Global stability of coupled Markovian switching reaction–diffusion systems on networks☆

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1. Introduction

Coupled systems on networks (CSNs) are composed of a large number of highly interconnected dynamical nodes. Each node is a unit with specific contents [1]. In the real world, many systems are modeled as coupled systems on networks (CSNs), such as communication networks, social networks, power grids, cellular networks, World Wide Web, metabolic systems, food webs, disease transmission networks, etc. [2,3]. When studying complex networks' dynamics, one problem is learning how a large ensemble of dynamical systems can behave collectively [4]. Therefore, it is very necessary to construct a relation between the stability criteria of a CSNs and some topology property of the network [5–7]. Li and Shuai [8] have considered global stability for the general CSNs based on graph theory, without discussing the stochastic effects.

Many large-scale dynamical systems from science and engineering often are represented as stochastic coupled systems on networks (SCSNs) [9–11], which could be described in a directed graph. Kao, Sun and Cao [12] have considered stability of coupled stochastic systems with time delay on networks without reaction–diffusion effects. However, for many realistic networks, the node state in CSNs is seriously dependent on the time and space. Hence, in order to describe more accurately the dynamic changes of CSNs, reaction–diffusion effects should also be considered [13–18]. Kao and Wang have considered global stability analysis for stochastic coupled reaction–diffusion systems on networks [19]. Markovian jump systems,

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introduced by Krasovskii and Lidskii [20] in 1961, have received increasing attentions [21–38], because the hybrid systems driven by continuous-time Markov chains have been used to model many practical systems such as components failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances, please see [22] and the references therein. The CSNs may be driven by continuous-time Markov chains. To the best of the authors' knowledge, stability analysis for Markovian switching reaction–diffusion coupled systems on networks (MSRDCSNs) has not been properly addressed, which still remains important and challenging.

Motivated by the above discussions, in this paper, we propose the MSRDCSNs model. In Section 2, some preliminaries are presented. In Section 3, some new stability principles for the global stability of MSRDCSNs are established. We provide a systematic method to construct the global Lyapunov function of MSRDCSNs by combining graph theory and the Lyapunov second method. The findings show that, if each vertex system has a globally stable equilibrium and possesses a global Lyapunov function V_k, then the global Lyapunov function for the MSRDCSNs can be systematically produced by individual V_k. An example is provided in Section 4. Section 5 is conclusion.

Notations: for convenience, we sometimes write $\mathbf{v}$, $\mathbf{v_k}$ and $\mathbf{v_j}$ as $\mathbf{v}(t, \mathbf{x})$, $\mathbf{v_k}(t, \mathbf{x})$ and $\mathbf{v_j}(t, \mathbf{x})$, respectively.

2. Preliminaries

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq t_0}$ satisfying the usual conditions. $W(\cdot)$ be a m-dimensional Brownian motion defined on the complete probability space. Let $\{\gamma(t), t \geq 0\}$ be a right-continuous Markov process on the probability space which takes values in the finite space $\mathbb{S} = \{1, 2, \ldots, \tilde{N}\}$ with generator $\Gamma = (\pi_{kj})(k, j \in \mathbb{S})$ given by

$$P\{\gamma(t+\Delta) = j|\gamma(t) = k\} = \begin{cases} \pi_{kj}\Delta + o(\Delta) & \text{if } k \neq j, \\ 1 + \pi_{kk}\Delta + o(\Delta) & \text{if } k = j, \end{cases}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\pi_{kj} \geq 0$ is the transition rate from k to j if $k \neq j$ and $\pi_{kk} = -\sum_{j\neq k}\pi_{kj}$. We assume that the Markov chain $\gamma(\cdot)$ is independent of the Brownian motion $W(\cdot)$. A general stochastic reaction–diffusion system with Markovian switching reads

$$\begin{aligned} d\mathbf{v}(t, \mathbf{x}) &= [\rho(\gamma(t))\Delta\mathbf{v}(t, \mathbf{x}) + f(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}), \gamma(t))]dt \\ &\quad + g(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}), \gamma(t))dW(t), \quad (t, \mathbf{x}, \gamma(t)) \in \mathbb{R}^+_{t_0} \times G \times \mathbb{S} \\ \mathbf{v}(t_0, \mathbf{x}) &= \varphi(\mathbf{x}), \quad \mathbf{x} \in G \\ \frac{\partial\mathbf{v}(t, \mathbf{x})}{\partial\mathscr{N}} &= 0, \quad (t, \mathbf{x}) \in \mathbb{R}^+_{t_0} \times \partial G \end{aligned} \tag{1}$$

where $\Delta\mathbf{v}(t, \mathbf{x}) \triangleq (\sum_{k=1}^{r} \frac{\partial}{\partial x_k}[D_{1k}(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}))\frac{\partial u_i}{\partial x_k}], \ldots, \sum_{k=1}^{r} \frac{\partial}{\partial x_k}[D_{nk}(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}))\frac{\partial u_i}{\partial x_k}])^T$, $G = \{\mathbf{x} = (x_1, x_2, \ldots, x_r)^T : \|\mathbf{x}\|l < +\infty\} \subset \mathbb{R}^r$, $\rho = \mathtt{diag}(\rho_1, \rho_2, \ldots, \rho_n)$ is the diffusion-matrix, $\rho_n \geq 0$ is constant; $f : \mathbb{R}_+ \times G \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \times G \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times m}$ are both Borel measurable functions. $D_{ik}(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x})) \geq 0$ is smooth enough. $\mathscr{N}$ is the unit normal vector of ∂G. Initial data $\mathbf{v}(t_0, \mathbf{x}) = \mathbf{v}_0 = \boldsymbol{\varphi}(\mathbf{x})$ is suitably smooth known function and $\gamma(t_0) = \gamma_0$, where γ_0 is an $\mathbb{S}$-valued $\mathscr{F}_{t_0}$-measurable random variable. $\|\cdot\|$ stands for vector norm.

(*Assumption* 1) Function $g(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}), \gamma(t))$ satisfies the integral linear growth condition and f, g meet Lipschitz condition, that is, there exists a constant $L > 0$ such that for $\forall i \in \mathbb{S}$

$$\begin{aligned} &\|g(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}), i)\|_G \leq L(1 + \|\mathbf{v}\|) \\ &\|g(t, \mathbf{x}, \mathbf{v_1}(t, \mathbf{x}), i) - g(t, \mathbf{x}, \mathbf{v_2}(t, \mathbf{x}), i)\|_G \leq L\|\mathbf{v_1} - \mathbf{v_2}\|_G \\ &\|f(t, \mathbf{x}, \mathbf{v_1}(t, \mathbf{x}), i) - f(t, \mathbf{x}, \mathbf{v_2}(t, \mathbf{x}), i)\|_G \leq L\|\mathbf{v_1} - \mathbf{v_2}\|_G \end{aligned} \tag{2}$$

where $\|\mathbf{v}(\cdot, \mathbf{x})\|_G \triangleq |\int_G \mathbf{v}(\cdot, \mathbf{x})d\mathbf{x}|$.

The existence and uniqueness of the solution for system (1) can refer to the relevant conclusions in [22], because stochastic reaction–diffusion systems can be transformed, via semi-group method, into abstract differential systems in Banach space including infinite linear operator and nonlinear term. Suppose that $f(t, \mathbf{x}, 0, i) \equiv 0$ and $g(t, \mathbf{x}, 0, i) \equiv 0$, $t \geq t_0$, which means $\mathbf{v}(t, \mathbf{x}) = 0$ is a trivial solution of (1).

Let the mathematical expectation with respect to the given probability measure P be denoted by $\mathbb{E}(\cdot)$. Let $|\cdot|$ denote the Euclidean norm for vectors or the trace norm for matrices. We shall use the notations $\mathbb{S}^n_\delta = \{\xi : G \to \mathbb{R}^n : |\int_G \xi(\mathbf{x})d\mathbf{x}| < \delta\}$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_k > 0, i = 1, 2, \ldots, n\}$. Some definitions on the stochastic stability of the trivial solution are given as follows.

