

# Finite-time stability analysis and stabilization for linear discrete-time system with time-varying delay

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## Abstract

The problem of finite-time stability for linear discrete-time systems with time-varying delay is studied in this paper. In order to deal with the time delay, the original system is firstly transformed into two interconnected subsystems. By constructing a delay-dependent Lyapunov–Krasovskii functional and using a two-term approximation of the time-varying delay, sufficient conditions of finite-time stability are derived and expressed in terms of linear matrix inequalities (LMIs). The derived stability conditions can be applied into analyzing the finite-time stability and deriving the maximally tolerable delay. Compared with the existing results on finite-time stability, the derived stability conditions are less conservative. In addition, for the stabilization problem, we design the state-feedback controller. Finally, numerical examples are used to illustrate the effectiveness of the proposed method.

## 1. Introduction

The phenomenon of time delay is very common in practical engineering systems, such as biological systems, chemical systems, mechanical systems and networked control systems

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(NCSs) [1]. The existence of time delay might result to the performance deterioration or even instability of system [2–8]. What is more, in many practical systems, the time delay is not constant but time-varying. It should be noted that the stability of the system with time-varying delay is generally poorer than that of the system with a constant time delay and the same upper bound [9]. In addition, if the time-varying delay is frequently changing and the range of variation is large, the effect of time delay to the system stability will be significant. The problem of time-varying delay has drawn considerable attention during the past few decades; see the filtering problems [10–17], the controller design [18–23] and the stability analysis [24–27]. Though there is much research on time-delay systems, the existing results are only sufficient conditions [28]. Persistent works are devoted to reduce the conservatism of the conditions.

The direct Lyapunov method is a well-known approach which has been widely used to reduce to conservatism of aforementioned conditions; see [29–33] and the references therein. Recently, another effective but indirect approach named input–output (IO) approach has attracted increasing attention. In this method, the original system is transformed into two interconnected subsystems. By studying the interconnected subsystems, the obtained stability results are much less conservative [34]. This approach was firstly studied for the system with constant time delays in [35], then applied into various systems with time-varying delays; see [36–38] and the references therein. The main idea of the IO approach is to approximate the time-varying delay and make the approximation error as small as possible [34]. A new model transformation is proposed in [39], which introduces a two-term approximation method. It is proved that the approximation error of this two-term approximation method is smaller than that of the one-term ones [40], which means that the conservatism is much less.

In many practical systems, the main concern is the behavior of the system in a finite-time interval [41]. In such case, the traditional Lyapunov method is not applicable. Therefore, the finite-time stability (FTS) method is introduced. The initial condition of a system is given, then the system is said to be finite-time stable, if the state variable of system does not exceed a certain bound in a prescribed time interval [42]. Compared with the traditional Lyapunov method, the FTS method is more practical and less conservative.

The FTS approach was firstly proposed in [43]. From then on, as the development of the LMI method, FTS approach has been applied into various systems. In [44], the finite-time stability of system with second-order sliding modes is studied. The FTS method is applied into a linear singular system in [45]. In addition, various nonlinear systems have been discussed by using FTS approach; see [46–50] and the references therein.

As the time delay is unavoidable in many practical systems, the aforementioned FTS method should be used to investigate the time-delay systems; see [51] and the references therein. However, in the literature [51], the time delay is assumed to be a constant, which would make the results more conservative. Inspired by recent works on the FTS theory and two-term approximation method, we derive some new results on the analysis of finite-time stability for discrete-time systems with time-varying delay in this paper. The original system is firstly transformed into two interconnected subsystems. Then, the two-term approximation is used to approximate the time-varying delay. The stability of system is analyzed by using FTS theory. Then the state-feedback controller is designed to stabilize an instable system. Finally, a comparison is given to illustrate the effectiveness of the approach.

*Notation:* In this paper,  $R^n$  means the  $n$ -dimensional Euclidean space. The superscripts  $-1$  and  $T$  stand for the inverse and the transpose of matrix, respectively. In addition,  $\text{sym}(A)$  indicates  $(A + A^T)$  for convenience, and  $*$  is used to describe the symmetry terms in a symmetry matrix. The real matrix  $X > 0$  or  $X < 0$  respectively denote that  $X$  is positive definite or negative definite.

$\lambda_{\min}(X)$  and  $\lambda_{\max}(X)$  stand for the minimum and maximum eigenvalues of matrix  $X$ , respectively. The formula  $a \approx b$  refers that  $a$  and  $b$  are very approximate.

## 2. Problem formulation and preliminaries

Consider the following time-delay system:

$$(S) : x(k+1) = Ax(k) + A_d x(k-h(k)). \quad (1)$$

The initial condition is defined as

$$\varphi(\theta) = x(\theta), \theta \in \{-h_2, -h_2+1, \dots, 0\}, \quad (2)$$

$$\sup_{\theta \in \{-h_2, -h_2+1, \dots, -1\}} [\varphi(\theta+1) - \varphi(\theta)]^T [\varphi(\theta+1) - \varphi(\theta)] < \mu, \quad (3)$$

where  $x(k) \in R^n$  is the state variable,  $A, A_d \in R^{n \times n}$  are constant matrices,  $h(k)$  represents the time-varying delay and satisfies

$$h_1 \leq h(k) \leq h_2, \quad (4)$$

where  $h_1$  and  $h_2$  denote the lower bound and upper bound of  $h(k)$ , respectively.

In addition, system (1) can be transformed into the following interconnected subsystems:

$$(S_1) : y(k) = \mathbf{G}w(k), (S_2) : w(k) = \mathbf{\Delta}y(k). \quad (5)$$

In this paper, the purpose is to derive sufficient conditions which guarantee the finite-time stability of system (1). Before proceeding, we introduce the definition of FTS as follow.

**Definition 1.** (Finite-Time Stability [51]). The linear system in (1) is said to be finite-time stable with respect to  $(\alpha, \beta, N)$ , where  $0 < \alpha < \beta$ , if the state variables satisfy

$$x^T(k)x(k) < \beta, \forall k \in \{1, 2, \dots, N\}, \quad (6)$$

under the following initial conditions:

$$\sup_{\theta \in \{-h_2, -h_2+1, \dots, 0\}} \varphi^T(\theta)\varphi(\theta) < \alpha. \quad (7)$$

## 3. Main results

In this section, we aim to: 1) transform system (1) into two interconnected subsystems (5) by using the approach proposed in [39]; 2) derive the sufficient conditions which can guarantee system (1) finite-time stable; and 3) consider an instable system, design a state-feedback controller to stabilize the system.

### 3.1. Model transformation

This approach of model transformation has been proposed in [39], we will recall it as follows.

