

**Partial Differential Equations** — *Local versus nonlocal interactions in a reaction-diffusion system of population dynamics*, by FABIO PUNZO and TETIANA SAVITSKA

ABSTRACT. — We study existence of patterns for a reaction-diffusion system of population dynamics with nonlocal interaction. We address the system as a bifurcation problem (the bifurcation parameter being the diffusivity of one species), and investigate the possibility of patterns bifurcating out of a constant steady state solution via Turing destabilization. It is shown that the nonlocal character of the interaction enhances the possibility that patterns exist with respect to the case of the companion problem with local interaction.

KEY WORDS: Patterns, Turing destabilization, bifurcation point, asymptotically stable solutions, nonlocal term.

## 1. INTRODUCTION

Reaction-diffusion systems with nonlocal interactions arise in a variety of applications, particularly in models of mathematical biology (*e.g.*, see [4, 6, 7, 10, 11, 19, 20] and references therein), the motivation for their introduction depending on the context. For instance, in epidemiological models it is conceivable that the presence of infectives at some point influences some surrounding region as far as the spread of epidemics is concerned, whereas in population dynamics one can think of a population whose individuals communicate by chemical means, or compete for some resource which can rapidly redistribute itself, *e.g.* by convection. Nonlocal terms in equations modelling population dynamics can also arise by very different factors (*e.g.*, see [2, 5, 18]), or derive by some limiting procedure (as in the “shadow system” associated to some reaction-diffusion system with local interaction [12, 13, 16, 17]).

In this note we address the following reaction-diffusion system with nonlocal interaction (see [8, 9]):

$$(1.1) \quad \begin{cases} u_t = u_{xx} + u(1 - u) - uv & \text{in } \Omega \times \mathbb{R}_+ \\ v_t = \lambda v_{xx} - \chi(u_x v)_x - \beta v + \delta \frac{\langle u, v \rangle}{\langle 1, v \rangle} v - \gamma \frac{uv}{1 + \tau v} & \text{in } \Omega \times \mathbb{R}_+ \\ u_x = v_x = 0 & \text{in } \partial\Omega \times \mathbb{R}_+ \\ u = u_0, v = v_0 & \text{in } \Omega \times \{0\}. \end{cases}$$

Here  $\Omega \equiv (0, 1)$ ,  $\mathbb{R}_+ \equiv (0, \infty)$ ,  $\partial\Omega \equiv \{0, 1\}$ ,  $\beta, \gamma, \delta, \tau$  are positive constant coefficients,  $\lambda > 0$  and  $\chi \geq 0$  will be regarded as parameters, and

$$(1.2) \quad \langle u, v \rangle(t) := \int_0^1 u(x, t) v(x, t) dx, \quad \langle 1, v \rangle(t) := \int_0^1 v(x, t) dx \quad (t \in \mathbb{R}_+)$$

for any measurable  $u, v : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . The unknowns  $v = v(x, t)$  and  $u = u(x, t)$  denote the densities of a population of amoebae, feeding on bacteria, respectively of bacteria belonging to a virulent strain, which can kill amoebae by infecting them—a novel feature with respect to standard predator-prey interaction (in fact, amoebae are attacked by bacteria following a Holling type II functional response, with handling time  $\tau$  and attack rate  $\gamma$ ). However, the main feature of the model is that predation of the amoeboid population on bacteria is governed by

a nonlocal law through the integral term  $\delta \frac{\langle u, v \rangle}{\langle 1, v \rangle} v$ . This describes the fact that amoebae behave like a sole organism when food supply is low, so that food is redistributed among all cells (see [8] for a discussion of this point).

The question we want to address in this note is that of existence of *patterns*, namely, of space dependent stable equilibrium solutions of problem (1.1). Both experimental and numerical evidence support the existence of such solutions, which is related to the pathogenic action of bacteria [9]. Specifically, we wonder whether existence or nonexistence of patterns is affected by the nonlocal character of the interaction. Therefore, we also investigate existence of patterns for the companion problem

$$(1.3) \quad \begin{cases} u_t = u_{xx} + u(1 - u) - uv & \text{in } \Omega \times \mathbb{R}_+ \\ v_t = \lambda v_{xx} - \chi(u_x v)_x - \beta v + \delta uv - \gamma \frac{uv}{1 + \tau v} & \text{in } \Omega \times \mathbb{R}_+ \\ u_x = v_x = 0 & \text{in } \partial\Omega \times \mathbb{R}_+ \\ u = u_0, v = v_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where the *nonlocal term*  $\delta \frac{\langle u, v \rangle}{\langle 1, v \rangle} v$  is replaced by the *local interaction term*  $\delta uv$ .