Definition 1. If for every triple of $\forall\varepsilon_1 \in (0, 1)$, $\forall\varepsilon_2 > 0$ and $t_0 \geq 0$, $\ni \delta = \delta(\varepsilon_1, \varepsilon_2, t_0) > 0$ such that

$$\mathbb{P}\{\|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G < \varepsilon_2, t \geq t_0\} \geq 1 - \varepsilon_1$$

holds for any $(\mathbf{v}_0, i) \in \mathbb{S}_\delta \times \mathbb{S}$, the trivial solution of system (1) is said to be stochastically stable or stable in probability. Otherwise, the trivial solution is said to be unstable in probability.

Definition 2. The trivial solution of Eq. (1) is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for $\forall\varepsilon \in (0, 1)$ and $t_0 \geq 0$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$\mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G = 0\right\} \geq 1 - \varepsilon$$

for any $(\mathbf{v}_0, i) \in \mathbb{S}_{\delta_0} \times \mathbb{S}$.

Definition 3. The trivial solution of Eq. (1) is said to be stochastically globally asymptotically stable if it is stochastically stable and, moreover, for $\forall\delta > 0$ and $(\mathbf{v}_0, i) \in \mathbb{S}_\delta \times \mathbb{S}$

$$\mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G = 0\right\} = 1.$$

Definition 4. The trivial solution of system (1) is said to be almost surely exponentially stable if there is, for any $(\mathbf{v}_0, i) \in \mathbb{S}_\delta \times \mathbb{S}$

$$\lambda \triangleq \lim_{t\to\infty}\sup \frac{1}{t} \lg \|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G < 0 \quad \text{a.s.}$$

where λ is called as Lyapunov exponent of the solution for system (1). Therefore, the trivial solution of system (1) is almost surely exponentially stable if and only if $\lambda < 0$.

Definition 5. If $\mu(\cdot) \in C[[0, r], \mathbb{R}]$ is a strictly increasing function and $\mu(0) = 0$, function μ is said to be class $\mathcal{K}$ function. Denote $\mu \in \mathcal{K}$ concisely. If $\mu(\cdot) \in C[\mathbb{R}^+, \mathbb{R}^+]$ and $\mu \in \mathcal{K}$, $\lim_{r\to+\infty} \mu(r) = +\infty$, then $\mu \in \mathcal{K}R$.

A continuous function $V(t, \xi, i)$ is said to be positive-definite if $V(t, 0, i) = 0$ for $i \in \mathbb{S}$ and for some $\mu \in \mathcal{K}$, $V(t, \xi, i) \geq \mu(|\xi|)$. Write $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ for the family of all nonnegative functions $V(t, \xi, i)$ on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$ that are continuously twice differentiable in ξ and once in t for all $i \in \mathbb{S}$. If $V(t, \xi, i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$, then define an operator $\mathcal{L}V(t, \xi, i)$ from $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$ to $\mathbb{R}$ with respect to (1) by

$$\mathcal{L}V(t, \xi, i) = V_t(t, \xi, i) + V_\xi^T(t, \xi, i)f(t, \mathbf{x}, \xi, i) + \frac{1}{2}\mathrm{Trace}[g^T(t, \mathbf{x}, \xi, i)V_{\xi\xi}(t, \xi, i)g(t, \mathbf{x}, \xi, i)] + \sum_{j=1}^{N} \gamma_{kj}V(t, \xi, j) \tag{3}$$

where

$$V_t(t, \xi, i) = \frac{\partial V(t, \xi, i)}{\partial t}, \qquad V_\xi^T(t, \xi, i) = \left(\frac{\partial V(t, \xi, i)}{\partial \xi_1}, \dots, \frac{\partial V(t, \xi, i)}{\partial \xi_n}\right)$$

and

$$V_{\xi\xi}(t, \xi, i) = \left(\frac{\partial^2 V(t, \xi, i)}{\partial \xi_k \partial \xi_j}\right)_{n\times n}.$$

Applying the generalized Itô formula to $\int_G V(t, \mathbf{v}(t, \mathbf{x}), \gamma(t))d\mathbf{x}$ along system (1) gives for $\forall t \geq t_0$

$$\int_G V(t, \mathbf{v}(t, \mathbf{x}), \gamma(t))d\mathbf{x} = \int_G V(t_0, \mathbf{v}_0, \gamma_0)d\mathbf{x} + \int_G \int_0^t [\mathcal{L}V(s, \mathbf{v}(s, \mathbf{x}), \gamma(s)) + V_\mathbf{v}^T(s, \mathbf{v}, \gamma(s))\triangle\mathbf{v}(s, \mathbf{x})]dsd\mathbf{x} + \int_G \int_0^t V_\mathbf{v}^T(s, \mathbf{v})g(s, \mathbf{x}, \mathbf{v}(s, \mathbf{x}), \gamma(s))dW(s)d\mathbf{x}. \tag{4}$$

The existence of function $V(t, \mathbf{v}(t, \mathbf{x}), i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ and another conditions in the classical Lyapunov theorem on the stability of (1) are needed. For convenience, similarly, we give the following definitions:

Definition 6. Function $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+)$ is called as Lyapunov-A function for (1), if $\mathcal{L}\int_G V(t, \mathbf{v}, i)d\mathbf{x} \leq 0$, and is called as Lyapunov-B function for (1), if $\mathcal{L}\int_G V(t, \mathbf{v}, i)d\mathbf{x} \leq -b\int_G V(t, \mathbf{v}, i)d\mathbf{x}$ in which $b > 0$.

The following basic concepts and theorems on graph theory can be found in [7]. A directed graph $\mathcal{G} = (\mathcal{V}, E)$ contains a set $\mathcal{V} = \{1, 2, \dots, n\}$ of vertices and a set E of arcs (i, j) leading from initial vertex i to terminal vertex j. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc (j, i) is assigned a positive weight a_{kj}. Here $a_{kj} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \dots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \dots, m - 1\}$. If $i_m = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Given a weighted digraph $\mathcal{G}$ with n vertices, define the weight matrix $A = (a_{ij})_{n\times n}$ whose entry a_{kj} equals the weight of arc (j, i) if it exists, and 0 otherwise. Denote the directed graph with weight matrix A as $(\mathcal{G}, A)$. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. For a unicyclic graph $\mathcal{Q}$ with cycle $\mathcal{C}_{\mathcal{Q}}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_{\mathcal{Q}}$ with $-\mathcal{C}_{\mathcal{Q}}$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L = \begin{pmatrix} \sum_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\neq n} a_{nk} \end{pmatrix}.$$

Let c_k denote the cofactor of the ith diagonal element of L.

Lemma 1 (*Kirchhoff's Matrix Tree Theorem*). *Assume $n \geq 2$. Then*

$$c_k = \sum_{\mathcal{T}\in\mathbb{T}_k} W(\mathcal{T}), \quad k = 1, 2, \ldots, N \tag{5}$$

where $\mathbb{T}_k$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ that are rooted at vertex i. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_k > 0$ for $c_k = 1, 2, \ldots, n$.

Lemma 2 ([8]). *Assume $n \geq 2$. Let c_k be given in (5). Then the following identity holds:*

$$\sum_{k,j=1}^{n} c_k a_{kj} F_{kj}(x_k, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(k,j)\in E(C_{\mathcal{Q}})} F_{kj}(x_k, x_j). \tag{6}$$

Here $F_{kj}(x_k, x_j)$, $1 \leq k, j \leq n$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $C_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$.

Lemma 3. *Under Assumption* 1,

$$\mathbb{P}\{\|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G \neq 0,\ t \geq t_0\} = 1$$

for all $(t_0, \mathbf{v}_0, \gamma_0) \in \mathbb{R}_+ \times (\mathbb{R}^n - 0) \times \mathbb{S}$. That is, almost any trajectory of (1) starting from a non-zero state will never be zero.