In order to obtain  $G$  and  $\mathbf{\Delta}$  which are proposed in (5), we express  $x(k-h(k))$  as follows:

$$x(k-h(k)) = \frac{1}{2}x(k-h_1) + \frac{1}{2}x(k-h_2) + \frac{h_{12}}{2}w_d(k), \quad (8)$$

where  $h_{12} = h_2 - h_1$  and  $(1/2)x(k-h_1)$ ,  $(1/2)x(k-h_2)$  are used to approximate  $x(k-h(k))$ , moreover,  $(h_{12}/2)w_d(k)$  represents the approximation error. Therefore, system (1) can be rewritten as

$$x(k+1) = Ax(k) + \frac{A_d}{2}x(k-h_1) + \frac{A_d}{2}x(k-h_2) + \frac{h_{12}}{2}A_d w_d(k). \quad (9)$$

By defining a new variable  $\eta(k) = x(k+1) - x(k)$ ,  $w_d(k)$  can be rewritten as

$$\begin{aligned} w_d(k) &= \frac{2}{h_{12}} \left[ x(k-h(k)) - \frac{1}{2}x(k-h_1) - \frac{1}{2}x(k-h_2) \right] \\ &= \frac{1}{h_{12}} \left[ \sum_{i=k-h_2}^{k-h(k)-1} \eta(i) - \sum_{i=k-h(k)}^{k-h_1-1} \eta(i) \right] = \frac{1}{h_{12}} \left[ \sum_{i=k-h_2}^{k-h_1-1} \delta(i)\eta(i) \right], \end{aligned} \quad (10)$$

where

$$\delta(i) = \begin{cases} 1, & i \leq k-h(k)-1, \\ -1, & i \geq k-h(k). \end{cases}$$

Therefore, subsystems  $S_1$  and  $S_2$  can be obtained as follows:

$$(S_1) : \begin{bmatrix} x(k+1) \\ \eta(k) \end{bmatrix} = \begin{bmatrix} A & \frac{1}{2}A_d & \frac{1}{2}A_d & \frac{h_{12}}{2}A_d \\ A-I & \frac{1}{2}A_d & \frac{1}{2}A_d & \frac{h_{12}}{2}A_d \end{bmatrix} \begin{bmatrix} \zeta(k) \\ w_d(k) \end{bmatrix}, \quad (11)$$

$$(S_2) : w_d(k) = \Delta \eta(k), \quad (12)$$

where  $\zeta(k) = [x^T(k) \ x^T(k-h_1) \ x^T(k-h_2)]^T$ , and subsystem (12) represents the simplification of (10).

It should be noted that the interconnected subsystems (11) and (12) are equivalent to the system (1).

### 3.2. FTS analysis and stability conditions

In this section, we aim to analyze the finite-time stability of systems (11) and (12).

**Theorem 1.** *The systems (11) and (12) are finite-time stable with respect to  $(\alpha, \beta, N)$ , where  $0 < \alpha < \beta$ , if there exist positive definite symmetric matrices  $P, Q_1, Q_2$  and  $Z$ , positive scalars  $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$  and  $\gamma \geq 1$ , such that the following inequalities hold:*

$$\begin{bmatrix} -2\gamma P + Q_1 + Q_2 & 0 & 0 & 0 & 2h_{12}(A-I)^T Z & 2A^T P \\ * & -Q_1 & 0 & 0 & h_{12}A_d^T Z & A_d^T P \\ * & * & -Q_2 & 0 & h_{12}A_d^T Z & A_d^T P \\ * & * & * & -2h_{12}^2 Z & h_{12}^2 A_d^T Z & h_{12}A_d^T P \\ * & * & * & * & -2Z & 0 \\ * & * & * & * & * & -2P \end{bmatrix} < 0, \quad (13)$$

$$\begin{bmatrix} -\gamma^{-N}\beta\theta_1 & \sqrt{\alpha}\theta_2 & \sqrt{\alpha h_1}\theta_3 & \sqrt{\alpha h_2}\theta_4 & \sqrt{\varepsilon}\theta_5 \\ * & -\theta_2 & 0 & 0 & 0 \\ * & * & -\theta_3 & 0 & 0 \\ * & * & * & -\theta_4 & 0 \\ * & * & * & * & -\theta_5 \end{bmatrix} < 0, \quad (14)$$

$$\theta_1 I < P < \theta_2 I, \quad 0 < Q_1 < \theta_3 I, \quad 0 < Q_2 < \theta_4 I, \quad 0 < Z < \theta_5 I, \quad \varepsilon = [\mu h_{12}^2 (h_1 + h_2 + 1)]/2. \quad (15)$$

**Proof.** Choose the Lyapunov–Krasovskii functional candidates as follows:

$$\begin{aligned} V(k) &= V_1(k) + V_2(k) + V_3(k), \\ V_1(k) &= x^T(k)Px(k), \\ V_2(k) &= \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-h_2}^{k-1} x^T(i)Q_2x(i), \\ V_3(k) &= h_{12} \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \eta^T(j)Z\eta(j). \end{aligned} \quad (16)$$

Then, the difference of the above Lyapunov function is described by

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &= \left[ Ax(k) + \frac{1}{2}A_d x(k-h_1) + \frac{1}{2}A_d x(k-h_2) + \frac{h_{12}}{2}A_d w_d(k) \right]^T \\ &\quad \times P \left[ Ax(k) + \frac{1}{2}A_d x(k-h_1) + \frac{1}{2}A_d x(k-h_2) + \frac{h_{12}}{2}A_d w_d(k) \right] - x^T(k)Px(k) \\ &= x^T(k)(A^T P A - P)x(k) + \frac{1}{4}x^T(k-h_1)A_d^T P A_d x(k-h_1) \\ &\quad + \frac{1}{4}x^T(k-h_2)A_d^T P A_d x(k-h_2) + \frac{h_{12}^2}{4}w_d^T(k)A_d^T P A_d w_d(k) \\ &\quad + \text{sym} \left[ \frac{1}{2}x^T(k)A^T P A_d x(k-h_1) + \frac{1}{2}x^T(k)A^T P A_d x(k-h_2) + \frac{h_{12}}{2}x^T(k)A^T P A_d w_d(k) \right. \\ &\quad \left. + \frac{1}{4}x^T(k-h_1)A_d^T P A_d x(k-h_2) \right. \\ &\quad \left. + \frac{h_{12}}{4}x^T(k-h_1)A_d^T P A_d w_d(k) + \frac{h_{12}}{4}x^T(k-h_2)A_d^T P A_d w_d(k) \right], \end{aligned} \quad (17)$$

$$\begin{aligned} \Delta V_2(k) &= \sum_{i=k-h_1+1}^k x^T(i)Q_1x(i) - \sum_{i=k-h_1}^{k-1} x^T(i)Q_1x(i) + \sum_{i=k-h_2+1}^k x^T(i)Q_2x(i) + \sum_{i=k-h_2}^{k-1} x^T(i)Q_2x(i) \\ &= x^T(k)Q_1x(k) - x^T(k-h_1)Q_1x(k-h_1) + x^T(k)Q_2x(k) - x^T(k-h_2)Q_2x(k-h_2), \end{aligned} \quad (18)$$

$$\begin{aligned} \Delta V_3(k) &= h_{12} \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i+1}^k \eta^T(j)Z\eta(j) - h_{12} \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \eta^T(j)Z\eta(j) \\ &= h_{12} \sum_{i=-h_2}^{-h_1-1} [\eta^T(k)Z\eta(k) - \eta^T(k+i)Z\eta(k+i)] \end{aligned}$$

$$\begin{aligned}
&= h_{12}^2 \eta^T(k) Z \eta(k) - h_{12} \sum_{i=-h_2}^{-h_1-1} \eta^T(k+i) Z \eta(k+i) \delta^2(i) = h_{12}^2 \eta^T(k) Z \eta(k) \\
&\quad - h_{12} \sum_{i=k-h_2}^{k-h_1-1} [\delta(i) \eta(i)]^T Z [\delta(i) \eta(i)]. \tag{19}
\end{aligned}$$

