega_1) &= V_{sl(2)}(2\omega_1) \oplus V_{sl(2)}(0), \end{aligned}$$

and arguing as in the proof of Theorem 6.6 we see that $\mathcal{U} \cdot W_3$ cannot contain primitive vectors of conformal weight ≤ 3 . Since the conformal weight of all elements of W_3 equals the conformal weight of w_3 which is $\frac{7}{2}$ and the conformal weight of all elements of \mathcal{U} is $\frac{3}{2}$, we conclude that

$$\mathcal{U}_{(n)}W_3 = 0 \quad (n \geq 1).$$

This implies that w_3 is a non-trivial singular vector in $W_{-5/2}(so(8), \theta)$. A contradiction. Therefore $W_3 = 0$ and $A \cong L(\lambda_1)^{\otimes 3}$.

We will now show that $\mathcal{V}_{-5/2}(\mathfrak{g}^{\natural})$ is simple. If not, since a quotient of $V^{-1/2}(sl(2))$ is either simple or $V^{-1/2}(sl(2))$ itself, we have

$$\mathcal{V}_{-5/2}(\mathfrak{g}^{\natural}) = V^{-1/2}(sl(2)) \otimes \tilde{V}_{-1/2}(sl(2)) \otimes \tilde{\tilde{V}}_{-1/2}(sl(2)),$$

where $\tilde{V}_{-1/2}(sl(2))$ and $\tilde{\tilde{V}}_{-1/2}(sl(2))$ are certain quotients of $V^{-1/2}(sl(2))$.

Set

$$w_4 = v_{\lambda_4} \otimes \mathbf{1} \otimes \mathbf{1}, \quad W_4 = U(\mathfrak{g}^\natural)w_4 \subset \mathcal{V}_{-5/2}(\mathfrak{g}^\natural).$$

By using fusion rules again we see that $\mathcal{U}_{(1)}W_4 = W_3 = 0$. So w_4 is a singular vector in $W_{-5/2}(so(8), \theta)$, a contradiction.

Therefore $\mathcal{V}_{-5/2}(\mathfrak{g}^\natural) = V_{-1/2}(sl(2))^{\otimes 3}$. By using fusion rules (5.3) we easily get that

$$\mathcal{V}_{-5/2}(\mathfrak{g}^\natural) \oplus A = V_{-1/2}(sl(2))^{\otimes 3} \oplus L(\lambda_1)^{\otimes 3}$$

is a vertex subalgebra of $W_{-5/2}(so(8), \theta)$. Since this subalgebra contains all generators of $W_{-5/2}(so(8), \theta)$, the claim follows. \square

6.3.2. *Proof of Theorem 6.9, case (2).* The proof is similar to case (1).

We claim that, as $\widehat{sl(2)}$ -modules,

$$\begin{aligned} & W_{1/2}(osp(4|2), \theta) \\ &= L_{sl(2)}(-\frac{1}{2}\Lambda_0) \otimes L_{sl(2)}(-\frac{5}{4}\Lambda_0) \oplus L_{sl(2)}(-\frac{3}{2}\Lambda_0 + \Lambda_1) \otimes L_{sl(2)}(-\frac{9}{4}\Lambda_0 + \Lambda_1). \end{aligned}$$

Let \mathcal{U}, A be as in Subsection 6.3.1. Then A is a highest weight $\mathcal{V}_{1/2}(\mathfrak{g}^\natural)$ -module with highest weight vector $v_{\lambda_1} \otimes v_{\tilde{\lambda}_1}$ where $\tilde{\lambda}_1 = -\frac{9}{4}\Lambda_0 + \Lambda_1$. Assume now that $V^{-1/2}(sl(2))v_{\lambda_1}$ is not simple. Then

$$A \cong N_{sl(2)}(\lambda_1) \otimes \tilde{L}_{sl(2)}(\tilde{\lambda}_1),$$

where $\tilde{L}_{sl(2)}(\tilde{\lambda}_1)$ is certain a quotient of $N_{sl(2)}(\tilde{\lambda}_1)$.

Set

$$w_3 = v_{\lambda_3} \otimes v_{\tilde{\lambda}_1}, \quad W_3 = U(\mathfrak{g}^\natural)w_3 \subset A.$$

The same argument of case (1), using fusion rules and evaluation of conformal weights, shows that w_3 is a non-trivial singular vector in $W_{1/2}(osp(4|2), \theta)$, a contradiction. Therefore $W_3 = 0$ and

$$A \cong L(\lambda_1) \otimes \tilde{L}(\tilde{\lambda}_1).$$

Assume next that $V^{-1/2}(sl(2)) \cdot \mathbf{1}$ is not simple. Then

$$\mathcal{V}_{1/2}(\mathfrak{g}^\natural) = V^{-1/2}(sl(2)) \otimes \tilde{V}^{-5/4}(sl(2))$$

where $\tilde{V}^{-5/4}(sl(2))$ is a certain quotient of $V^{-5/4}(sl(2))$.

Set

$$w_4 = v_{\lambda_4} \otimes \mathbf{1}, \quad W_4 = U(\mathfrak{g}^\natural)w_4 \subset \mathcal{V}_{1/2}(\mathfrak{g}^\natural).$$

By using fusion rules again we see that $\mathcal{U}_{(1)}W_4 = W_3 = 0$. So w_4 is a singular vector in $W_{1/2}(osp(4|2), \theta)$. A contradiction.

Therefore $\mathcal{V}_{1/2}(\mathfrak{g}^\natural) = V_{-1/2}(sl(2)) \otimes \tilde{V}^{-5/4}(sl(2))$. By using fusion rules (5.3) we easily get that

$$\mathcal{V}_{1/2}(\mathfrak{g}^\natural) \oplus A = W_{1/2}(osp(4|2), \theta).$$

In particular $V_{1/2}(\mathfrak{g}^\natural)$ is a simple vertex algebra and A is its simple module. The claim follows. \square

6.3.3. *Proof of Theorem 6.9, case (3).* In case (3) we have

$$W_{1/2}(D(2, 1; 1/4), \theta) = L_{sl(2)}(\Lambda_0) \otimes L_{sl(2)}(-\frac{7}{5}\Lambda_0) \oplus L_{sl(2)}(\Lambda_1) \otimes L_{sl(2)}(-\frac{12}{5}\Lambda_0 + \Lambda_1),$$

and the proof is completely analogous to that of case (2).

7. THE VERTEX ALGEBRA $\mathcal{R}^{(3)}$ AND PROOF OF THEOREM 6.5

In this section we will present an explicit realization of the vertex algebra $W_k(sl(4), \theta)$ and prove that it is isomorphic to the vertex algebra $\mathcal{R}^{(3)}$ from [3]. In this way we prove Conjecture 2 from [3]. Then we apply this new realization to construct explicitly infinitely many singular vectors in each charge component $W_k^{(i)}$, proving Theorem 6.5.