Proof. If the lemma was false, there would exist some $t_0 \geq 0$, $\mathbf{v}_0 \neq 0$ and $i \in \mathbb{S}$ such that $\mathbb{P}\{\tau < \infty\} > 0$, where τ is the first time to reach zero for the corresponding solution, that is,

$$\tau = \inf\left\{t \geq t_0, \int_G \mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, \gamma_0)d\mathbf{x} = 0\right\}.$$

It is not difficult to find an integer $k > t_0 \vee (1 + \|\mathbf{v}_0\|)$ sufficiently large for $\mathbb{P}\{B\} > 0$, where $B = \{\tau < T, T$ is the time upper bound and $\|\mathbf{v}(t, \mathbf{x})\| \leq k - 1$, for $\forall(t, \mathbf{x}) \in [t_0, \tau] \times G\}$. While in view of the linear growth conditions there is positive constant L_k such that, for any $\|\mathbf{v}(t, \mathbf{x})\| \geq k$, $\forall(t, \mathbf{x}, i) \in [t_0, \tau] \times G \times \mathbb{S}$,

$$\|f(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}), i)\|_G \vee \|g(t, \mathbf{x}, \mathbf{v}(t, \mathbf{x}), i)\|_G \leq L_k(1 + \|\mathbf{v}\|).$$

Let $V(t, \mathbf{v}, i) = \|\mathbf{v}\|^{-1}$, we calculate, for $\|\mathbf{v}(t, \mathbf{x})\| \geq k$, $\forall(t, \mathbf{x}, i) \in [t_0, \tau] \times G \times \mathbb{S}$,

$$\begin{aligned} \mathcal{L}\int_G V(t, \mathbf{v}(t, \mathbf{x}), i)d\mathbf{x} &= \int_G \mathcal{L}V(t, \mathbf{v}(t, \mathbf{x}), i)d\mathbf{x} \\ &= \int_G [-\|\mathbf{v}\|^{-3}\mathbf{v}^T f(t, \mathbf{x}, \mathbf{v}, i) - \|\mathbf{v}\|^{-3}\mathbf{v}^T \rho(i)\triangle\mathbf{v}(t, \mathbf{x}) \\ &\quad + \frac{1}{2}\left(-\|\mathbf{v}\|^{-3}\|g(t, \mathbf{x}, \mathbf{v}, i)\|^2 + 3\|\mathbf{v}\|^{-5}\|\mathbf{v}^T g(t, \mathbf{x}, \mathbf{v}, i)\|^2\right)]d\mathbf{x} \\ &\leq \int_G \left[\|\mathbf{v}\|^{-1}\|f(t, \mathbf{x}, \mathbf{v}, i)\| + \|\mathbf{v}\|^{-1}\|g(t, \mathbf{x}, \mathbf{v}, i)\|^2\right] d\mathbf{x} \\ &\leq \int_G \left[L_k\|\mathbf{v}\|^{-1} + L_k^2\|\mathbf{v}\|^{-1}\right] d\mathbf{x} = |G|L_k(1 + L_k)\|\mathbf{v}\|^{-1}. \end{aligned}$$

For any $\varepsilon \in (0, \|\mathbf{v}_0\|)$, define a stopping time $\tau_\varepsilon = \inf\{t \geq t_0 : \|\mathbf{v}\| \notin (\varepsilon, k)\}$. Applying the Itô formula gives

$$\mathbb{E}\left[e^{-L_k(1+L_k)(\tau_\varepsilon \wedge k)} \int_G V(\tau_\varepsilon \wedge k, \mathbf{v}(\tau_\varepsilon \wedge k, \mathbf{x}), i) d\mathbf{x}\right] = \|\mathbf{v}_0\|^{-1} e^{-L_k(1+L_k)t_0} |G|$$
$$+ \mathbb{E} \int_{t_0}^{\tau_\varepsilon \wedge k} e^{-L_k(1+L_k)s} \left[-|G| L_k(1+L_k)\|\mathbf{v}(s,\mathbf{x})\|^{-1} + \int_G V(s, \mathbf{v}(s,\mathbf{x}), \gamma(s)) d\mathbf{x}\right] ds \leq \|\mathbf{v}_0\|^{-1} e^{-L_k(1+L_k)t_0} |G|.$$

Note that for $\omega \in B$ we have $\tau_\varepsilon \leq k$ and $\|\mathbf{v}(\tau_\varepsilon \leq k, \mathbf{x})\| = \varepsilon$, then we obtain

$$\mathbb{E}\left[e^{-L_k(1+L_k)k} \varepsilon^{-1} |G| \mathcal{X}_B\right] \leq \|\mathbf{v}_0\|^{-1} e^{-L_k(1+L_k)t_0} |G|.$$

Thus,

$$\mathbb{P}\{B\} \leq \varepsilon \|\mathbf{v}_0\|^{-1} e^{-L_k(1+L_k)(k-t_0)} |G|.$$

Letting $\varepsilon \to 0$ yields $\mathbb{P}\{B\} = 0$, which is in contradiction with the definition of $\mathbb{P}\{B\} > 0$. The proof is complete.

3. Global stability analysis for Markovian switching coupled reaction–diffusion systems on networks

To begin with our main results, we will give a MSRDCSNs represented by digraph $\mathcal{G}$ with N vertices, $N \geq 2$. In ith vertex it is assigned a stochastic reaction–diffusion system with Markovian switching

$$d\mathbf{v}_k(t,\mathbf{x}) = [\rho_k(\gamma(t)) \Delta \mathbf{v}_k(t,\mathbf{x}) + f_k(t, \mathbf{x}, \mathbf{v}_k(t,\mathbf{x}), \gamma(t))]dt$$
$$+ g_k(t, \mathbf{x}, \mathbf{v}_k(t,\mathbf{x}), \gamma(t)) dW_k(t), \quad (t, \mathbf{x}, \gamma(t)) \in \mathbb{R}^+_{t_0} \times \Omega \times \mathbb{S} \tag{7}$$

where $\mathbf{v}_k(t,\mathbf{x}) \in \mathbb{R}^{n_k}$, $f_k : \mathbb{R}_+ \times G \times \mathbb{R}^{n_k} \times \mathbb{S} \to \mathbb{R}^{n_k}$ and $g_k : \mathbb{R}_+ \times G \times \mathbb{R}^{n_k \times m_i} \times \mathbb{S} \to \mathbb{R}^{n_k \times m}$. If these systems are coupled, let

$$H_{kj} : \mathbb{R}^{n_k} \times \mathbb{R}^{n_j} \times \mathbb{R} \times \mathbb{S} \to \mathbb{R}^{m_i}, \qquad N_{kj} : \mathbb{R}^{n_k} \times \mathbb{R}^{n_j} \times \mathbb{R} \times \mathbb{S} \to \mathbb{R}^{n_k \times m}, \quad k, j = 1, 2, \ldots, N$$

represent the influence of vertex j on vertex i, and $H_{kj} = N_{kj} = 0$ if there exists no arc from j to i in $\mathcal{G}$. Then, by replacing f_k and g_k with $f_k + \sum_{j=1}^n H_{kj}$ and $g_k + \sum_{j=1}^n N_{kj}$, we get the following stochastic coupled system on graph $\mathcal{G}$:

$$d\mathbf{v}_k(t,\mathbf{x}) = \left[\rho_k(i)\Delta \mathbf{v}_k(t,\mathbf{x}) + f_k(t,\mathbf{x},\mathbf{v}_k(t,\mathbf{x}), i) + \sum_{j=1}^N H_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\right] dt$$
$$+ \left[g_k(t,\mathbf{x},\mathbf{v}_k(t,\mathbf{x}), i) + \sum_{j=1}^N N_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\right] dW(t), \quad (t, \mathbf{x}, i) \in \mathbb{R}^+_{t_0} \times G \times \mathbb{S} \tag{8}$$
$$\mathbf{v}_k(t_0, \mathbf{x}) = \boldsymbol{\varphi}_{\mathbf{k}(\mathbf{x})}, \quad \mathbf{x} \in G, \ \gamma(t_0) = \gamma_0$$
$$\frac{\partial \mathbf{v}_k(t,\mathbf{x})}{\partial \mathcal{N}} = 0, \quad (t, \mathbf{x}) \in \mathbb{R}^+_{t_0} \times \partial G.$$