By recalling the system (12) and using Jensen's inequality [52], the inequality (19) can be regarded as

$$\begin{aligned}
\Delta V_3(k) &\leq h_{12}^2 \eta^T(k) Z \eta(k) - \left[ \sum_{i=k-h_2}^{k-h_1-1} \delta(i) \eta(i) \right]^T Z \left[ \sum_{i=k-h_2}^{k-h_1-1} [\delta(i) \eta(i)] \right] \\
&\leq h_{12}^2 \eta^T(k) Z \eta(k) - h_{12}^2 w_d^T(k) Z w_d(k). \tag{20}
\end{aligned}$$

Therefore, by using system (11), the above inequality can be rewritten as

$$\begin{aligned}
\Delta V_3(k) &\leq x^T(k) (A-I)^T h_{12}^2 Z (A-I) x(k) + x^T(k-h_1) A_d^T \frac{h_{12}^2}{4} Z A_d x(k-h_1) \\
&\quad + x^T(k-h_2) A_d^T \frac{h_{12}^2}{4} Z A_d x(k-h_2) \\
&\quad + w_d^T(k) \left( A_d^T \frac{h_{12}^4}{4} Z A_d - h_{12}^2 Z \right) w_d(k) + \text{sym} \left[ x^T(k) (A-I)^T \frac{h_{12}^2}{2} Z A_d x(k-h_1) \right. \\
&\quad + x^T(k) (A-I)^T \frac{h_{12}^2}{2} Z A_d x(k-h_2) \\
&\quad + x^T(k) (A-I)^T \frac{h_{12}^3}{2} Z A_d w_d(k) + x^T(k-h_1) A_d^T \frac{h_{12}^2}{4} Z A_d x(k-h_2) \\
&\quad \left. + x^T(k-h_1) A_d^T \frac{h_{12}^3}{4} Z A_d w_d(k) + x^T(k-h_2) A_d^T \frac{h_{12}^3}{4} Z A_d w_d(k) \right]. \tag{21}
\end{aligned}$$

Hence, it is concluded that

$$\begin{aligned}
\Delta V(k) &\leq x^T(k) \left[ A^T P A - P + Q_1 + Q_2 + h_{12}^2 (A-I)^T Z (A-I) \right] x(k) \\
&\quad + x^T(k-h_1) \left[ \frac{1}{4} A_d^T P A_d + \frac{h_{12}^2}{4} A_d^T Z A_d - Q_1 \right] x(k-h_1) \\
&\quad + x^T(k-h_2) \left[ \frac{1}{4} A_d^T P A_d + \frac{h_{12}^2}{4} A_d^T Z A_d - Q_2 \right] x(k-h_2) \\
&\quad + w_d^T(k) \left[ \frac{h_{12}^2}{4} A_d^T P A_d + \frac{h_{12}^4}{4} A_d^T Z A_d - h_{12}^2 Z \right] w_d(k) \\
&\quad + \text{sym} \left\{ x^T(k) \left[ \frac{1}{2} A^T P A_d + \frac{h_{12}^2}{2} (A-I)^T Z A_d \right] x(k-h_1) \right. \\
&\quad + x^T(k) \left[ \frac{1}{2} A^T P A_d + \frac{h_{12}^2}{2} (A-I)^T Z A_d \right] x(k-h_2) \\
&\quad + x^T(k) \left[ \frac{h_{12}}{2} A^T P A_d + \frac{h_{12}^3}{2} (A-I)^T Z A_d \right] w_d(k) \\
&\quad \left. + x^T(k-h_1) \left[ \frac{1}{4} A_d^T P A_d + \frac{h_{12}^2}{4} A_d^T Z A_d \right] x(k-h_2) \right\}
\end{aligned}$$

$$\begin{aligned}
& +x^T(k-h_1) \left[ \frac{h_{12}}{4} A_d^T P A_d + \frac{h_{12}^3}{4} A_d^T Z A_d \right] w_d(k) \\
& +x^T(k-h_2) \left[ \frac{h_{12}}{4} A_d^T P A_d + \frac{h_{12}^3}{4} A_d^T Z A_d \right] w_d(k) \Big\}, \tag{22}
\end{aligned}$$

i.e.,

$$\Delta V(k) \leq \xi^T(k) \Omega \xi(k), \tag{23}$$

where

$$\begin{aligned}
\xi(k) &= \begin{bmatrix} x^T(k) & x^T(k-h_1) & x^T(k-h_1) & w_d^T(k) \end{bmatrix}^T, \\
\Omega &= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix}, \\
\Omega_{11} &= A^T P A - P + Q_1 + Q_2 + h_{12}^2 (A-I)^T Z (A-I), \Omega_{12} = \Omega_{13} = \frac{1}{2} A^T P A_d \\
&+ \frac{h_{12}^2}{2} (A-I)^T Z A_d, \\
\Omega_{14} &= \frac{h_{12}}{2} A^T P A_d + \frac{h_{12}^3}{2} (A-I)^T Z A_d, \Omega_{22} = \frac{1}{4} A_d^T P A_d + \frac{h_{12}^2}{4} A_d^T Z A_d - Q_1, \\
\Omega_{23} &= \frac{1}{4} A_d^T P A_d + \frac{h_{12}^2}{4} A_d^T Z A_d, \\
\Omega_{24} &= \Omega_{34} = \frac{h_{12}}{4} A_d^T P A_d + \frac{h_{12}^3}{4} A_d^T Z A_d, \Omega_{33} = \frac{1}{4} A_d^T P A_d + \frac{h_{12}^2}{4} A_d^T Z A_d - Q_2, \\
\Omega_{44} &= \frac{h_{12}^2}{4} A_d^T P A_d + \frac{h_{12}^4}{4} A_d^T Z A_d - h_{12}^2 Z. \tag{24}
\end{aligned}$$