7.1. **Definition of $\mathcal{R}^{(3)}$.** Let us first recall the definition of the vertex algebra $\mathcal{R}^{(3)}$ introduced in Section 12 of [3]. Let $V_L = M(1) \otimes \mathbb{C}[L]$ be the generalized lattice vertex algebra (cf. [14], [18]) associated to the (non-integral) lattice

$$L = \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\delta + \mathbb{Z}\varphi,$$

with non-zero inner products

$$\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \quad \langle \delta, \delta \rangle = -\langle \varphi, \varphi \rangle = \frac{2}{3}.$$

Set $\alpha_1 = \alpha + \beta$, $\alpha_2 = \frac{3}{2}(\delta + \varphi)$, $\alpha_3 = \frac{3}{2}(\delta - \varphi)$, and

$$D = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3.$$

Then D is an even integral lattice. We choose a bi-multiplicative 2-cocycle ε such that for every $\gamma_1, \gamma_2 \in D$ we have

$$\varepsilon(\gamma_1, \gamma_2)\varepsilon(\gamma_2, \gamma_1) = (-1)^{\langle \gamma_1, \gamma_2 \rangle}.$$

We fix the following choice of the cocycle:

$$\begin{aligned} \varepsilon(\alpha_1, \alpha_i) &= \varepsilon(\alpha_i, \alpha_1) = \varepsilon(\alpha_i, \alpha_i) = 1 \quad (i = 1, 2, 3) \\ \varepsilon(\alpha_2, \alpha_3) &= -\varepsilon(\alpha_3, \alpha_2) = 1. \end{aligned}$$

This cocycle can be extended to a 2-cocycle on L by bimultiplicativity. Then we have

$$\begin{aligned} \varepsilon(\alpha + \beta - 3\delta, \alpha_3) &= \varepsilon(\alpha_1 - \alpha_2 - \alpha_3, \alpha_3) = 1, \\ \varepsilon(\alpha + \beta - 3\delta, \alpha_2) &= \varepsilon(\alpha_1 - \alpha_2 - \alpha_3, \alpha_2) = -1. \end{aligned}$$

Let $C_\varepsilon[D]$ be the twisted group algebra associated to the lattice D and cocycle ε . We consider the lattice type vertex algebra

$$V_D^{ext} = M(1) \otimes C_\varepsilon[D],$$

which is realized as a vertex subalgebra of V_L . (Note that V_D^{ext} contains the complete Heisenberg vertex subalgebra $M(1)$ of V_L , and that the lattice D has three generators.) All calculations below will be done in this vertex algebra.

For $\gamma \in D$ we define the following elements of the Heisenberg vertex algebra $M(1)$:

$$S_2(\gamma) = \frac{1}{2}((\gamma_{(-1)})^2 + \gamma_{(-2)}), \quad S_3(\gamma) = \frac{1}{6}(\gamma_{(-1)}^3 + 3\gamma_{(-1)}\gamma_{(-2)} + 2\gamma_{(-3)}).$$

First we recall that the vertex subalgebra \mathcal{V} of V_D^{ext} , generated by

$$(7.1) \quad \begin{aligned} e &= e^{\alpha+\beta}, \\ h &= -2\beta + \delta, \\ f &= \left(-\frac{2}{3}(\alpha_{(-1)}^2 + \alpha_{(-2)}) - \alpha_{(-1)}\delta_{(-1)} + \frac{1}{3}\alpha_{(-1)}\beta_{(-1)}\right)e^{-\alpha-\beta}, \\ j &= \varphi, \end{aligned}$$

is an affine vertex algebra. More precisely, it is isomorphic to $M_\varphi(-2/3) \otimes V_{-5/3}(sl(2))$ (Note that $k = -5/3$ is a generic level, i.e. $V_{-5/3}(sl(2)) = V^{-5/3}(sl(2))$, cf. [24]).

Let $Q = e_{(0)}^{\alpha+\beta-3\delta}$ be the screening operator (cf. [3]). Note that Q is a derivation of the vertex algebra V_D^{ext} . We also have that the Sugawara Virasoro vector $\omega_{sug}^{\mathcal{V}}$ of \mathcal{V} maps to

$$\left(\frac{1}{2}(\alpha_{(-1)}^2 - \alpha_{(-2)} - \beta_{(-1)}^2 + \beta_{(-2)}) + \frac{3}{4}(\delta_{(-1)}^2 - 2\delta_{(-2)} - \varphi_{(-1)}^2)\right) \mathbf{1}.$$

We define $\mathcal{R}^{(3)}$ to be the vertex subalgebra of V_D^{ext} generated by the generators of \mathcal{V} and the following four even vectors of conformal weight $3/2$:

$$\begin{aligned} E^1 &= e^{\frac{3}{2}(\delta+\varphi)}, \\ E^2 &= Qe^{\frac{3}{2}(\delta-\varphi)} = S_2(\alpha + \beta - 3\delta)e^{-\frac{3}{2}(\delta+\varphi)+\alpha+\beta}, \\ F^1 &= f_{(0)}E^1 = -\alpha_{(-1)}e^{-\alpha-\beta+\frac{3}{2}(\delta+\varphi)}, \\ F^2 &= f_{(0)}E^2 = (-\alpha_{(-1)}S_2(\alpha + \beta - 3\delta) + S_3(\alpha + \beta - 3\delta))e^{-\frac{3}{2}(\delta+\varphi)}. \end{aligned}$$

The vertex algebra $\mathcal{R}^{(3)}$ satisfies the following properties:

- $\mathcal{R}^{(3)}$ is integrable, as a module over $sl(2)$.
- $\mathcal{R}^{(3)}$ has finite-dimensional weight spaces with respect to $(\omega_{sug}^{\mathcal{V}})_0$. The conformal weights lie in $\frac{1}{2}\mathbb{Z}_{\geq 0}$.
- $\mathcal{R}^{(3)}$ is contained in the following subalgebra of V_D^{ext} :

$$M \otimes \Pi(0),$$

where M is the Weyl vertex algebra (i.e., the algebra of symplectic bosons [28]) generated by

$$a = e^{\alpha+\beta}, a^* = -\alpha_{(-1)}e^{-\alpha-\beta},$$

and $\Pi(0)$ is the "half-lattice" vertex algebra

$$\Pi(0) = M_{\delta,\varphi}(1) \otimes \mathbb{C}[\mathbb{Z} \frac{3(\delta+\varphi)}{2}]$$

containing the Heisenberg vertex algebra $M_{\delta,\varphi}(1)$ generated by δ and φ (cf. [3]).

Let $(M \otimes \Pi(0))^{int}$ denote the maximal $sl(2)$ -integrable submodule of $M \otimes \Pi(0)$. It is clear that it is a vertex subalgebra of $M \otimes \Pi(0)$.

We shall prove the following result.

Theorem 7.1.

- (1) *There is a vertex algebra homomorphism $W^k(sl(4), \theta) \rightarrow \mathcal{R}^{(3)}$.*
- (2) *$\mathcal{R}^{(3)}$ is a simple vertex algebra, i.e., $W_k(sl(4), \theta) = \mathcal{R}^{(3)}$.*
- (3) *$\mathcal{R}^{(3)} \cong (M \otimes \Pi(0))^{int}$.*

Remark 7.1. Theorem 7.1 gives a positive answer to Conjecture 2 from [3]. The representation theory of $\mathcal{R}^{(p)}$ for $p > 3$ and its relation with C_2 -cofinite vertex algebras appearing in LCFT (such as triplet vertex algebras) will be studied in [4].