Without loss of generality, we suppose that functions f_k, g_k, H_{kj} and N_{kj} are such that initial-value problems to (7) and (8) have unique solution and trivial solution $\mathbf{v}(t,\mathbf{x}) = (\mathbf{v}_1, \ldots, \mathbf{v}_n) = 0$. Functions f_k and g_k meet Lipschitz conditions and linear growth conditions with constant $L > 0$. For $V_k(t, \mathbf{v}_k, i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{n_k} \times \mathbb{S}; \mathbb{R}_+)$, define a differential operator $\mathcal{L}V_k(t, \mathbf{v}_k, i)$ associated with the ith equation of (8) by

$$\mathcal{L}V_k(t, \mathbf{v}_k, i) \triangleq \frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial t} + \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k}\right)^T \left[f_k(t, \mathbf{x}, \mathbf{v}_k(t,\mathbf{x}), i) + \sum_{j=1}^N H_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\right]$$
$$+ \frac{1}{2} \mathrm{Trace}\left\{\left[g_k(t,\mathbf{x},\mathbf{v}_k(t,\mathbf{x}), i) + \sum_{j=1}^N N_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\right]^T \left(V_k(t, \mathbf{v}_k)\right)''_{\mathbf{v}_k \mathbf{v}_j}\right.$$
$$\left.\times \left[g_k(t,\mathbf{x},\mathbf{v}_k(t,\mathbf{x}), i) + \sum_{j=1}^N N_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\right]\right\} + \sum_{j=1}^N \gamma_{kj} V(t, \mathbf{v}, j). \tag{9}$$

Theorem 1. *Suppose that the following conditions hold.*

A1. *Assume that there exist positive functions $V_k(t, \xi, i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{n_k} \times \mathbb{S}; \mathbb{R}_+)$, functions $F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t)$ and constants $a_{kj} \geq 0$ satisfying*

(I) *there exist* $\mu_1, \mu_2 \in \mathcal{KR}$ *such that*

$$\begin{aligned}
&\mu_1(\|\mathbf{v}_k\|) \le \int_G V_k(t, \mathbf{v}_k, i)d\mathbf{x} \le \mu_2(\|\mathbf{v}_k\|),\\
&\mathcal{L}\int_G V_k(t, \mathbf{v}_k, i\,)d\mathbf{x} \le \sum_{j=1}^{n} a_{kj}F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t),\\
&t \ge t_0, \quad k = 1, 2, \ldots, N,
\end{aligned} \tag{10}$$

hold for $\forall(t, \mathbf{v}_k(t, x), i) \in [t_0, \infty) \times \mathbb{s}_h^{n_k} \times \mathbb{S}$, *where* $\mathbf{v}_k(t, \cdot) \in \mathbb{s}_h^{n_k} = \{\zeta : G \to \mathbb{R}^{n_k} \,\|\, \int_G \zeta(\mathbf{x})d\mathbf{x}| < h\}$;

(II) $V_k(t, \xi, i)$ *is separated as to variables* $\xi(k \in N)$ *for* $i \in \mathbb{S}$;

(III) $\frac{\partial^2 V_k(t,\xi,i)}{\partial \xi^2} \ge 0,\ k \in N, (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$ *for* $i \in \mathbb{S}$;

A2. *Along each directed cycle* $\mathcal{C}$ *of the weighted digraph* $(\mathcal{G}, A)$ *in which* $A = (a_{kj})_{n\times n}$, *there is*

$$\sum_{(i,j)\in E(\mathcal{C})} F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t) \le 0, \quad t \ge t_0. \tag{11}$$

Then function $V(t, \mathbf{v}, i) \triangleq \sum_{i=1}^{n} c_k V_k(t, \mathbf{v}_k, i\,)$ *is a Lyapunov-A function for* (8), *in which* c_k *is defined as* (5). *Furthermore, the trivial solution of* (8) *is stochastically stable. In addition, if more condition is added.*

A3. *If* $V_k(t, \xi, i)$ *is radial unbounded, then the trivial solution of* (8) *is globally asymptotically stable in probability.*

Proof. We first prove that the trivial solution of system (8) is stochastically stable.

For $\forall\varepsilon_1 \in (0, 1)$ and $\varepsilon_2 \ge 0$, suppose $\varepsilon_2 < h$. Since $V_k(t, \xi, i\,)$ is continuous and $V_k(t_0, 0, i) = 0$, there is $\delta = \delta(\varepsilon_1, \varepsilon_1, t_0) > 0$, such that

$$\frac{1}{\varepsilon_1}\sup_{\mathbf{v}(t,\mathbf{x})\in\mathbb{S}_\delta^n}\int_G V(t_0, \mathbf{v}(t, \mathbf{x}), i)d\mathbf{x} \le \mu^*(\varepsilon_2), \tag{12}$$

where $n = n_1 + n_2 + \cdots + n_N$, $\mu^* \in \mathcal{KR}$.

It follows from Condition (I) and (12) that $\delta < \varepsilon_2$. Denote $\bar{\mathbf{v}}(t) = \int_G \mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i\,)d\mathbf{x}$ for $\forall(\mathbf{v}_0, i\,) \in \mathbb{S}_\delta^n \times \mathbb{S}$. Let τ be the first escape time of $\bar{\mathbf{v}}(t)$ from $\mathbb{S}_{\varepsilon_2}^n$, that is

$$\tau = \inf\{t \ge t_0 | \bar{\mathbf{v}}(t) \notin \mathbb{S}_{\varepsilon_2}^n\}.$$

Applying the Itô formula to $\int_G V_k(t, \mathbf{v}_k(t, \mathbf{x}), i)d\mathbf{x}$ along system (8) gives for $\forall t \ge t_0$

$$\begin{aligned}
\int_G V_k(\tau\wedge t, \mathbf{v}_k(\tau\wedge t, \mathbf{x}), \gamma(\tau\wedge t))d\mathbf{x} &= \int_G V_k(t_0, \mathbf{v}_{k0}, \gamma_0)d\mathbf{x}\\
&+ \int_{t_0}^{\tau\wedge t}\int_G \mathcal{L}V_k(s, \mathbf{v}_k(s, \mathbf{v}, t_0, \mathbf{v}_{k0}, i))d\mathbf{x}ds\\
&+ \int_{t_0}^{\tau\wedge t}\int_G \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k}\right)^T \triangle\mathbf{v}_k(s, \mathbf{x})d\mathbf{x}ds\\
&+ \int_{t_0}^{\tau\wedge t}\int_G \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k}\right)^T \Bigg[g_k(t, \mathbf{x}, \mathbf{v}_k(t, \mathbf{x}, t_0, \mathbf{v}_{k0}, i))\\
&+ \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\Bigg] d\mathbf{x}dW(t).
\end{aligned} \tag{13}$$

By Condition A1(II), we have $\frac{\partial^2 V_k(t,\xi)}{\partial\xi_k\partial\xi_j} = 0$, $(\,k \ne j,\ k, j \in N)$. Employing integration by parts and combining Condition A1(II), A1(III) and boundary condition of (8), we obtain

$$\int_G \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k}\right)^T \triangle\mathbf{v}_k(t, \mathbf{x})d\mathbf{x} \le 0. \tag{14}$$