In addition, it is assumed that

$$\Pi = \Omega - \begin{bmatrix} (\gamma-1)P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & \Pi_{34} \\ * & * & * & \Pi_{44} \end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned}
\Pi_{11} &= A^T P A - \gamma P + Q_1 + Q_2 + h_{12}^2 (A-I)^T Z (A-I), \Pi_{12} = \Omega_{12}, \Pi_{13} = \Omega_{13}, \\
\Pi_{14} &= \Omega_{14}, \Pi_{22} = \Omega_{22}, \Pi_{23} = \Omega_{23}, \Pi_{24} = \Omega_{24}, \Pi_{33} = \Omega_{33}, \Pi_{34} = \Omega_{34}, \Pi_{44} = \Omega_{44}.
\end{aligned}$$

It should be noted that the matrix variable  $P$  is coupled with  $A$  and  $A_d$  in inequality (25). Therefore, in order to decouple the triple terms, inequality (25) is rewritten as

$$\Pi = \begin{bmatrix} A^T P A - \gamma P + Q_1 + Q_2 & \frac{1}{2} A^T P A_d & \frac{1}{2} A^T P A_d & \frac{h_{12}}{2} A^T P A_d \\ * & -Q_1 + \frac{1}{4} A_d^T P A_d & \frac{1}{4} A_d^T P A_d & \frac{h_{12}}{4} A_d^T P A_d \\ * & * & -Q_2 + \frac{1}{4} A_d^T P A_d & \frac{h_{12}}{4} A_d^T P A_d \\ * & * & * & -h_{12}^2 Z + \frac{h_{12}^2}{4} A_d^T P A_d \end{bmatrix}$$

$$\begin{aligned}
& + \begin{bmatrix} h_{12}^2(A-I)^T Z(A-I) & \frac{h_{12}^2}{2}(A-I)^T Z A_d & \frac{h_{12}^2}{2}(A-I)^T Z A_d & \frac{h_{12}^3}{2}(A-I)^T Z A_d \\ * & \frac{h_{12}^2}{4} A_d^T Z A_d & \frac{h_{12}^2}{4} A_d^T Z A_d & \frac{h_{12}^3}{4} A_d^T Z A_d \\ * & * & \frac{h_{12}^2}{4} A_d^T Z A_d & \frac{h_{12}^3}{4} A_d^T Z A_d \\ * & * & * & \frac{h_{12}^4}{4} A_d^T Z A_d \end{bmatrix} \\
= & \begin{bmatrix} A^T P A - \gamma P + Q_1 + Q_2 & \frac{1}{2} A^T P A_d & \frac{1}{2} A^T P A_d & \frac{h_{12}}{2} A^T P A_d \\ * & -Q_1 + \frac{1}{4} A_d^T P A_d & \frac{1}{4} A_d^T P A_d & \frac{h_{12}}{4} A_d^T P A_d \\ * & * & -Q_2 + \frac{1}{4} A_d^T P A_d & \frac{h_{12}}{4} A_d^T P A_d \\ * & * & * & -h_{12}^2 Z + \frac{h_{12}^2}{4} A_d^T P A_d \end{bmatrix} \\
- & \begin{bmatrix} h_{12}(A-I)^T Z \\ \frac{h_{12}}{2} A_d^T Z \\ \frac{h_{12}}{2} A_d^T Z \\ \frac{h_{12}^2}{2} A_d^T Z \end{bmatrix} (-Z)^{-1} \begin{bmatrix} h_{12}(A-I)^T Z \\ \frac{h_{12}}{2} A_d^T Z \\ \frac{h_{12}}{2} A_d^T Z \\ \frac{h_{12}^2}{2} A_d^T Z \end{bmatrix}^T < 0. \tag{26}
\end{aligned}$$

By applying Schur complement lemma, the inequality (26) is equivalent to

$$\begin{bmatrix} A^T P A - \gamma P + Q_1 + Q_2 & \frac{1}{2} A^T P A_d & \frac{1}{2} A^T P A_d & \frac{h_{12}}{2} A^T P A_d & h_{12}(A-I)^T Z \\ * & -Q_1 + \frac{1}{4} A_d^T P A_d & \frac{1}{4} A_d^T P A_d & \frac{h_{12}}{4} A_d^T P A_d & \frac{h_{12}}{2} A_d^T Z \\ * & * & -Q_2 + \frac{1}{4} A_d^T P A_d & \frac{h_{12}}{4} A_d^T P A_d & \frac{h_{12}}{2} A_d^T Z \\ * & * & * & -h_{12}^2 Z + \frac{h_{12}^2}{4} A_d^T P A_d & \frac{h_{12}^2}{2} A_d^T Z \\ * & * & * & * & -Z \end{bmatrix} < 0. \tag{27}$$

Then, by using Schur complement lemma one more time, the inequality (27) is equivalent to:

$$\begin{bmatrix} -\gamma P + Q_1 + Q_2 & 0 & 0 & 0 & h_{12}(A-I)^T Z & A^T P \\ * & -Q_1 & 0 & 0 & \frac{h_{12}}{2} A_d^T Z & \frac{1}{2} A_d^T P \\ * & * & -Q_2 & 0 & \frac{h_{12}}{2} A_d^T Z & \frac{1}{2} A_d^T P \\ * & * & * & -h_{12}^2 Z & \frac{h_{12}^2}{2} A_d^T Z & \frac{h_{12}}{2} A_d^T P \\ * & * & * & * & -Z & 0 \\ * & * & * & * & * & -P \end{bmatrix} < 0. \tag{28}$$

By introducing two substitutions  $P \leftrightarrow 2P$  and  $Z \leftrightarrow 2Z$ , the inequality (28) can be regarded as inequality (13). Hence, if the inequality (13) holds, it concludes that  $\Pi < 0$ . Then, we have

$$\begin{aligned}
\Delta V(k) & \leq \xi^T(k) \Omega \xi(k) = \xi^T(k) \left\{ \Pi + \begin{bmatrix} (\gamma-1)P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \xi(k) < \xi^T(k) \begin{bmatrix} (\gamma-1)P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\xi(k) & = (\gamma-1)x^T(k) P x(k) < (\gamma-1)V(k), \tag{29}
\end{aligned}$$



i.e.,

$$V(k) < \gamma V(k-1). \quad (30)$$

Therefore, it infers that

$$V(k) < \gamma V(k-1) < \gamma^2 V(k-2) < \dots < \gamma^k V(0). \quad (31)$$

Furthermore, the initial value of Lyapunov function can be described by

$$\begin{aligned} V(0) &= x^T(0)Px(0) + \sum_{i=-h_1}^{-1} x^T(i)Q_1x(i) + \sum_{i=-h_2}^{-1} x^T(i)Q_2x(i) + h_{12} \sum_{i=-h_2}^{-h_1-1} \sum_{j=i}^{-1} \eta^T(j)Z\eta(j) \\ &< \lambda_{\max}(P)x^T(0)x(0) + \lambda_{\max}(Q_1) \sum_{i=-h_1}^{-1} x^T(i)x(i) \\ &\quad + \lambda_{\max}(Q_2) \sum_{i=-h_2}^{-1} x^T(i)x(i) + h_{12}\lambda_{\max}(Z) \sum_{i=-h_2}^{-h_1-1} \sum_{j=i}^{-1} \eta^T(j)\eta(j). \end{aligned} \quad (32)$$