7.2. λ -brackets for $\mathcal{R}^{(3)}$.

Proposition 7.2. *We have the following λ -brackets:*

$$\begin{aligned}
[E^i{}_{\lambda}E^i] &= [F^i{}_{\lambda}F^i] = 0 \quad (i = 1, 2), \\
[E^1{}_{\lambda}E^2] &= 3(\partial e + 3 : je :) + 6\lambda e, \\
[F^1{}_{\lambda}F^2] &= -3(\partial f + 3 : jf :) - 6\lambda f, \\
[E^1{}_{\lambda}F^1] &= 0, \\
[E^1{}_{\lambda}F^2] &= -3(\omega_{sug}^{\mathcal{V}} + \frac{1}{2}(\partial h + 3 : jh : -6 : jj : -5\partial j)) \\
&\quad + 3\lambda(-h + 5j) + 5\lambda^2, \\
[E^2{}_{\lambda}F^1] &= -3(\omega_{sug}^{\mathcal{V}} + \frac{1}{2}(\partial h - 3 : jh : -6 : jj : +5\partial j)) \\
&\quad - 3\lambda(h + 5j) + 5\lambda^2, \\
[E^2{}_{\lambda}F^2] &= 0.
\end{aligned}$$

Proof. The proof uses the standard computations in lattice vertex algebras [28]. Let us discuss the calculation of $[E^1{}_{\lambda}F^2]$ and of $[E^2{}_{\lambda}F^1]$.

For $[E^1{}_{\lambda}F^2]$, the only difficult part is to compute $E^1{}_{(0)}F^2$. We have

$$\begin{aligned}
E^1{}_{(2)}F^2 &= 10, \\
E^1{}_{(1)}F^2 &= -h + 5\varphi = -h + 5j, \\
E^1{}_{(0)}F^2 &= -9\alpha_{(-1)}(\delta + \varphi) - 3\alpha_{(-1)}(\alpha + \beta - 3\delta), \\
&\quad + 10S_2(\frac{3}{2}(\delta + \varphi))\mathbf{1} + 9(\alpha_{(-1)} + \beta_{(-1)} - 3\delta_{(-1)})(\delta + \varphi) \\
&\quad + 3S_2(\alpha + \beta - 3\delta)\mathbf{1} \\
&= -3(\omega_{sug} + 1/2(h_{(-2)} + 3\varphi_{(-1)}h_{(-1)} - 6\varphi_{(-1)}^2 - 5\varphi_{(-2)})\mathbf{1}).
\end{aligned}$$

For the calculation of $[E^2_\lambda F^1]$, we shall use the fact that Q is a derivation in the lattice vertex algebra V_D . Set

$$\overline{E}^1 = e^{\frac{3}{2}(\delta-\varphi)},$$

$$\overline{F}^2 = QF^1 = -(-\alpha_{(-1)}S_2(\alpha + \beta - 3\delta) + S_3(\alpha + \beta - 3\delta))e^{-\frac{3}{2}(\delta-\varphi)}.$$

Note that the minus sign in front of the r.h.s. of the formula above comes from the cocycle computation

$$\varepsilon(\alpha + \beta - 3\delta, -\alpha - \beta + \frac{3}{2}(\delta + \varphi)) = \varepsilon(\alpha_1 - \alpha_2 - \alpha_3, -\alpha_1 + \alpha_2) = -1.$$

Next, we have

$$[E^2_\lambda F^1] = Q[e^{\frac{3}{2}(\delta-\varphi)}_\lambda F^1] - [e^{\frac{3}{2}(\delta-\varphi)}_\lambda QF^1] = -[\overline{E}^1_\lambda \overline{F}^2].$$

The calculation of $[\overline{E}^1_\lambda \overline{F}^2]$ is essentially the same as for $[E^1_\lambda F^2]$ (we just replace j by $-j$). Now we have

$$\begin{aligned} [E^2_\lambda F^1] &= -[\overline{E}^1_\lambda \overline{F}^2] \\ &= (-3(\omega_{sug} + \frac{1}{2}(\partial h - 3 : jh : -6 : jj : +5\partial j))) + 3\lambda(-h - 5j) + 5\lambda^2 \\ &= -3(\omega_{sug} + \frac{1}{2}(\partial h - 3 : jh : -6 : jj : +5\partial j))) + 3\lambda(-h - 5j) + 5\lambda^2 \end{aligned}$$

The claim follows.

7.3. The homomorphism $\Phi : W^k(sl(4), \theta) \rightarrow \mathcal{R}^{(3)}$. Recall from Example 3.1 that the vertex algebra $W^k(sl(4), \theta)$ is generated by the Virasoro vector ω of central charge $c(k) = 15k/(k+4) - 6k$, four even generators $J^{\{e_{2,3}\}}, J^{\{e_{3,2}\}}, J^{\{e_{2,2}-e_{3,3}\}}, J^{\{c\}}$ of conformal weight 1, and four even vectors $G^{\{e_{2,1}\}}, G^{\{e_{3,1}\}}, G^{\{e_{4,2}\}}, G^{\{e_{4,3}\}}$ of conformal weight $3/2$.

By comparing λ -brackets from Proposition 7.2 and λ -brackets for the vertex algebra $W^{-8/3}(sl(4), \theta)$ we get the following result:

Proposition 7.3. *Let $k = -8/3$. There is a vertex algebra homomorphism*

$$\Phi : W^k(sl(4), \theta) \rightarrow \mathcal{R}^{(3)}$$

such that

$$\begin{aligned} J^{\{e_{2,3}\}} &\mapsto e, & J^{\{e_{3,2}\}} &\mapsto f, & J^{\{e_{2,2}-e_{3,3}\}} &\mapsto h, & J^{\{c\}} &\mapsto j, \\ G^{\{e_{2,1}\}} &\mapsto \frac{\sqrt{2}}{3}E^1, & G^{\{e_{3,1}\}} &\mapsto \frac{\sqrt{2}}{3}F^1, & G^{\{e_{4,3}\}} &\mapsto \frac{\sqrt{2}}{3}E^2, & G^{\{e_{4,2}\}} &\mapsto -\frac{\sqrt{2}}{3}F^2, \\ \omega &\mapsto \omega_{sug}^\vee. \end{aligned}$$

Proof. It is enough to check λ -brackets from Example 3.1 in the case $k = -8/3$. In particular, taking into account that

$$\begin{aligned}\omega = \omega_{sug} &= \frac{3}{2}(2 : J^{\{e_{2,3}\}} J^{\{e_{3,2}\}} : -\partial J^{\{e_{2,2-e_{3,3}\}}}) \\ &+ \frac{1}{2} : J^{\{e_{2,2-e_{3,3}\}} J^{\{e_{2,2-e_{3,3}\}}}) : - \frac{3}{4} : J^{\{c\}} J^{\{c\}} :, \end{aligned}$$

we get

$$\begin{aligned}[G^{\{e_{2,1}\}} \lambda G^{\{e_{4,2}\}}] &= \frac{2}{3}\omega - 2 : J^{\{c\}} J^{\{c\}} : + \frac{1}{3}\partial J^{\{e_{2,2-e_{3,3}\}} - \frac{5}{3}\partial J^{\{c\}} \\ &+ : J^{\{c\}} J^{\{e_{2,2-e_{3,3}\}} : + \frac{2}{3}\lambda(-5J^{\{c\}} + J^{\{e_{2,2-e_{3,3}\}}}) - \frac{10}{9}\lambda^2 \\ &= -\frac{2}{9}(-3(\omega_{sug}^{\mathcal{V}} + \frac{1}{2}(\partial h + 3 : jh : -6j^2 - 5\partial j))) + 3\lambda(-h + 5j) + 5\lambda^2. \\ &= -\frac{2}{9}[E^1 \lambda F^2]. \end{aligned}$$

All other λ -brackets are checked similarly.