Furthermore, due to the continuity of $\frac{\partial V_k(t,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}$ on $[t_0, \tau\wedge t] \times \mathbb{S}_h^{\mathbb{R}^{n_k}} \times \mathbb{S}$, there must exist constant $L_1 > 0$ such that $\|(\frac{\partial V_k(t,\mathbf{v}_k,i)}{\partial \mathbf{v}_k})^T\| \le L_1$ holds for $(t, \mathbf{v}, i) \in [t_0, \tau\wedge t] \times \mathbb{S}_h^{\mathbb{R}^{n_k}} \times \mathbb{S}$, in addition, since $g_k(t, \mathbf{x}, \mathbf{v}_k(t, \mathbf{x}), i)$ meets integral linear growth condition, we obtain, for $(t, \mathbf{v}(t, \mathbf{x})) \in [t_0, \tau\wedge t] \times \mathbb{S}_h^{\mathbb{R}^{n_k}}$

$$\left\|\left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k}\right)^T g_k(t, \mathbf{x}, \mathbf{v}_k(t, \mathbf{x}), i)\right\|_G \le L_1 L(1 + \|\mathbf{v}_k(t, \mathbf{x})\|_G) \le L_1 L(1 + h)$$

From Theorem 1.45 of Ref. ([22], pp. 49), we have

$$\mathbb{E}\left[\int_{t_0}^{\tau\wedge t}\int_G\left(\frac{\partial V_k(t,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}\right)^T\left(g_k(t,\mathbf{x},\mathbf{v}_k(t,\mathbf{x},t_0,\mathbf{v}_{k0}),i)+\sum_{j=1}^N N_{kj}(\mathbf{v}_k,\mathbf{v}_j,t,i)\right)d\mathbf{x}dW(t)\right]=0. \tag{15}$$

On the other hand, by (10), it is derived that

$$\mathcal{L}\int_G V(t,\mathbf{v},i)d\mathbf{x}=\sum_{i=1}^n c_k\mathcal{L}\int_G V_k(t,\mathbf{v}_k,i)d\mathbf{x}\le\sum_{k,j=1} c_k a_{kj}F_{kj}(\mathbf{v}_k,\mathbf{v}_j,t). \tag{16}$$

Making use of Lemma 2 with weighted digraph $(\mathcal{G},A)$, it yields

$$\sum_{k,j=1} c_k a_{kj}F_{kj}(\mathbf{v}_k,\mathbf{v}_j,t)=\sum_{\mathcal{Q}\in\mathbb{Q}}W(\mathcal{Q})\sum_{(i,j)\in E(C_{\mathcal{Q}})}F_{kj}(\mathbf{v}_k,\mathbf{v}_j,t). \tag{17}$$

In view of Condition A2 and the fact $W(\mathcal{Q})>0$, we get

$$\mathcal{L}\int_G V(t,\mathbf{v},i)d\mathbf{x}\le\sum_{\mathcal{Q}\in\mathbb{Q}}W(\mathcal{Q})\sum_{(i,j)\in E(C_{\mathcal{Q}})}F_{kj}(\mathbf{v}_k,\mathbf{v}_j,t)\le 0. \tag{18}$$

Thus $V(t,\mathbf{v},i)$ is a Lyapunov-A function for (8). Taking the mathematical expectation at the two sides of Eq. (13) and using (15), (16) and (19) we have

$$\mathbb{E}\left[\int_G V(\tau\wedge t,\mathbf{v}(\tau\wedge t,\mathbf{x}),\gamma(\tau\wedge t))d\mathbf{x}\right]\le\int_G V(t_0,\mathbf{v}_0,\gamma_0)d\mathbf{x}. \tag{19}$$

Recalling the fact that $\|\mathbf{v}(\tau\wedge t,\mathbf{x})\|_G=\|\mathbf{v}(\tau,\mathbf{x})\|_G=\varepsilon_2$ if $\tau\le t$, it is derived

$$\mathbb{E}\left[\int_G V(\tau\wedge t,\mathbf{v}(\tau\wedge t,\mathbf{x}),\gamma(\tau\wedge t))d\mathbf{x}\right]\ge\mathbb{E}\left[I_{\{\tau<t\}}\int_G V(\tau,\mathbf{v}(\tau,\mathbf{x}),\gamma(\tau))d\mathbf{x}\right]\ge\mu^*(\varepsilon_2)\mathbb{P}(\tau\le t).$$

Combining (12) and (19) we get $\mathbb{P}(\tau\le t)\le\varepsilon_1$. Letting $t\to\infty$, we infer that

$$\mathbb{P}(\tau\le\infty)\le\varepsilon_1.$$

That is,

$$\mathbb{P}(|\bar{\mathbf{v}}(t)|<\varepsilon_2,\ \forall t\ge t_0)\ge 1-\varepsilon_1.$$

Namely,

$$\mathbb{P}(\|\mathbf{v}(t,\mathbf{x},t_0,\mathbf{v}_0,i)\|_G<\varepsilon_2,t\ge t_0)\ge 1-\varepsilon_1$$

which implies that the trivial solution of system (8) is stable in probability.

We then prove that the trivial solution of system (8) is globally asymptotically stable in probability. In the following, we only need to prove for $\forall\mathbf{v}_0$,

$$\mathbb{P}\left(\lim_{t\to\infty}\|\mathbf{v}(t,\mathbf{x},t_0,\mathbf{v}_0,i)\|_G=0\right)=1. \tag{20}$$

For $\forall\varepsilon\in(0,1)$, since $V_k(t,\xi,i)$ is radial unbounded, we can find $h>\|\mathbf{v}_0\|_G$ for $i\in\mathbb{S}$ satisfying

$$\inf_{t\ge t_0,\|\mathbf{v}\|_G\ge h}\int_G V(t_0,\mathbf{v}(t,\mathbf{x}),i)d\mathbf{x}\ge\frac{4}{\varepsilon}\int_G V(t_0,\mathbf{v}_0,i)d\mathbf{x}. \tag{21}$$

Define stop-time $\tau_h=\inf\{t\ge t_0,\|\bar{\mathbf{v}}(t)\|\ge h\}$. Applying Itô formula, similar to get (19), we obtain for $t\ge t_0$

$$\mathbb{E}\left[\int_G V(\tau_h\wedge t,\mathbf{v}(\tau_h\wedge t,\mathbf{x}),\gamma(\tau\wedge t))d\mathbf{x}\right]\le\int_G V(t_0,\mathbf{v}_0,\gamma_0)d\mathbf{x}. \tag{22}$$

From (21), we get

$$\mathbb{E}\left[\int_G V(\tau_h\wedge t,\mathbf{v}(\tau_h\wedge t,\mathbf{x}),\gamma(\tau\wedge t))d\mathbf{x}\right]\ge\frac{4}{\varepsilon}\int_G V(t_0,\mathbf{v}_0,\gamma_0)d\mathbf{x}\mathbb{P}\{\tau_h\le t\}. \tag{23}$$

Then, it follows that

$$\mathbb{P}\{\tau_h\le t\}\le\frac{\varepsilon}{4}.$$

Letting $t \to \infty$ results in

$$\mathbb{P}\{\tau_h \leq \infty\} \leq \frac{\varepsilon}{4}.$$

Therefore

$$\mathbb{P}(|\bar{\mathbf{v}}(t)| < h, \forall t \geq t_0) \geq 1 - \frac{\varepsilon}{4}.$$

From Theorem 4.2.3 of Ref. ([39], pp. 112–114),

$$\mathbb{P}(|\bar{\mathbf{v}}(t)| = 0) \geq 1 - \varepsilon.$$

Thus, (20) holds owing to the arbitrariness of ε. The proof is complete. □

Note that if $(\mathcal{G}, A)$ is balanced, then

$$\sum_{k,j=1} c_k a_{kj} F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t) = \frac{1}{2} \sum_{\mathcal{Q} \in \mathbb{Q}} W(\mathcal{Q}) \sum_{(k,j) \in E(C_{\mathcal{Q}})} \left[F_{j\,k}(\mathbf{v}_j, \mathbf{v}_k, t) + F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t)\right].$$