Then, by recalling the initial conditions in Eqs. (3) and (7), inequality (32) can be regarded as

$$\begin{aligned} V(0) &< \lambda_{\max}(P)\alpha + \lambda_{\max}(Q_1) \sum_{i=-h_1}^{-1} \alpha + \lambda_{\max}(Q_2) \sum_{i=-h_2}^{-1} \alpha + h_{12}\lambda_{\max}(Z) \sum_{i=-h_2}^{-h_1-1} \sum_{j=i}^{-1} \mu \\ &= \lambda_{\max}(P)\alpha + \lambda_{\max}(Q_1)h_1\alpha + \lambda_{\max}(Q_2)h_2\alpha + \lambda_{\max}(Z)[\mu h_{12}^2(h_1 + h_2 + 1)]/2. \end{aligned} \quad (33)$$

By applying Schur complement lemma, the inequality (14) is equivalent to

$$\begin{aligned} -\gamma^{-N}\beta\theta_1 - \begin{bmatrix} \sqrt{\alpha}\theta_2 & \sqrt{\alpha h_1}\theta_3 & \sqrt{\alpha h_2}\theta_4 & \sqrt{\varepsilon}\theta_5 \end{bmatrix} \begin{bmatrix} -\theta_2^{-1} & 0 & 0 & 0 \\ 0 & -\theta_5^{-1} & 0 & 0 \\ 0 & 0 & -\theta_4^{-1} & 0 \\ 0 & 0 & 0 & -\theta_5^{-1} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha}\theta_2 \\ \sqrt{\alpha h_1}\theta_3 \\ \sqrt{\alpha h_2}\theta_4 \\ \sqrt{\varepsilon}\theta_5 \end{bmatrix} \\ = -\gamma^{-N}\beta\theta_1 + \alpha\theta_2 + \alpha h_1\theta_3 + \alpha h_2\theta_4 + \varepsilon\theta_5 < 0. \end{aligned} \quad (34)$$

Therefore, via inequalities in Eqs. (15) and (34), the inequality (33) can be regarded as

$$V(0) < \alpha\theta_2 + \alpha h_1\theta_3 + \alpha h_2\theta_4 + \varepsilon\theta_5 < \gamma^{-N}\beta\theta_1 < \gamma^{-N}\beta\lambda_{\min}(P). \quad (35)$$

In addition, by combining inequalities (31) and (35), we can obtain

$$\begin{aligned} \lambda_{\min}(P)x^T(k)x(k) &< x^T(k)Px(k) < V(k) < \gamma^k V(0) \leq \gamma^N V(0) < \gamma^N [\gamma^{-N}\beta\lambda_{\min}(P)] \\ &= \beta\lambda_{\min}(P), \quad k \in \{1, 2, \dots, N\}. \end{aligned} \quad (36)$$

Hence, it is concluded that

$$x^T(k)x(k) < \beta, \quad \forall k \in \{1, 2, \dots, N\}. \quad (37)$$

i.e., the system composed of Eqs. (11) and (12) is finite-time stable, which means that system (1) is finite-time stable. This completes the proof.  $\square$

**Remark 1.** The inequalities (13)–(15) are LMIs, which can be easily calculated by the LMI Toolbox in MATLAB. In addition, the results are related to the lower and upper bounds of time delay.

**Remark 2.** It should be noted that the inequalities  $\theta_1 I < P < \theta_2 I$ ,  $Q_1 < \theta_3 I$ ,  $Q_2 < \theta_4 I$  and  $Z < \theta_5 I$  in Eq. (15) are conservative. The approach to reduce the conservativeness is to minimize the positive scalar  $\gamma_{\min}$ . Then, it concludes that  $\lambda_{\min}(P) \approx \theta_1$ ,  $\lambda_{\max}(P) \approx \theta_2$ ,  $\lambda_{\max}(Q_1) \approx \theta_3$ ,  $\lambda_{\max}(Q_2) \approx \theta_4$ ,  $\lambda_{\max}(Z) \approx \theta_5$ , which can make the inequalities in Eq. (15) less conservative.

**Remark 3.** In inequalities (13) and (14), the upper bound  $h_2$  of time delay is not given, which is needed to be calculated. The computation method for  $h_2$  is described by the following steps. Choose an initial value for  $h_2$ , which should be larger than  $h_1$ . Then amplify the value of  $h_2$ , until the LMIs having infeasible solutions. For example, if the LMIs have infeasible solutions for  $h_2 = 11$ , while have feasible solutions for  $h_2 = 10$ , then we choose the upper bound of time delay as  $h_2 = 10$ .

In above analysis, it is assumed that the time delay is time-varying. On the other hand, if the time delay is constant, we will obtain the following corollary.

**Corollary 1.** *The system in Eq. (1) with  $h(k) = h$  is finite-time stable with respect to  $(\alpha, \beta, N)$ , where  $0 < \alpha < \beta$ , if there exist positive definite symmetric matrices  $P$  and  $Q$ , positive scalars  $\theta_1, \theta_2, \theta_3$  and  $\gamma \geq 1$ , such that the following inequalities hold:*

$$\begin{bmatrix} -\gamma P + Q & 0 & A^T P \\ * & -Q & A_d^T P \\ * & * & -P \end{bmatrix} < 0, \quad (38)$$

$$\begin{bmatrix} -\gamma^{-N} \beta \theta_1 & \sqrt{\alpha} \theta_2 & \sqrt{\alpha h} \theta_3 \\ * & -\theta_2 & 0 \\ * & * & -\theta_3 \end{bmatrix} < 0, \quad (39)$$

$$\theta_1 I < P < \theta_2 I, \quad 0 < Q < \theta_3 I. \quad (40)$$

**Proof.** Consider the Lyapunov–Krasovskii functional candidates as follows:

$$\begin{aligned} V(k) &= V_1(k) + V_2(k), \\ V_1(k) &= x^T(k) P x(k), \\ V_2(k) &= \sum_{i=k-h}^{k-1} x^T(i) Q x(i). \end{aligned} \quad (41)$$

Since  $h(k) = h$ , we have

$$x(k+1) = Ax(k) + A_d x(k-h). \quad (42)$$

With similar steps in Theorem 1, inequalities (38)–(40) can be easily obtained. This completes the proof.  $\square$

### 3.3. Controller design

In this section, a state-feedback controller is designed, which can make the instable system finite-time stable.