Proposition 7.3 implies that $\mathcal{R}^{(3)}$ is conformally embedded into a certain quotient of $W^k(sl(4), \theta)$. In the following subsection we will prove that $\mathcal{R}^{(3)}$ is isomorphic to $W_k(sl(4), \theta)$.

7.4. Simplicity of $\mathcal{R}^{(3)}$ and proof of Theorem 7.1. Our proof of simplicity is similar to the proof of simplicity of the $N = 4$ superconformal vertex algebra realized in [3]. As a tool we shall use the theory of Zhu algebras associated to the Neveu-Schwarz sector of $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded vertex algebras. Let V is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded vertex algebra and $A(V) = V/O(V)$ the associated Zhu algebra. Let $[a] = a + O(V)$ (cf. Subsection 2.4).

Lemma 7.4.

- (1) The Zhu algebra $A(\mathcal{R}^{(3)})$ is isomorphic to a quotient of $U(gl(2))$.
- (2) In the Zhu algebra $A(\mathcal{R}^{(3)})$ the following relation holds:

$$[e]([\omega] + \frac{2}{3} - \frac{3}{2}[j]^2) = 0.$$

Proof. Since $\mathcal{R}^{(3)}$ is a quotient of $W^k(sl(4), \theta)$, the first assertion follows from Proposition 4.3. Let us prove the second assertion. We notice that

$$\begin{aligned} : E^1 E^2 := & (S_2(\alpha + \beta - 3\delta) + 6S_2(\frac{3}{2}(\delta + \varphi)) + \frac{9}{2}(\alpha + \beta - 3\delta)_{(-1)}(\delta + \varphi)_{(-1)})e^{\alpha+\beta}. \\ : e\omega := & (-\alpha_{(-1)} + \beta_{(-1)})\beta_{(-1)} + \beta_{(-2)} + \frac{3}{4}(\delta_{(-1)}^2 - 2\delta_{(-2)} - \varphi_{(-1)}^2)e^{\alpha+\beta}. \end{aligned}$$

By direct calculation we get the following relation:

$$\begin{aligned} & : E^1 E^2 : + 3 : e \omega := \\ & (e_{(-3)} + \frac{3}{2}(h_{(-1)}e_{(-2)} - h_{(-2)}e_{(-1)}) + \frac{9}{2}((\varphi_{(-1)}^2 + \varphi_{(-2)})e_{(-1)} + \varphi_{(-1)}e_{(-2)))\mathbf{1}. \end{aligned}$$

We have

$$\begin{aligned} E^1 \circ E^2 &= (E_{(-1)}^1 + E_{(0)}^1)E^2 = \\ & -3e_{(-1)}\omega + e_{(-3)} + \frac{3}{2}(h_{(-1)}e_{(-2)} - h_{(-2)}e_{(-1)}) + \frac{9}{2}(\varphi_{(-1)}^2 + \varphi_{(-2)})e_{(-1)} \\ & + \frac{9}{2}\varphi_{(-1)}e_{(-2)} + 3e_{(-2)} + 9\varphi_{(-1)}e_{(-1)}. \end{aligned}$$

This gives the following relation in the Zhu algebra:

$$-3[e][\omega] - 2[e] + \frac{9}{2}([j]^2 + [j] - [j])[e] = -3[e][\omega] - 2[e] + \frac{9}{2}[j]^2[e] = 0.$$

The claim follows.

Proposition 7.5.

- (1) $\mathcal{R}^{(3)}$ is a simple vertex algebra.
- (2) $\mathcal{R}^{(3)} \cong (M \otimes \Pi(0))^{int}$.

Proof. By using the fact that $\mathcal{R}^{(3)}$ is a subalgebra of $M \otimes \Pi(0)$, we conclude that if w_{sing} is a singular vector for $W^k(sl(4), \theta)$ in $\mathcal{R}^{(3)}$, it must have $gl(2)$ -weight $(n\omega_1, m)$ for $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$. This means that

$$h_{(0)}w_{sing} = nw_{sing}, \quad \varphi_{(0)}w_{sing} = mw_{sing}.$$

This leads to the relation

$$L_{(0)}w_{sing} = \left(\frac{3n(n+2)}{4} - \frac{3}{4}m^2\right)w_{sing}.$$

On the other hand, w_{sing} generates a submodule whose lowest component must be a module for the Zhu algebra. Now Lemma 7.4 implies that $U(gl(2))w_{sing}$ is annihilated by $[e](\omega + \frac{2}{3} - \frac{3}{2}[j]^2)$. If $n > 0$, we get

$$\frac{3n(n+2)}{4} - \frac{3}{4}m^2 - \frac{3}{2}m^2 = \frac{3n(n+2) - 9m^2}{4} = -\frac{2}{3},$$

which gives a contradiction since $m \in \mathbb{Z}$. So $n = 0$. Then the fact that conformal weight must be positive implies that $m = 0$. Therefore w_{sing} must be proportional to the vacuum vector. We deduce that there are no non-trivial singular vectors, and therefore $\mathcal{R}^{(3)}$ is a simple vertex algebra. This proves (1). The proof of assertion (2) is completely analogous. \square

Proof of Theorem 7.1. Apply Propositions 7.3 and 7.5.

7.5. $\widehat{gl(2)}$ -singular vectors in $\mathcal{R}^{(3)}$.

Lemma 7.6. *Let $\ell \in \mathbb{Z}$.*

- (1) *If $\ell \geq 0$, then for every $j \geq 0$*

$$v_{\ell,j} = Q^j e^{\frac{3\ell}{2}(\delta+\varphi)+3j\delta}$$

is a non-trivial singular vector in $\mathcal{R}^{(3)}$.

- (2) *If $\ell \leq 0$, then for every $j \geq 0$*

$$v_{\ell,j} = Q^{j-\ell} e^{-\frac{3\ell}{2}(\delta-\varphi)+3j\delta}$$

is a non-trivial singular vector in $\mathcal{R}^{(3)}$.

In particular, the set $\{v_{\ell,j} \mid j \geq 0\}$ provides an infinite family of linearly independent $\widehat{gl(2)}$ -singular vectors in the ℓ -eigenspace of $\varphi_{(0)}$.