In this case, Condition A2 is replaced by the following:

$$\sum_{(k,j) \in E(C_{\mathcal{Q}})} \left[F_{j\,k}(\mathbf{v}_j, \mathbf{v}_k, t) + F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t)\right] \leq 0. \tag{24}$$

Consequently, we get the following corollary:

Corollary 1. *Suppose that $(\mathcal{G}, A)$ is balanced. Then the conclusion of Theorem 1 holds if (11) is replaced by (24).*

Remark 1. The MSRDCSNs is so complicated that it is very difficult to give the analytical solution. It is important to work on the qualitative analysis of the system. Therefore, how to construct appropriate Lyapunov function is of great importance. The proof shows that, if each vertex system of (8) has a globally stable trivial solution and possesses a Lyapunov function V_k, then the Lyapunov function for (8) can be systematically constructed by using individual V_k. In special, when $\rho_k = 0, m = 1$ some examples are given in [12]. Moreover, when $g_k = 0$ and $N_{kj} = 0$ $(k, j = 1, 2, \ldots, n)$, some examples are given in [8].

Theorem 2. *Suppose that the following conditions hold.*

B1. *Assume that there exist positive functions $V_k(t, \xi, i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{n_k} \times \mathbb{S}; \mathbb{R}_+)$, satisfying*

(I) *there exist $\mu_1, \mu_2 \in \mathcal{KR}$ such that*

$$\mu_1(\|\mathbf{v}_k\|) \leq \int_G V_k(t, \mathbf{v}_k, i) d\mathbf{x} \leq \mu_2(\|\mathbf{v}_k\|) \tag{25}$$

hold for $\forall (t, \mathbf{v}_k(t, \mathbf{x})) \in [t_0, \infty) \times \mathfrak{s}_h^{n_k}$, where $\mathbf{v}_k(t, \cdot) \in \mathfrak{s}_h^{n_k} = \{\zeta : G \to \mathbb{R}^{n_k} \parallel \int_G \zeta(\mathbf{x}) d\mathbf{x}| < h\}$;

(II) *$V_k(t, \xi, i)$ is separated as to variables ξ $(k \in N)$ for $i \in \mathbb{S}$;*

(III) *$\frac{\partial^2 V_k(t,\xi,i)}{\partial \xi^2} \geq 0,\ k \in N, (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$ for $i \in \mathbb{S}$.*

B2. *There exist functions $F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t)$ and constants $a_{kj} \geq 0$, $b_k > 0$ such that*

$$\mathcal{L} \int_G V_k(t, \mathbf{v}_k, i) d\mathbf{x} \leq -b_k \int_G V_k(t, \mathbf{v}_k, i) d\mathbf{x} + \sum_{j=1}^{n} a_{kj} F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t), \tag{26}$$

$t \geq t_0$, for $k = 1, 2, \ldots, N$.

B3. *Condition A2 holds, or if $(\mathcal{G}, A)$ is balanced and (24) holds.*

Then function $V(t, \mathbf{v}, i) \triangleq \sum_{i=1}^{n} c_k V_k(t, \mathbf{v}_k, i)$ is a Lyapunov-B function for (8), in which c_k is defined as (5). Consequently, the trivial solution of (8) is asymptotically stable in probability.

In addition, if more condition is added.

B4. *Each $V_k(x, t, i)$ satisfies*

$$\lim_{\|\mathbf{v}_k\| \to \infty} \inf_{t \geq t_0} \int_G V_k(t, \mathbf{v}_k, i) d\mathbf{x} = \infty.$$

Then the trivial solution of (8) is globally asymptotically stable in probability.

Proof. We can show in the same way as in the proof of Theorem 1 that

$$\mathcal{L} \int_G V(t, \mathbf{v}, i) d\mathbf{x} = \sum_{i=1}^{n} c_k \mathcal{L} \int_G V_k(t, \mathbf{v}_k, i) d\mathbf{x} \leq -b \int_G V(t, \mathbf{v}, i) d\mathbf{x}, \tag{27}$$

where $b = \min\{b_1, b_2, \ldots, b_n\}$. Hence, we conclude that function $V(t, \mathbf{v}, i)$ is a Lyapunov-B function for (8). Let $C = \max\{c_1, c_2, \ldots, c_n\}$, it follows easily that

$$\int_G V(t, \mathbf{v}, i)d\mathbf{x} = \sum_{k=1}^{n} c_k \int_G V_k(t, \mathbf{v}_k, i)d\mathbf{x} \le \sum_{k=1}^{n} C\mu_2(\|\mathbf{v}_k\|) \le nC\mu_2(\|\mathbf{v}\|) \tag{28}$$

where $\|\mathbf{v}\| = \sum_{i=1}^{n} \|\mathbf{v}_k\|$, obviously, $\|\mathbf{v}\| \ge \|\mathbf{v}_k\|$. By Theorem 4.2.3 in (Mao (1997) [39]), the trivial solution of (8) is stochastically asymptotically stable. Furthermore, making use of Condition B4 yields

$$\lim_{\|\mathbf{v}\|\to\infty} \inf_{t\ge 0} \int_G V(t, \mathbf{v}, i)d\mathbf{x} = \lim_{\|\mathbf{v}_k\|\to\infty} \inf_{t\ge 0} \left(\sum_{i=1}^{n} c_k \int_G V_k(t, \mathbf{v}_k, i)d\mathbf{x} \right) = \infty.$$

Then, the trivial solution of (8) is globally asymptotically stable in probability. The proof is complete. □

Theorem 3. *Suppose that the following conditions hold.*

C1. *Assume that there exist positive functions $V_k(t, \mathbf{v}_k, i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{n_k} \times \mathbb{S}; \mathbb{R}_+)$, and constants $p > 0, q_1 > 0, q_2 \ge 0$ satisfying*
   (I) *$V_k(t, \mathbf{v}, i)$ is separated as to variables $\mathbf{v}_k (k \in N)$ for $i \in \mathbb{S}$;*
   (II)
$$q_1 \|\mathbf{v}_k\|_G^p \le \left| \int_G V_k(t, \mathbf{v}_k, i)d\mathbf{x} \right| \tag{29}$$
   hold for $\forall (t, \mathbf{v}_k(t, \mathbf{x})) \in [t_0, \infty) \times s_h^{n_k}$, where $\mathbf{v}_k(t, \cdot) \in s_h^{n_k} = \{\zeta : G \to \mathbb{R}^{n_k} \,\|\, \int_G \zeta(\mathbf{x})d\mathbf{x}| < h\}$;
   (III) *$\| \left(\frac{\partial V_k(t,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}\right)^T \left(g_k(t, \mathbf{x}, \mathbf{v}_k(t, \mathbf{x}, t_0, \mathbf{v}_{k0}), i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i)\right) \|_G^2 \ge q_2 \|V_k(t, \mathbf{v}_k, i)\|_G^2$, for any $\mathbf{v}_k \ne 0,\ k \in N,\ (t, \mathbf{x}_k) \in \mathbb{R}_+ \times G$ for $i \in \mathbb{S}$.*

C2. *There exist functions $F_{kj}(\mathbf{v}, \mathbf{v}_j, t)$ and constants $a_{kj} \ge 0$, $b_k > 0$ such that*
$$\mathcal{L} \int_G V_k(t, \mathbf{v}_k, i)d\mathbf{x} \le -b_k \int_G V_k(t, \mathbf{v}_k, i)d\mathbf{x} + \sum_{j=1}^{n} a_{kj} F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t), \tag{30}$$
   $t \ge t_0$, for $k = 1, 2, \ldots, N$. $i \in \mathbb{S}$. Here LV_k is defined in (9).