The system is given by

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-h(k)) + Bu(k), \\ u(k) &= Kx(k). \end{aligned} \quad (43)$$

**Corollary 2.** *The closed-loop system in Eq. (43) is finite-time stable with respect to  $(\alpha, \beta, N)$ , where  $0 < \alpha < \beta$ , if there exist matrices  $W$  and  $H_6$ , positive definite symmetric matrices  $M, N, U$ ,*

$V, H_1, H_2, H_3, H_4, H_5$  and positive scalar  $\gamma \geq 1$ , such that the following inequalities hold:

$$\begin{bmatrix} -2\gamma M + U + V & 0 & 0 & 0 & 2h_{12}[M(A-I)^T + W^T B^T] & 2(MA^T + W^T B^T) \\ * & -U & 0 & 0 & h_{12}MA_d^T & MA_d^T \\ * & * & -V & 0 & h_{12}MA_d^T & MA_d^T \\ * & * & * & -2h_{12}^2 N & h_{12}^2 NA_d^T & h_{12}NA_d^T \\ * & * & * & * & -2N & 0 \\ * & * & * & * & * & -2M \end{bmatrix} < 0, \quad (44)$$

$$\begin{bmatrix} -\gamma^{-N}\beta H_1 & \sqrt{\alpha}H_2 & \sqrt{\alpha h_1}H_3 & \sqrt{\alpha h_2}H_4 & \sqrt{\varepsilon}H_6 \\ * & -H_2 & 0 & 0 & 0 \\ * & * & -H_3 & 0 & 0 \\ * & * & * & -H_4 & 0 \\ * & * & * & * & -H_5 \end{bmatrix} < 0, \quad (45)$$

$$H_1 < M < H_2, 0 < U < H_3, 0 < V < H_4, 0 < N < H_5, H_6 > 0,$$

$$\varepsilon = [\mu h_{12}^2(h_1 + h_2 + 1)]/2. \quad (46)$$

Further, if the LMIs in Eqs. (44)–(46) have feasible solutions, the control gain matrix is given by

$$K = WM^{-1}. \quad (47)$$

**Proof.** The closed-loop system (43) can be rewritten as

$$x(k+1) = (A + BK)x(k) + A_d x(k-h(k)). \quad (48)$$

Then, by applying Theorem 1, we can obtain

$$\begin{bmatrix} -2\gamma P + Q_1 + Q_2 & 0 & 0 & 0 & 2h_{12}(A_K - I)^T Z & 2A_K^T P \\ * & -Q_1 & 0 & 0 & h_{12}A_d^T Z & A_d^T P \\ * & * & -Q_2 & 0 & h_{12}A_d^T Z & A_d^T P \\ * & * & * & -2h_{12}^2 Z & h_{12}^2 A_d^T Z & h_{12}A_d^T P \\ * & * & * & * & -2Z & 0 \\ * & * & * & * & * & -2P \end{bmatrix} < 0, \quad (49)$$

where

$$A_K = A + BK.$$

It should be noted that the inequality (49) contains matrix coupled terms which make the inequality (49) non-LMI. In order to convert it into an LMI, perform a Schur complement to the inequality (49) with

$$\text{diag}\{P^{-1}, P^{-1}, P^{-1}, Z^{-1}, Z^{-1}, P^{-1}\}.$$

Then we have

$$\begin{bmatrix}
 -2\gamma P^{-1} + P^{-1}Q_1P^{-1} + P^{-1}Q_2P^{-1} & 0 & 0 & 0 & 2h_{12}[P^{-1}(A-I)^T + P^{-1}K^TB^T] & 2(P^{-1}A^T + P^{-1}K^TB^T) \\
 * & -P^{-1}Q_1P^{-1} & 0 & 0 & h_{12}P^{-1}A_d^T & P^{-1}A_d^T \\
 * & * & -P^{-1}Q_1P^{-1} & 0 & h_{12}P^{-1}A_d^T & P^{-1}A_d^T \\
 * & * & * & -2h_{12}^2Z^{-1} & h_{12}^2Z^{-1}A_d^T & h_{12}Z^{-1}A_d^T \\
 * & * & * & * & -2Z^{-1} & 0 \\
 * & * & * & * & * & -2P^{-1}
 \end{bmatrix} < 0. \tag{50}$$

In addition, new matrix variables are defined as follows:

$$M = P^{-1}, N = Z^{-1}, U = P^{-1}Q_1P^{-1}, V = P^{-1}Q_2P^{-1}, W = KP^{-1}.$$

Therefore, the inequality Eq. (50) is equivalent to Eq. (44).

The inequality Eq. (14) can be regarded as

$$\begin{bmatrix} -\gamma^{-N}\beta\theta_1I & \sqrt{\alpha}\theta_2I & \sqrt{ah_1}\theta_3I & \sqrt{ah_2}\theta_4I & \sqrt{\varepsilon}\theta_5I \\ * & -\theta_2I & 0 & 0 & 0 \\ * & * & -\theta_3I & 0 & 0 \\ * & * & * & -\theta_4I & 0 \\ * & * & * & * & -\theta_5I \end{bmatrix} < 0. \quad (51)$$

Again, performing a Schur compliment to the inequality (51) with the matrix  $\text{diag}\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, Z^{-1}\}$  gives

$$\begin{bmatrix} -\gamma^{-N}\beta P^{-1}(\theta_1I)P^{-1} & \sqrt{\alpha}P^{-1}(\theta_2I)P^{-1} & \sqrt{ah_1}P^{-1}(\theta_3I)P^{-1} & \sqrt{ah_2}P^{-1}(\theta_4I)P^{-1} & \sqrt{\varepsilon}P^{-1}(\theta_5I)Z^{-1} \\ * & -P^{-1}(\theta_2I)P^{-1} & 0 & 0 & 0 \\ * & * & -P^{-1}(\theta_3I)P^{-1} & 0 & 0 \\ * & * & * & -P^{-1}(\theta_4I)P^{-1} & 0 \\ * & * & * & * & -Z^{-1}(\theta_5I)Z^{-1} \end{bmatrix} < 0. \quad (52)$$

In addition, some new matrix variables are defined as follows:

$$H_1 = P^{-1}(\theta_1I)P^{-1}, H_2 = P^{-1}(\theta_2I)P^{-1}, H_3 = P^{-1}(\theta_3I)P^{-1}, \\ H_4 = P^{-1}(\theta_4I)P^{-1}, H_5 = Z^{-1}(\theta_5I)Z^{-1}, H_6 = P^{-1}(\theta_5I)Z^{-1}.$$

Hence, inequalities (52) and (15) are equivalent to Eqs. (45) and (46), respectively.

In consequence, via [Theorem 1](#), the closed-loop system (43) is finite-time stable with respect to  $(\alpha, \beta, N)$ , if inequalities (44)–(46) hold. In addition, the control gain matrix  $K$  is given by

$$K = WM^{-1}.$$

This completes the proof.  $\square$

#### 4. Numerical example

In this section, two examples are provided to demonstrate the effectiveness and less conservatism of the proposed method in this paper.