Proof. The non-triviality of the singular vectors $v_{\ell,j}$ is well known (cf. [5]). The assertions now follow from the fact $v_{\ell,j}$ belongs to a maximal $sl(2)$ -integral part of $M \otimes \Pi(0)$. \square

7.6. Proof of Theorem 6.5. Since we have proved that $W_k(sl(4), \theta)$ is isomorphic to the simple vertex algebra $\mathcal{R}^{(3)}$, Lemma 7.6 shows that each $W_k(sl(4), \theta)^{(i)}$ contains infinitely many linearly independent singular vectors.

Remark 7.2. Assertion (3) of Theorem 7.1 implies that $W_k(sl(4), \theta) = \mathcal{R}^{(3)}$ is an object of the category KL_{k+1} of $V_{k+1}(sl(2))$ -module. In particular, each $W_k(sl(4), \theta)^{(i)}$ is an object this category. Since $k+1 = -5/3$ is a generic level for $\widehat{sl(2)}$, and the category KL_{k+1} is semisimple (this follows easily from [34], we skip details), we have that $W_k(sl(4), \theta)^{(i)}$ is completely reducible. So we actually proved that each $W_k(sl(4), \theta)^{(i)}$ is a direct sum of infinitely many irreducible $V_{k+1}(gl(2))$ -modules.

8. EXPLICIT DECOMPOSITIONS FROM THEOREM 6.4: \mathfrak{g}^{\natural} IS A LIE ALGEBRA

In Theorem 6.4 we proved a semisimplicity result for conformal embeddings of $\mathcal{V}_k(\mathfrak{g}^{\natural})$ in $W_k(\mathfrak{g}, \theta)$ where $\mathfrak{g} = sl(n)$ or $\mathfrak{g} = sl(2|n)$. But this semisimplicity result does not identify highest weights of the components $W_k(\mathfrak{g}, \theta)^{(i)}$. In this section we shall identify these components in certain cases and prove that then $W_k(\mathfrak{g}, \theta)$ is a simple current extension of $\mathcal{V}_k(\mathfrak{g}^{\natural})$.

Recall from Section 2.7 that F_n denotes a rank one lattice vertex algebra and F_n^i , $i = 0, \dots, n-1$, denote its irreducible modules. The following result refines Theorem 6.4.

Theorem 8.1. *(1) If $\mathfrak{g} = sl(2n)$ and $k = \frac{1}{2} - n$, $n \geq 2$, then*

$$\text{Com}(V_{k+1}(sl(2n-2)), W_k(\mathfrak{g}, \theta)) \cong F_{4n(n-1)}.$$

Moreover, we have the following decomposition of $W_k(\mathfrak{g}, \theta)$ as a $V_{k+1}(sl(2n-2)) \otimes F_{4n(n-1)}$ -module:

$$(8.1) \quad W_k(\mathfrak{g}, \theta) \cong \bigoplus_{i=0}^{2n-3} L_{sl(2n-2)}(k\Lambda_0 + \Lambda_i) \otimes F_{4n(n-1)}^{2in}.$$

(2) If $\mathfrak{g} = sl(2|n)$ and $k = -\frac{2}{3}h^\vee \notin \mathbb{Z}$, then

$$\text{Com}(V_{-k-1}(sl(n)), W_k(\mathfrak{g}, \theta)) \cong F_{3n}.$$

Moreover, we have the following decomposition of $W_k(\mathfrak{g}, \theta)$ as a $V_{-k-1}(sl(n)) \otimes F_{3n}$ -module:

$$(8.2) \quad W_k(\mathfrak{g}, \theta) \cong \bigoplus_{i=0}^{n-1} L_{sl(n)}(-(k+2)\Lambda_0 + \Lambda_i) \otimes F_{3n}^{3i}.$$

Proof. (1) Let $\alpha = 2nJ^{\{c\}}$ and note that

$$\mathcal{V}_k(\mathfrak{g}^{\natural}) = V_{k+1}(sl(2n-2)) \otimes M_\alpha(4n(n-1)).$$

By Theorem 6.4 we have that each $W_k(\mathfrak{g}, \theta)^{(i)}$ is an irreducible $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module, and, by checking the action of $J_{(0)}^{\{c\}}$, we see that there is a weight Λ such that

$$W_k(\mathfrak{g}, \theta)^{(i)} = L_{sl(2n-2)}(\Lambda) \otimes M_\alpha(4n(n-1), 2in).$$

Since

$$\begin{aligned} W_k(\mathfrak{g}, \theta)^{(1)} &\cong L_{sl(2n-2)}(k\Lambda_0 + \Lambda_1) \otimes M_\alpha(4n(n-1), 2n), \\ W_k(\mathfrak{g}, \theta)^{(-1)} &\cong L_{sl(2n-2)}(k\Lambda_0 + \Lambda_{2n-3}) \otimes M_\alpha(4n(n-1), -2n), \end{aligned}$$

the fusion rules result from Proposition 5.1 and the fusion rules (2.5) imply that

$$(8.3) \quad W_k(\mathfrak{g}, \theta)^{(i)} \cong L_{sl(2n-2)}(k\Lambda_0 + \Lambda_{\bar{i}}) \otimes M_\alpha(4n(n-1), 2in),$$

where

$$\bar{i} \in \{0, \dots, 2n-3\}, \quad i \equiv \bar{i} \pmod{2n-2}.$$

Since

$$\text{Com}(V_{k+1}(sl(2n-2)), W_k(\mathfrak{g}, \theta)) = \{v \in W_k(\mathfrak{g}, \theta) \mid J_{(n)}^{\{u\}}v = 0, n \geq 0, u \in \mathfrak{g}^{\natural}\},$$

we get that

$$\text{Com}(V_{k+1}(sl(2n-2)), W_k(\mathfrak{g}, \theta)) \cong \bigoplus_{i \in \mathbb{Z}} M_\alpha(4n(n-1), 4in(n-1))$$

as a $M_\alpha(4n(n-1))$ -module. Now Proposition 2.2 implies that

$$\text{Com}(V_{k+1}(sl(2n-2)), W_k(\mathfrak{g}, \theta)) \cong F_{4n(n-1)}.$$

The decomposition (8.1) now easily follows from (8.3). This proves (1).

The proof of (2) is based on Proposition 5.2 and it is completely analogous to the proof of assertion (1). \square

Remark 8.1. Decompositions (8.1) and (8.2), together with the fusion rules result from Propositions 5.1 and 5.2 imply that the minimal W -algebras from Theorem 8.1 are finite simple current extensions of the tensor product of an admissible affine vertex algebra with a rank one lattice vertex algebra. It is also interesting to notice that the r.h.s. of (8.1) and (8.2) have sense for the cases $\mathfrak{g} = sl(n)$, n odd, and $\mathfrak{g} = sl(2|n)$, $k = -\frac{2}{3}h^\vee \in \mathbb{Z}$. But Corollary 8.3 shows that most likely we won't get lattice vertex subalgebra in these cases.