C3. *Condition A2 holds, or if $(\mathcal{G}, A)$ is balanced and (24) holds.*

Then

$$\lim_{t\to\infty} \sup \frac{1}{t} \lg \|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G \le \frac{-2\sum_{k=1}^{N} c_k b_k - q_2}{2p} \quad a.s. \tag{31}$$

holds. The function $V(t, \mathbf{v}, i) \triangleq \sum_{i=1}^{n} c_k V_k(t, \mathbf{v}_k, i)$ is a Lyapunov-B function for (8), in which c_k is defined as (5). Particularly, the trivial solution of (8) is almost surely exponentially stable.

Proof. For any $\mathbf{v}_{k0} \ne 0$, denote $\mathbf{v}_k(t, \mathbf{x}) \triangleq \mathbf{v}_k(t, \mathbf{x}, t_0, \mathbf{v}_{k0})$. It follows from Lemma 3 that $\mathbf{v}_k(t, \mathbf{x}) \ne 0$ holds almost surely for all, $(t, \mathbf{x}) \in (t_0, \infty) \times G$, $i \in \mathbb{S}$. Applying the Itô formula to (8) gives

$$\begin{aligned} d\left(\int_G V_k(t, \mathbf{v}_k(t, \mathbf{x}), \gamma(t))d\mathbf{x} \right) &= \left[\int_G \mathcal{L}V_k(t, \mathbf{v}_k(t, \mathbf{x}), i)d\mathbf{x} + \int_G \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k} \right)^T \triangle \mathbf{v}_k(t, \mathbf{x})d\mathbf{x} \right] dt \\ &+ \int_G \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k} \right)^T \left[g_k(t, \mathbf{x}, \mathbf{v}_k(t, \mathbf{x}), i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k, \mathbf{v}_j, t, i) \right] d\mathbf{x}dW(t). \end{aligned} \tag{32}$$

By condition C1(I), we have $\frac{\partial^2 V_k(t,\mathbf{v}_k)}{\partial \mathbf{v}_k \partial \mathbf{v}_j} = 0$, $(k \ne j,\ i, j \in N)$. From integration by parts and condition C1. (III) together with boundary condition, we obtain

$$\begin{aligned} \int_G \left(\frac{\partial V_k(t, \mathbf{v}_k, i)}{\partial \mathbf{v}_k} \right)^T \triangle \mathbf{v}_k(t, \mathbf{x})d\mathbf{x} &= \left(\sum_{m=1}^{n_k} \sum_{j=1}^{r} \frac{\partial V_k}{\partial v_{km}} D_{mj}(t, \mathbf{x}, \mathbf{v}) \frac{\partial v_{km}}{\partial x_j} \right)_{\partial G} \\ &- \int_G \sum_{m=1}^{n_k} \sum_{j=1}^{r} D_{mj}(t, \mathbf{x}, \mathbf{v}) \frac{\partial^2 V_k}{\partial v_{km}^2} \left(\frac{\partial v_{km}}{\partial x_j} \right)^2 d\mathbf{x} \le 0 \end{aligned} \tag{33}$$

where $\mathbf{v}_k = (v_{k1}, \ldots, v_{kn_k})^T$. By conditions C2 and C3, we can show in the same way as in the proof of Theorem 1 that

$$\mathcal{L}\int_G V(t,\mathbf{v},i)d\mathbf{x} = \sum_{i=1}^{n} c_k \mathcal{L}\int_G V_k(t,\mathbf{v}_k,i)d\mathbf{x} \le -\sum_{i=1}^{n} c_k b_k \int_G V_k(t,\mathbf{v}_k,i)d\mathbf{x} \le -b\int_G V(t,\mathbf{v},i)d\mathbf{x}, \tag{34}$$

where $b = \min\{b_1, b_2, \ldots, b_n\}$. Hence, we conclude that function $V(t,\mathbf{v},i)$ is a Lyapunov-B function for (8). From (32)–(34), we get

$$\log\left(\int_G V_k(t,\mathbf{v}_k(t,\mathbf{x}),i)d\mathbf{x}\right) \le \log\left(\int_G V_k(t_0,\mathbf{v}_0,\gamma_0)d\mathbf{x}\right) - \sum_{i=1}^{n} c_k b_k (t-t_0) + M(t)$$
$$-\frac{1}{2}\int_{t_0}^{t}\frac{\left|\int_G \left(\frac{\partial V_k(s,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}\right)^T \left[g_k(s,\mathbf{x},\mathbf{v}_k(s,\mathbf{x}),i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k,\mathbf{v}_j,s,i)\right]d\mathbf{x}\right|^2}{\left(\int_G V_k(s,\mathbf{v}_k(s,\mathbf{x}),i)d\mathbf{x}\right)^2}ds \tag{35}$$

where

$$M(t) = \int_{t_0}^{t}\frac{\int_G \left(\frac{\partial V_k(s,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}\right)^T \left[g_k(s,\mathbf{x},\mathbf{v}_k(s,\mathbf{x}),i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k,\mathbf{v}_j,s,i)\right]d\mathbf{x}}{\int_G V_k(s,\mathbf{v}_k(s,\mathbf{x}),i)d\mathbf{x}}dW(s).$$

It is obvious that $M(t)$ is a continuous martingale with initial value $M(t_0) = 0$ when $i = \gamma_0$. Assign $\varepsilon \in (0,1)$ arbitrarily and let $n = 1, 2, \ldots$. It can be deduced from the exponential martingale inequality that

$$\mathbb{P}\left\{\sup_{t_0\le t\le t_0+n}\left[M(t) - \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{\left|\int_G \left(\frac{\partial V_k(s,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}\right)^T \left[g_k(s,\mathbf{x},\mathbf{v}_k(s,\mathbf{x}),i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k,\mathbf{v}_j,s,i)\right]d\mathbf{x}\right|^2}{\left(\int_G V_k(s,\mathbf{v}_k(s,\mathbf{x}),i)d\mathbf{x}\right)^2}ds\right] > \frac{2}{\varepsilon}\log n\right\}$$
$$\le \frac{1}{n^2}.$$

From Lemma Borel–Cantelli [40], it is easy to see there is a corresponding integer $n_0 = n_0(\omega)$ such that if $n \ge n_0$

$$M(t) \le \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{\left|\int_G \left(\frac{\partial V_k(s,\mathbf{v}_k,i)}{\partial \mathbf{v}_k}\right)^T \left[g_k(s,\mathbf{x},\mathbf{v}_k(s,\mathbf{x}),i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k,\mathbf{v}_j,s,i)\right]d\mathbf{x}\right|^2}{\left(\int_G V_k(s,\mathbf{v}_k(s,\mathbf{x}),i)d\mathbf{x}\right)^2}ds + \frac{2}{\varepsilon}\log n$$

holds for all $t_0 \le t \le t_0 + n$. Substituting the above inequality to (35) and noting condition C1(III), we obtain that

$$\log\left(\int_G V_k(t,\mathbf{v}_k(t,\mathbf{x}),i)d\mathbf{x}\right) \le \log\left(\int_G V_k(t_0,\mathbf{v}_0,\gamma_0)d\mathbf{x}\right) - \frac{(1-\varepsilon)q_2 + 2\sum_{i=1}^{n} c_k b_k}{2}(t-t_0) + \frac{2}{\varepsilon}\log n \tag{36}$$

holds for all $t_0 \le t \le t_0 + n$ and $n \ge n_0$ almost surely. Thus, for almost all $\omega \in \Omega$, when $t_0 + n - 1 \le t \le t_0 + n$, $n \ge n_0$

$$\frac{1}{t}\log\left(\int_G V_k(t,\mathbf{v}_k(t,\mathbf{x}),i)d\mathbf{x}\right) \le -\frac{(t-t_0)}{2t}\left[(1-\varepsilon)q_2 + 2\sum_{i=1}^{n} c_k b_k\right]$$
$$+\frac{\log\left(\int_G V_k(t_0,\mathbf{v}_0,\gamma_0)d\mathbf{x}\right) + \frac{2}{\varepsilon}\log n}{t_0 + n - 1} \quad \text{a.s.} \tag{37}$$

This implies

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\int_G V_k(t,\mathbf{v}_k(t,\mathbf{x}),i)d\mathbf{x}\right) \le -\frac{(1-\varepsilon)q_2 + 2\sum_{i=1}^{n} c_k b_k}{2} \quad a.s. \tag{38}$$

Combining condition C1(I), we infer

$$\lim_{t\to\infty}\sup \frac{1}{t}\log\|V_k(t,\mathbf{v}_k(t,\mathbf{x}),i)\| \le -\frac{(1-\varepsilon)q_2+2\sum\limits_{i=1}^{n}c_kb_k}{2p} \quad \text{a.s.} \tag{39}$$

and the required assertion (31) follows since $\varepsilon > 0$ is arbitrary. The proof is complete. □

Remark 2. If in C2, $b_k > 0$ is changed into $b_k \in \mathbb{R}$ such that $2\sum_{k=1}^{N} c_k b_k < q_2$, obviously, the trivial solution of (8) is almost surely exponentially stable.