**Example 1.** Consider the following time-delay system which has been used in [51]:

$$x(k+1) = \begin{bmatrix} 0.60 & 0.00 \\ 0.35 & 0.70 \end{bmatrix} x(k) + \begin{bmatrix} 0.10 & 0.00 \\ 0.20 & 0.10 \end{bmatrix} x(k-h(k)). \quad (53)$$

- i) Firstly, the time delay is assumed to be time-varying. In addition, we choose  $(\alpha, \beta, N) = (2.1, 80, 100)$ ,  $\mu = 1.1$  and  $h_1 = 2$ . Then, via [Theorem 1](#), by choosing the scalar  $\gamma = 1.000012$ , matrix variables in LMIs (13)–(15) and the upper bound  $h_2$  of time-varying delay can be

calculated as follows:

$$P = \begin{bmatrix} 656.848 & -58.714 \\ -58.714 & 126.553 \end{bmatrix}, Q_1 = \begin{bmatrix} 98.780 & 14.534 \\ 14.534 & 11.013 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 69.322 & 14.534 \\ 14.534 & 8.916 \end{bmatrix},$$

$$Z = \begin{bmatrix} 12.768 & 3.242 \\ 3.242 & 1.731 \end{bmatrix}, \sup(h_2) = 10,$$

$$\theta_1 = 120.107, \theta_2 = 664.130, \theta_3 = 101.557, \theta_4 = 72.724, \theta_5 = 13.654.$$

Therefore, the corresponding eigenvalues of the above matrices can be calculated as follows:

$$\lambda(P) = \{120.130, 663.262\}, \lambda(Q_1) = \{8.669, 101.124\},$$

$$\lambda(Q_2) = \{5.601, 72.637\}, \lambda(Z) = \{0.849, 13.649\}.$$

Moreover, it can be obviously seen that

$$\theta_1 < \lambda_{\min}(P), \theta_2 > \lambda_{\max}(P), \theta_3 > \lambda_{\max}(Q_1), \theta_4 > \lambda_{\max}(Q_2), \theta_5 > \lambda_{\max}(Z).$$

Hence, it concludes that the system (53) with time-varying delay  $2 \leq h(k) \leq 10$  is finite-time stable with respect to (2.1, 80, 100).

In order to show the trajectory of the state variable, we choose initial values as follows:

$$\begin{aligned} \varphi(\theta)_{\theta \in \{-10, -9, \dots, 0\}} &= [\varphi(-10), \varphi(-9), \dots, \varphi(0)] \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}. \end{aligned} \quad (54)$$

In addition, it can be seen that the initial values satisfy the following conditions:

$$\begin{aligned} \sup_{\theta \in \{-10, -9, \dots, 0\}} \varphi^T(\theta)\varphi(\theta) &< \alpha = 2.1, \\ \sup_{\theta \in \{-10, -9, \dots, -1\}} [\varphi(\theta+1) - \varphi(\theta)]^T [\varphi(\theta+1) - \varphi(\theta)] &< \mu = 1.1. \end{aligned}$$

The curves of state variables and the random time delays are shown in Figs. 1 and 2, respectively.

Hence, it can be seen that, under the initial conditions in Eq. (54), the state responses of system satisfy the following condition, which infers that the system is finite-time stable with respect to (2.1, 80, 100):

$$x^T(k)x(k) < \beta = 80, \forall k \in \{1, 2, \dots, 100\}.$$

- ii) In order to compare the results in this paper with those in [51], we assume that the time delay is constant. In addition, we choose  $(\alpha, \beta, N) = (2.1, 20, 140)$ . Therefore, based on Corollary 1, by choosing the scalar  $\gamma = 1.00024$ , the matrix variables in LMIs Eqs. (38)–(40) and the maximum of time delay are calculated as follows:

$$P = \begin{bmatrix} 61.484 & -3.217 \\ -3.217 & 25.972 \end{bmatrix}, Q = \begin{bmatrix} 5.150 & 1.434 \\ 1.434 & 1.149 \end{bmatrix},$$

$$\sup(h) = 29, \theta_1 = 25.350, \theta_2 = 64.628, \theta_3 = 5.718.$$

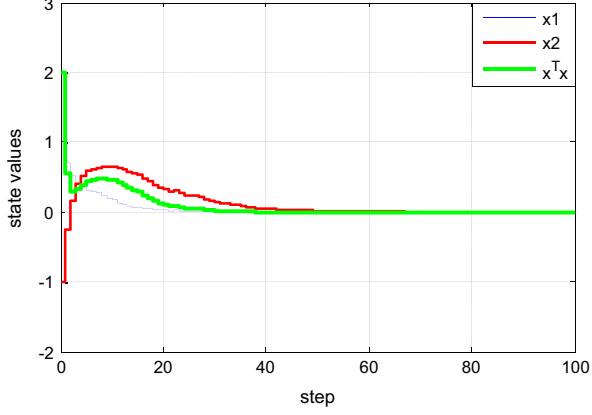


Fig. 1. State responses of the system with time-varying delay  $2 \leq h(k) \leq 10$ .

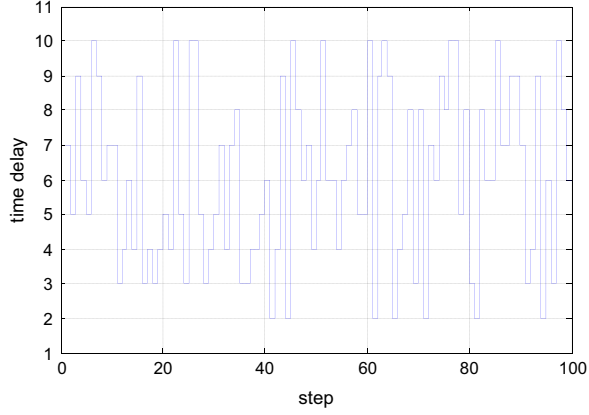


Fig. 2. Randomized time-varying delay  $2 \leq h(k) \leq 10$ .

Hence, the corresponding eigenvalues of the aforementioned matrices can be calculated as follows:

$$\lambda(P) = \{25.682, 61.773\}, \lambda(Q) = \{0.688, 5.611\}.$$

Therefore, it can be seen that

$$\theta_1 < \lambda_{\min}(P), \theta_2 > \lambda_{\max}(P), \theta_3 > \lambda_{\max}(Q).$$

In consequence, based on [Corollary 1](#), system (53) with the constant time delay  $h(k) = h = 29$  is finite time stable with respect to  $(2.1, 20, 100)$ .

In order to show the trajectory of the state variable, the initial conditions are given as follows:

$$\begin{aligned} \varphi(\theta)_{\theta \in \{-29, -28, \dots, 0\}} &= [\varphi(-29), \varphi(-28), \dots, \varphi(0)] \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & -1 \end{bmatrix}. \end{aligned} \quad (55)$$

Fig. 3 shows the curves of state variables under the initial conditions in Eq. (55).