Remark 8.2. The computation of the explicit decompositions in Theorem 6.4 when $\mathcal{V}_k(\mathfrak{g})$ does not contain an admissible vertex algebra of type A needs a subtler analysis. Our approach motivates the study of the following non-admissible affine vertex algebras:

- $V_{k'}(sl(2n+1))$ for $k' = -n$,
- $V_{k'}(sl(n))$ for $k' = -\frac{2n+1}{3}$,
- $V_{k'}(sl(3n+2))$ for $k' = -2n-1$,
- $V_{k'}(sl(n))$ for $k' = -\frac{n+1}{2}$, $n \geq 4$.

Their representation theory is known only for $V_{-1}(sl(3))$ (cf. [8]):

Proposition 8.2. [8] For $s \in \mathbb{Z}_{\geq 0}$ set

$$U_s = L_{sl(3)}(-(1+s)\Lambda_0 + s\Lambda_1), \quad U_{-s} = L_{sl(3)}(-(1+s)\Lambda_0 + s\Lambda_2).$$

- The set $\{U_s \mid s \in \mathbb{Z}\}$ provides a complete list of irreducible $V_{-1}(sl(3))$ modules from the category KL_{-1} .
- The following fusion rules hold in the category KL_{-1} :

$$U_{s_1} \times U_{s_2} = U_{s_1+s_2} \quad (s_1, s_2 \in \mathbb{Z}).$$

By using this proposition we get the following refinement of Theorem 6.4 (1) for the case $n = 5$:

Corollary 8.3. We have the following isomorphism of $V_{-2,3/5}(\mathfrak{g}^\natural)$ -modules:

$$W_{-2}(sl(5), \theta) \cong \bigoplus_{s \in \mathbb{Z}} U_s \otimes M(3/5, s).$$

Proof. Set $\alpha = J^{\{c\}}$. Then $\mathcal{V}_{-2}(\mathfrak{g}^\natural) = V_{-1}(sl(3)) \otimes M_\alpha(3/5)$. By Theorem 6.4, we have that each $W_k(\mathfrak{g}, \theta)^{(i)}$ is an irreducible $\mathcal{V}_k(\mathfrak{g}^\natural)$ -module, and, by checking the action of $J_{(0)}^{\{c\}}$, we see that there is a weight Λ such that

$$W_k(\mathfrak{g}, \theta)^{(i)} = L_{sl(3)}(\Lambda) \otimes M_\alpha(3/5, i).$$

The assertion follows as in the proof of Theorem 8.1 from the fusion rules result from Proposition 8.2.

Remark 8.3. In [28], the vertex algebra U_0 and its modules U_s from Proposition 8.2 are realized inside of the Weyl vertex algebra M_3 of rank three. It was proved in [8] that

$$M_3 \cong \bigoplus_{s \in \mathbb{Z}} U_s \otimes M(-3, s).$$

Note that although $W_{-2}(sl(5), \theta)$ admits an analogous decomposition, one can easily see that this W -algebra is not isomorphic to any subalgebra of M_3 .

We also believe that the modules which appear in the decomposition of $W_k(\mathfrak{g}, \theta)$ in Theorem 6.4 (3) are also simple currents, so one can also expect the decomposition like in Corollary 8.3. Indeed, we can show that such decomposition holds but instead of applying fusion rules (which we don't know yet), we will apply results from our previous papers [9] and [10]. In [9] we proved that the affine vertex algebra $V_{-\frac{n+1}{2}}(sl(n+1))$ ($n \geq 4$) is semisimple as a $V_{-\frac{n+1}{2}}(gl(n))$ -module and identified highest weights of all modules appearing in the decomposition. In [10] we proved that $V_{-\frac{n+1}{2}}(sl(n+1))$ ($n \geq 4, n \neq 5$) is embedded in the tensor product vertex algebra $W_k(sl(2|n), \theta) \otimes F_{-1}$. An application of these results will give the branching rules.

For $s \in \mathbb{Z}_{\geq 0}$, we set

$$U_s^{(n)} = L_{sl(n)}(-(\frac{n+1}{2} + s)\Lambda_0 + s\Lambda_1), \quad U_{-s}^{(n)} = L_{sl(n)}(-(\frac{n+1}{2} + s)\Lambda_0 + s\Lambda_n).$$

Theorem 8.4. *Let $\mathfrak{g} = sl(2|n)$, $k = \frac{1-h^\vee}{2}$, $n = 4$ or $n \geq 6$. We have an isomorphism as $V_k(\mathfrak{g}^{\mathfrak{h}})$ -modules:*

$$W_k(\mathfrak{g}, \theta) \cong \bigoplus_{s \in \mathbb{Z}} U_s^{(n)} \otimes M\left(\frac{n}{n-2}, s\right).$$

Proof. We first consider the Heisenberg vertex algebra $M_\alpha(\frac{n}{n-2}) \otimes M_\varphi(-1)$ generated by the Heisenberg fields $\alpha = J^{\{c\}}$ and φ such that

$$[\alpha_\lambda \alpha] = \frac{n}{n-2} \lambda, \quad [\varphi_\lambda \varphi] = -\lambda.$$

Define

$$\bar{\varphi} = \alpha + \varphi, \quad \hat{\varphi} = \frac{2-n}{2} \alpha + \frac{n}{n-2} \varphi.$$

Then

$$M_\alpha\left(\frac{n}{n-2}\right) \otimes M_\varphi(-1) = M_{\hat{\varphi}}\left(-\frac{n}{2}\right) \otimes M_{\bar{\varphi}}\left(\frac{2}{n-2}\right).$$

Theorem 6.4 (3) implies that

$$W_k(\mathfrak{g}, \theta)^{(s)} \cong L_{sl(n)}(\Lambda^{(s)}) \otimes M_\alpha\left(\frac{n}{n-2}, s\right),$$

where $L_{sl(n)}(\Lambda^{(s)})$ is an irreducible highest weight module from $KL_{-\frac{n+1}{2}}$. It was proved in [10, Theorem 5.6] that

$$\begin{aligned}
& V_{-\frac{n+1}{2}}(sl(n+1)) \otimes M_{\widehat{\varphi}}\left(\frac{2}{n-2}\right) \\
= & \bigoplus_{s \in \mathbb{Z}} W_k(\mathfrak{g}, \theta)^{(s)} \otimes M_{\varphi}(-1, -s) \\
= & \bigoplus_{s \in \mathbb{Z}} L_{sl(n)}(\Lambda^{(s)}) \otimes M_{\alpha}\left(\frac{n}{n-2}, s\right) \otimes M_{\varphi}(-1, -s). \\
= & \left(\bigoplus_{s \in \mathbb{Z}} L_{sl(n)}(\Lambda^{(s)}) \otimes M_{\widehat{\varphi}}\left(-\frac{n}{2}, s\right) \right) \otimes M_{\widehat{\varphi}}\left(\frac{2}{n-2}\right).
\end{aligned}$$

This implies that

$$V_{-\frac{n+1}{2}}(sl(n+1)) \cong \bigoplus_{s \in \mathbb{Z}} L_{sl(n)}(\Lambda^{(s)}) \otimes M_{\widehat{\varphi}}\left(-\frac{n}{2}, s\right).$$

Now results from [9] (see in particular [9, Theorem 2.4, Theorem 5.1 (2)]) imply that

$$L_{sl(n)}(\Lambda^{(s)}) \cong U_s^{(n)}.$$

The claim follows.