Corollary 2. *Suppose that the following conditions hold.*

D1. *Assume that there exist positive functions $V_k(t,\mathbf{v}_k,i) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{n_k} \times \mathbb{S};\mathbb{R}_+)$, and constants $p > 0, \alpha > 0$ satisfying*
  (I) *$V_k(t,\mathbf{v},i)$ is separated as to variables $\mathbf{v}_k (k \in N)$ for $i \in \mathbb{S}$;*
  (II)
$$\alpha\|V_k(t,\mathbf{v}_k,i)\|_G^p \le \int_G V_k(t,\mathbf{v}_k,i)d\mathbf{x} \tag{40}$$
  hold for $\forall(t,\mathbf{v}_k(t,\mathbf{x})) \in [t_0,\infty) \times \mathfrak{s}_h^{n_k}$, where $\mathbf{v}_k(t,\cdot) \in \mathfrak{s}_h^{n_k} = \{\zeta : G \to \mathbb{R}^{n_k} \;\|\; \int_G \zeta(\mathbf{x})d\mathbf{x}| < h\}$;
  (III) *$\|(\frac{\partial V_k(t,\mathbf{v}_k,i)}{\partial \mathbf{v}_k})^T\left(g_k(t,\mathbf{x},\mathbf{v}_k(t,\mathbf{x},t_0,\mathbf{v}_{k0}),i) + \sum_{j=1}^{N} N_{kj}(\mathbf{v}_k,\mathbf{v}_j,t,i)\right)\|_G^2 \ge q_2\|V_k(t,\mathbf{v}_k,i)\|_G^2$, for any $\mathbf{v}_k \ne 0$, $k \in N$, $(t,\mathbf{x}_k) \in \mathbb{R}_+ \times G$ for $i \in \mathbb{S}$.*

D2. *There exist functions $F_{kj}(\mathbf{v},\mathbf{v}_j,t)$ and constants $a_{kj} \ge 0$, $\lambda > 0$ such that*
$$\mathcal{L}\int_G V_k(t,\mathbf{v}_k,i)d\mathbf{x} \le -\lambda\int_G V_k(t,\mathbf{v}_k,i)d\mathbf{x} + \sum_{j=1}^{n} a_{kj}F_{kj}(\mathbf{v}_k,\mathbf{v}_j,t), \tag{41}$$
  $t \ge t_0$, for $k = 1,2,\ldots,N$. $i \in \mathbb{S}$. Here LV_k is defined in (9).

D3. *Condition A2 holds, or if $(\mathcal{G},A)$ is balanced and (24) holds.*

Then

$$\lim_{t\to\infty}\sup \frac{1}{t}\lg\|\mathbf{v}(t,\mathbf{x},t_0,\mathbf{v}_0,i)\|_G \le \frac{-\lambda\sum\limits_{k=1}^{N}c_k}{p} \quad a.s. \tag{42}$$

holds. The function $V(t,\mathbf{v},i) \triangleq \sum_{i=1}^{n} c_kV_k(t,\mathbf{v}_k,i)$ is a Lyapunov-B function for (8), in which c_k is defined as (5). Particularly, the trivial solution of (8) is almost surely exponentially stable.

Proof. This corollary is the direct result of Theorem 3, if we choose $q_1 = \alpha$, $b_k = \lambda$ and $q_2 = 0$. □

Remark 3. In this paper, the novel methods of constructing a Lyapunov function to study stability conditions of the SCRDSNs are proposed. Ours are different from some other latest methods on constructing the Lyapunov function [41–46], because our methods build a relation between the stability criteria of a CSN and some topology property of the network. Topology property of the networks is of great importance to the dynamical analysis for the coupled networks. Therefore, for dealing with CSN, our methods are less conservative.

4. Example

Consider the 2-dimensional Itô SRDSMS (1) satisfying the bounded condition (2), and we assume $(\mathcal{G},A)$ is strongly connected and balanced. The Markov chain $\gamma(\cdot)$ is independent of the Brownian motion $W(\cdot)$.

$$\begin{cases} dv_1(t,x) = \left[\triangle v_1(t,x) + \alpha(\gamma(t))v_2(t,x) - \alpha(\gamma(t))v_1(t,x) - \alpha(\gamma(t))\sum\limits_{j=1}^{2} a_{1j}(v_1(t,x) - v_j(t,x))\right]dt, \\ dv_2(t,x) = \left[\triangle v_2(t,x) - \alpha(\gamma(t))v_1(t,x) - 2\alpha(\gamma(t))v_2(t,x) + \alpha(\gamma(t))\sum\limits_{j=1}^{2} a_{2j}(v_2(t,x) - v_j(t,x))\right]dt \\ \qquad + \sqrt{\alpha(\gamma(t))}v_2(t,x)dw(t). \end{cases} \tag{43}$$

Construct function $V = (\int_G v_1(t,x)dx)^2 + (\int_G v_2(t,x)dx)^2$, we have

$$\int_G v(t,x)dx \ge \frac{1}{\|G\|}\|v\|.$$

Besides,

$$\mathcal{L}\int_G Vdx = \int_G \mathcal{L}Vdx = \alpha(\gamma(t))\int_G [2v_1v_2 - 2v_1^2 - 2v_1v_2 - 4v_2^2 + v_2^2]dx + \sum_{j=1}^{2} a_{kj}F_{kj}(v_k, v_j)$$
$$= -2\alpha(\gamma(t))\int_G Vdx < 0$$

where $F_{kj}(v_k, v_j) = 2\int_G v_k^2 - v_j^2 dx$. It is easy to see that along every directed cycle C of the weighted digraph $(\mathcal{G}, A)$,

$$\sum_{(k,j)\in E(C_{\mathcal{Q}})} \left[F_{j\,k}(\mathbf{v}_j, \mathbf{v}_k, t) + F_{kj}(\mathbf{v}_k, \mathbf{v}_j, t)\right] = 0.$$

According to Corollary 1, we deduce

$$\lim_{t\to\infty}\sup \frac{1}{t} \lg \|\mathbf{v}(t, \mathbf{x}, t_0, \mathbf{v}_0, i)\|_G \le -2 \quad \text{a.s.}$$

Therefore, the trial solution of system (43) is almost surely exponentially stable.

5. Conclusions

In this paper, we have investigated some stabilities of some stochastic coupled reaction–diffusion systems on networks (MSRDCSNs). First, a MSRDCSNs model has been proposed. Then, a systematic method for constructing a global Lyapunov function for MSRDCSNs by using graph theory has been presented. This method overcomes the difficulty in finding Lyapunov function for a coupled system. At last, some sufficient conditions for stability of MSRDCSNs are obtained. Future work is to give a systematic approach to build a Lyapunov function for coupled impulsive reaction–diffusion systems on networks.

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