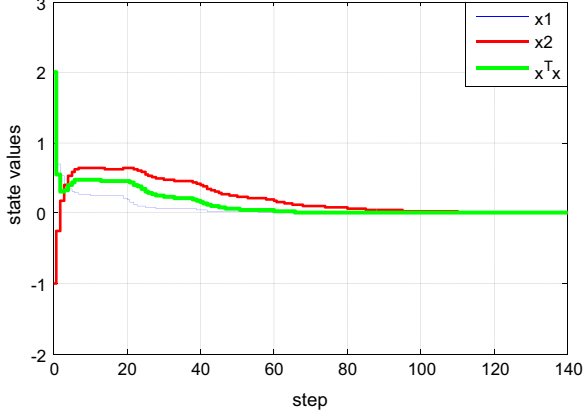


Fig. 3. State responses of the system with constant time delay  $h(k) = 29$ .

Table 1

Upper bound of time delay in [51] and this paper.

	$h_1 \leq h(k) \leq h_2$	$h(k) = h$
Upper bound of time delay in [51]	$2 \leq h(k) \leq 6$	$h(k) = h = 11$
Upper bound of time delay in this paper	$2 \leq h(k) \leq 10$	$h(k) = h = 29$

It infers from Fig. 3 that, under the initial conditions in Eq. (55), the state responses satisfy the following condition, which implies that the system is finite-time stable with respect to  $(2.1, 20, 140)$ :

$$x^T(k)x(k) < \beta = 20, \quad \forall k \in \{1, 2, \dots, 140\}.$$

The upper bounds of time delays in this paper and literature [51] are respectively shown in Table 1. It results from the above table that, the upper bounds of time delays in this paper are larger than those in [51], which means that the approach in this paper is less conservative than that in [51].

**Example 2.** Consider the following system with a state-feedback controller:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.80 & 0.30 \\ 0.80 & 0.70 \end{bmatrix} x(k) + \begin{bmatrix} 0.15 & 0.00 \\ 0.40 & 0.05 \end{bmatrix} x(k-h(k)) + \begin{bmatrix} 0.20 \\ 0.10 \end{bmatrix} u(k), \\ u(k) &= Kx(k). \end{aligned} \quad (56)$$

We choose  $(\alpha, \beta, N) = (18.1, 80, 50)$ ,  $\mu = 1.1$  and  $h_1 = 2$ . Then, via Corollary 2, by choosing the scalar  $\gamma = 1.00026$ , the control gain matrix  $K$  and the upper bound of time-varying delay are calculated as follows:

$$K = \begin{bmatrix} -1.0887 & -2.9660 \end{bmatrix}, \quad h_2 = 11.$$

In addition, initial conditions are given as follows:

$$\begin{aligned} \varphi(\theta)_{\theta \in \{-11, -10, \dots, 0\}} &= [\varphi(-11), \varphi(-10), \dots, \varphi(0)] \\ &= \begin{bmatrix} 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 \end{bmatrix}. \end{aligned} \quad (57)$$



Moreover, it can be seen that the initial values satisfy the following conditions:

$$\sup_{\theta \in \{-11, -10, \dots, 0\}} \varphi^T(\theta)\varphi(\theta) < \alpha = 18.1,$$

$$\sup_{\theta \in \{-11, -10, \dots, -1\}} [\varphi(\theta + 1) - \varphi(\theta)]^T [\varphi(\theta + 1) - \varphi(\theta)] < \mu = 1.1.$$

The curves of time delay, state responses with the controller and without controller are respectively shown in Figs. 4–6.

It results from Fig. 6 that, under initial conditions in Eq. (57), the system in Eq. (56) is instable without controller. In addition, it infers from Fig. 5 that, the state responses satisfy the following condition under the control gain matrix  $K = \begin{bmatrix} -1.0887 & -2.9660 \end{bmatrix}$ , which means that the system is finite-time stable with respect to  $(18.1, 80, 50)$ :

$$x^T(k)x(k) < \beta = 80, \quad \forall k \in \{1, 2, \dots, 50\}.$$

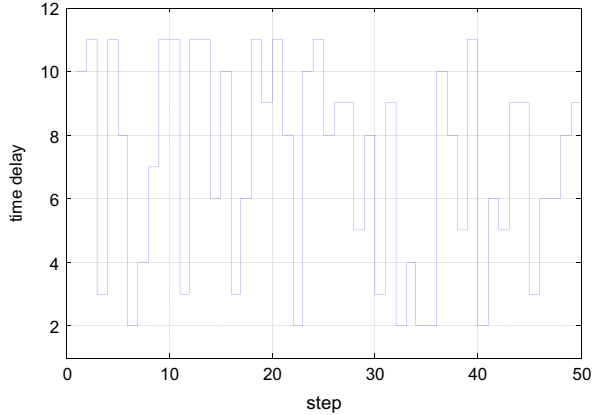


Fig. 4. Randomized time-varying delay  $2 \leq h(k) \leq 11$ .

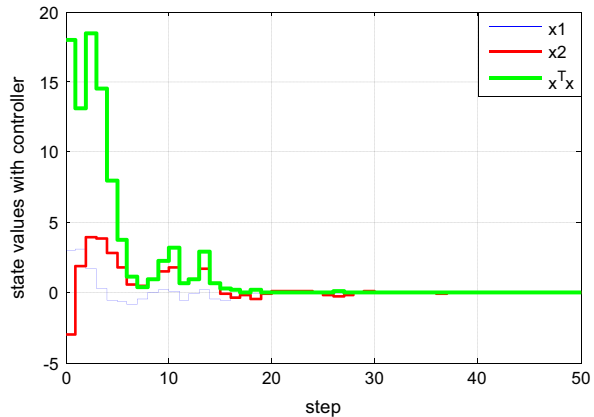


Fig. 5. State responses of the system with the controller.

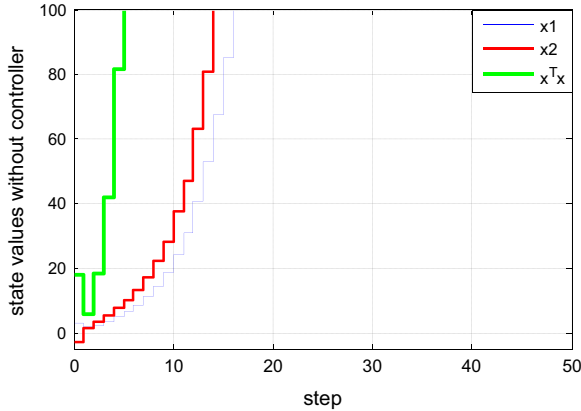


Fig. 6. State responses of the system without controller.

Finally, we can make the following conclusions:

- a) The upper bound of time delay in this paper is much larger than that in literature [51], which infers that the result in this paper is less conservative than that in [51].
- b) The controller guarantees the instable system finite-time stable, which infers that the controller is effective.

## 5. Conclusions

In this paper, the finite-time stability of systems with time-varying delays has been analyzed. By using the input–output method, the original system is transformed into two subsystems which are interconnected. A two-term approximation is used to approximate the time-varying delay. Then, based on a Lyapunov–Krasovskii formulation, the sufficient conditions of finite-time stability are derived. Moreover, a state-feedback controller is designed which can guarantee the instable system finite-time stable. Finally, the validity and advantages of this approach are illustrated through numerical examples.

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