9. EXPLICIT DECOMPOSITIONS FROM THEOREM 6.4: \mathfrak{g}^{\natural} IS NOT A LIE ALGEBRA

In this section we describe the decomposition of $W_{-2}(sl(n+5|n), \theta)$ as $\mathcal{V}_k(\mathfrak{g}^{\natural})$ -module. We obtain, similarly to the results of Section 8, that $W_{-2}(sl(n+5|n), \theta)$ is a simple current extension of $\mathcal{V}_k(\mathfrak{g}^{\natural})$. We expect this to hold in general.

9.1. Simple current $V_{k_0}(sl(m|n))$ -modules. Let us first recall a few details on simple current modules obtained by using the simple current operator $\Delta(\alpha, z)$.

Let V be a conformal vertex algebra with conformal vector ω . Let α be an even vector in V such that

$$\omega_n \alpha = \delta_{n,0} \alpha, \quad \alpha_{(n)} \alpha = \delta_{n,1} \gamma \mathbf{1} \quad (n \geq 0),$$

where γ is a complex number. Assume that $\alpha_{(0)}$ acts semisimply on V with eigenvalues in \mathbb{Z} . Let [19]

$$\Delta(\alpha, z) = z^{\alpha_{(0)}} \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_{(n)}}{-n} (-z)^{-n}\right).$$

Then [19]

$$(9.1) \quad (V^{(\alpha)}, Y_{\alpha}(\cdot, z)) := (V, Y(\Delta(\alpha, z)\cdot, z))$$

is a V -module, called a simple current V -module, and

$$(9.2) \quad Y_\alpha(\omega, z) = Y(\omega, z) + z^{-1}Y(\alpha, z) + 1/2\gamma z^{-2}.$$

When V is the simple affine vertex algebra associated to $sl(m|n)$, we will use this construction to produce simple current modules in a suitable category.

Let $\mathfrak{g} = sl(m|n)$ ($m \neq n$), $k_0 \in \mathbb{C}$. Let $e_{i,j}$ denote the standard matrix units in $sl(m|n)$; consider the following vector in $V_{k_0}(sl(m|n))$:

$$\alpha^{m,n} = \frac{1}{m-n}(ne_{1,1} + \cdots + ne_{m,m} + me_{m+1,m+1} + \cdots + me_{m+n,m+n})_{(-1)}\mathbf{1}.$$

Note that

$$(9.3) \quad \mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1, \quad \mathfrak{g}^i = \{x \in \mathfrak{g} \mid [\alpha^{m,n}, x] = ix\}.$$

In particular, $\mathfrak{g}^0 \cong sl(m) \times sl(n) \times \mathbb{C}\alpha^{m,n}$ is the even part of \mathfrak{g} and $\mathfrak{g}^{-1} + \mathfrak{g}^1$ is its odd part.

For $i \in \{-1, 0, 1\}$ and $n \in \mathbb{Z}$ set

$$\mathfrak{g}^i(n) = \mathfrak{g}^i \otimes t^n.$$

The decomposition (9.3) implies that $\alpha_{(0)}^{m,n}$ acts semi-simply on $V_{k_0}(sl(m|n))$ with integral eigenvalues. Moreover

$$[\alpha^{m,n} \lambda \alpha^{m,n}] = \frac{1}{(m-n)^2}(n^2mk_0 - m^2nk_0)\lambda = -\frac{nmk_0}{m-n}\lambda.$$

Set

$$U_s^{m,n} = V_{k_0}(\mathfrak{g})^{(s\alpha^{m,n})} \quad (s \in \mathbb{Z}).$$

By definition (9.1), we see that $U_s^{m,n}$ is obtained from $V_{k_0}(\mathfrak{g})$ by applying the automorphism π_s of $\widehat{\mathfrak{g}}$ (and $V_{k_0}(\mathfrak{g})$) uniquely determined by

$$(9.4) \quad \pi_s(x_{(r)}^\pm) = x_{(r \mp s)}^\pm \quad (x^\pm \in \mathfrak{g}^{\pm 1}),$$

$$(9.5) \quad \pi_s(x_{(r)}) = x_{(r)} \quad (x \in sl(m) \times sl(n) \subset \mathfrak{g}^0),$$

$$(9.6) \quad \pi_s(\alpha_{(r)}^{m,n}) = \alpha_{(r)}^{m,n} - \frac{nmk_0}{m-n}s\delta_{r,0},$$

where $r, s \in \mathbb{Z}$. Note that in $U_s^{m,n}$ we have

$$\mathfrak{g}^{\pm 1}(n \pm s).\mathbf{1} = 0 \quad (n \geq 0).$$

Theorem 9.1. *Assume that $m, n \geq 1$. We have:*

- (1) $U_s^{m,n}$, $s \in \mathbb{Z}$, are irreducible $V_{k_0}(\mathfrak{g})$ -modules from the category KL_{k_0} .
- (2) Let $s = \pm 1$. Then the lowest graded component of $U_s^{m,n}$ is, as a vector space, isomorphic to

$$\bigwedge (\mathfrak{g}^s(0)).\mathbf{1}.$$

It has conformal weight $-\frac{nmk_0}{m-n}$.

- (3) $U_s^{m,n}$, $s \in \mathbb{Z}$, are simple current $V_{k_0}(\mathfrak{g})$ -modules in KL_{k_0} and the following fusion rules holds in KL_{k_0} :

$$(9.7) \quad U_{s_1}^{m,n} \times U_{s_2}^{m,n} = U_{s_1+s_2}^{m,n} \quad (s_1, s_2 \in \mathbb{Z}).$$

Proof. (1) Since $U_s^{m,n} = \pi_s(V_{k_0}(\mathfrak{g}))$, we get that $U_s^{m,n}$ is irreducible $V_{k_0}(\mathfrak{g})$ -module. Relations (9.4) -(9.6) together with (9.2) imply that $U_s^{m,n}$ belongs to KL_{k_0} . In fact, the lowest graded component is contained in the vector space

$$\begin{aligned} \bigwedge (\mathfrak{g}^1(0) + \cdots + \mathfrak{g}^1(s-1)) \cdot \mathbf{1} \quad (s \geq 1) \\ \bigwedge (\mathfrak{g}^{-1}(0) + \cdots + \mathfrak{g}^{-1}(-s-1)) \cdot \mathbf{1} \quad (s \leq -1). \end{aligned}$$

For $s = \pm 1$ we get assertion (2). Assertion (3) follows from [36]. More precisely, for any irreducible $V_{k_0}(\mathfrak{g})$ -module (M, Y_M) from the category KL_{k_0} one can show that

$$(9.8) \quad (M_s, \tilde{Y}_M(\cdot, z)) = (M, Y_M(\Delta(\alpha^{m,n}, z) \cdot, z))$$

is also an irreducible $V_{k_0}(\mathfrak{g})$ -modules from the category KL_{k_0} (this follows from the fact that M_s is essentially obtained by applying the automorphism π_s). Then [36, Theorem 2.13] gives the fusion rules

$$M \times U_s^{m,n} = M_s.$$

In particular, this proves the fusion rules (9.7).

Remark 9.1. Let us consider the case $s = \pm 1$. Then the lowest weight component $U_s^{m,n}(0)$ of $U_s^{m,n}$ is a irreducible (sub)quotient of the Kac module $K_{m,n}^s(k_0)$ induced from the 1-dimensional $(\mathfrak{g}^0 + \mathfrak{g}^{-s})$ -module $\mathbb{C}\mathbf{1}$ with action

$$\begin{aligned} \mathfrak{g}^{-s} \cdot \mathbf{1} &= 0, \\ x \cdot \mathbf{1} &= 0 \quad (x \in \mathfrak{sl}(m) \times \mathfrak{sl}(n)), \\ \alpha^{m,n} \cdot \mathbf{1} &= -s \frac{nmk_0}{m-n} \mathbf{1}. \end{aligned}$$

As a vector space $K_{m,n}^s(k_0) \cong \bigwedge \mathfrak{g}^s$. If we take an odd coroot $\beta = e_{i,i} + e_{m+j,m+j}$, $1 \leq i \leq m$, $1 \leq j \leq n$, by direct calculation we get

$$(9.9) \quad \beta \cdot \mathbf{1} = -sk_0 \cdot \mathbf{1}.$$

This implies that $K_{m,n}^s(k_0)$ is typical iff $k_0 \notin \{-(m-1), \dots, n-1\}$.

Recall that a weight λ of a basic Lie superalgebra is said to be typical if $(\lambda + \rho)(\beta) \neq 0$ for each isotropic odd root β . To derive the above condition on k_0 , we make computations in a distinguished set of positive roots; we have

$$\rho = -\frac{s}{2} \left(\sum_{i=1}^m (m-n-2i+1)\epsilon_i + \sum_{j=1}^n (n+m-2j+1)\delta_j \right),$$

and from (9.9) we deduce that

$$(\lambda + \rho)(\beta) = -s(k_0 + m - i - j + 1),$$

which is non-zero if $k_0 \notin \{-(m-1), \dots, n-1\}$. Under this hypothesis, the lowest graded component $U_s^{m,n}(0)$ of $U_s^{m,n}$ is isomorphic to $K_{m,n}^s(k_0)$ as a \mathfrak{g} -module.

Now we specialize the previous construction to $\mathfrak{g} = sl(4|1)$ and $k_0 = -1$. So

$$\alpha = \alpha^{4,1} = \frac{1}{3}(e_{1,1} + \cdots + e_{4,4} + 4e_{5,5})_{(-1)} \mathbf{1} \in V_{-1}(sl(4|1)).$$

and

$$[\alpha_\lambda \alpha] = \frac{4}{3}\lambda, \quad U_s = U_s^{4,1} = V_{-1}(sl(4|1))^{(s\alpha)} \quad (s \in \mathbb{Z}).$$

Corollary 9.2. *We have:*

- (1) U_s , $s \in \mathbb{Z}$, are irreducible $V_{-1}(sl(4|1))$ -modules from the category KL_{-1} .
- (2) The lowest graded component of U_1 is isomorphic to $\mathbb{C}^{4|1}$ and that of U_{-1} is isomorphic to $(\mathbb{C}^{4|1})^*$.
- (3) U_s is an irreducible, simple current $V_{-1}(sl(4|1))$ -module and the following fusion rules hold:

$$U_{s_1} \times U_{s_2} = U_{s_1+s_2} \quad (s_1, s_2 \in \mathbb{Z}).$$

Proof. Proof follows from Theorem 9.1 and the fact that top component of U_1 (resp. U_{-1}) has the same highest weight as the $sl(4|1)$ -module $\mathbb{C}^{4|1}$ (resp. $(\mathbb{C}^{4|1})^*$). \square

Modules U_s are actually obtained from the vertex algebra $V_{k_0}(sl(4|1))$ by applying the spectral flow automorphism π_s of $\widehat{sl(4|1)}$ which leaves $\widehat{sl(4|1)}$ -invariant.

9.2. The decomposition for $W_k(sl(6|1), \theta)$, $k = -2$. We now consider the minimal W -algebra $W_k(\mathfrak{g}, \theta)$ for Lie superalgebra $\mathfrak{g} = sl(6|1)$ at the conformal, non-collapsing level $k = -2$. We shall prove that each $W_k(\mathfrak{g}, \theta)^{(i)}$ is a simple current $\mathcal{V}_k(\mathfrak{g}^\natural)$ -module. In order to see this, essentially it suffices to prove that $W_k(\mathfrak{g}, \theta)^{(\pm 1)}$ are the simple current modules described in previous section. Note that $\mathfrak{g}^\natural = sl(4|1) + \mathbb{C}$, and that

$$\mathcal{V}_k(\mathfrak{g}) = V_{k+1}(sl(4|1)) \otimes M_\beta\left(\frac{2h^\vee-4}{h^\vee}(k + h^\vee/2)\right),$$

where $\beta = J^{\{c\}}$, $[\beta_\lambda \beta] = \frac{2h^\vee-4}{h^\vee}(k + h^\vee/2) = \frac{3}{5}$.

By the irreducibility statement from Theorem 6.4 we see that there are weights Λ^\pm such that

$$W_k(\mathfrak{g}, \theta)^{(\pm 1)} \cong L_{sl(4|1)}(\Lambda^\pm) \otimes M_\beta\left(\frac{3}{5}, \pm 1\right).$$

The lowest graded component of $L_{sl(4|1)}(\Lambda^+)$ (resp. $L_{sl(4|1)}(\Lambda^-)$) is isomorphic as $sl(4|1)$ -module to $\mathbb{C}^{4|1}$ (resp. $(\mathbb{C}^{4|1})^*$) and it has conformal weight $h_1 = \frac{2}{3}$. By Corollary 9.2, we get that

$$W_k(\mathfrak{g}, \theta)^{(\pm 1)} \cong U_{\pm 1} \otimes M_\beta\left(\frac{3}{5}, \pm 1\right).$$

Since $U_{\pm 1}$ and $M_\beta\left(\frac{3}{5}, \pm 1\right)$ are simple current modules we get that $W_k(\mathfrak{g}, \theta)$ is a simple current extension. In this way we have proved the following result, which gives a super-analog of Corollary 8.3. (Arguments are essentially the same, only the proof that $W_k(\mathfrak{g}, \theta)^{(\pm 1)}$ are simple current modules uses different techniques).

Corollary 9.3. *Let $\mathfrak{g} = sl(6|1)$. We have the following isomorphism of $V_{-2,3/5}(\mathfrak{g}^{\natural})$ -modules:*

$$W_{-2}(\mathfrak{g}, \theta) \cong \bigoplus_{s \in \mathbb{Z}} U_s \otimes M_{\beta}(3/5, s).$$

Remark 9.2. By using similar arguments one can obtain analogous decompositions for $\mathfrak{g} = sl(n+5|n)$ and conformal level $k = -2$. For decompositions in the case of other conformal levels we need more precise fusion rules analysis. This and related questions will be discussed in our forthcoming papers.

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