Our approach is to treat both (1.1) and (1.3) as bifurcation problems, the bifurcation parameter being the *diffusivity*  $\lambda$  of amoebae, and investigate the possibility of patterns bifurcating out of a constant steady state solution. As we shall see, a second relevant parameter in this analysis is the strength  $\chi$  of the *chemotactic term*  $-\chi(u_x v)_x$ .

Both systems (1.1) and (1.3) are space dependent generalizations of the “lumped parameter” Cauchy problem

$$(1.4) \quad \begin{cases} \dot{u} = u(1 - u) - uv & \text{in } \mathbb{R}_+ \\ \dot{v} = -\beta v + \delta uv - \gamma \frac{uv}{1 + \tau v} & \text{in } \mathbb{R}_+ \\ u(0) = u_0, v(0) = v_0. \end{cases}$$

Steady state solutions of problem (1.4) are spatially homogeneous equilibria of both problems (1.1) and (1.3). In particular, we shall be interested in *coexistence equilibria* of (1.4)—namely, in steady state solutions  $\bar{U} \equiv (\bar{u}, \bar{v})$  such that  $\bar{u}, \bar{v} > 0$ . In the following we assume that there exists a coexistence steady state  $\bar{U}$ , with  $0 < \bar{u} < 1$ ,  $0 < \bar{v} < 1$ , which is asymptotically stable with respect to problem (1.4) (note that we can do it, since this occurs for a suitable choice of parameters  $\beta, \delta, \gamma, \tau$  (see [8])). Then we seek conditions on the parameters  $\lambda$  and  $\chi$  ensuring that:

- (i) the steady state  $\bar{U}$  becomes unstable with respect to solutions of problem (1.3)—namely, *Turing destabilization* of  $\bar{U}$  occurs;
- (ii) patterns of problem (1.3) bifurcate from  $\bar{U}$ .

Subsequently, the same question is addressed for problem (1.1), to study whether the conditions of Turing destabilization are affected by the nonlocal character of the interaction.

The main qualitative outcome of the above analysis is that the nonlocal interaction enhances the possibility that patterns exist with respect to the case of local interaction. In fact, in the local case patterns can only exist for small values of the parameter  $\chi$  (see assumption  $(A_1)$ ), a requirement which has no counterpart in the nonlocal case. Moreover, in the local case patterns can exist for a more limited range of values of the diffusivity  $\lambda$  than in the nonlocal case (this is apparent from the subsequent discussion, since the function  $\tilde{\psi}$  defined in (2.20) is always smaller than the function  $\psi$  defined in (2.9)). It is worth observing that these results are in agreement with those proven in [10] for a single reaction-diffusion equation with nonlocal interaction, showing that for such an equation patterns can exist in cases where this is impossible, if a local interaction is considered [3, 15].

## 2. RESULTS

**2.1. Well-posedness.** Let  $C^k(\bar{\Omega})$  denote the space of  $k$  times continuously differentiable functions  $u : \bar{\Omega} \rightarrow \mathbb{R}$ , endowed with the usual norm ( $k \in \mathbb{N} \cup \{0\}$ ;  $C(\bar{\Omega}) \equiv C^0(\bar{\Omega})$ ).

Solutions of problems (1.1) and (1.3) are always meant in the classical sense. The following well-posedness result for problem (1.1) is easily proven. A companion result holds for problem (1.3), whose formulation is left to the reader.

**THEOREM 2.1.** *For any  $u_0, v_0 \in C(\bar{\Omega})$ ,  $u_0 \geq 0$ ,  $v_0 \geq 0$  there exists a unique global solution  $(u, v)$  of problem (1.1). Moreover, there holds  $u > 0$ ,  $v > 0$  in  $\Omega \times \mathbb{R}_+$ .*

**2.2. Existence of patterns: local interaction.** Let us first address the simpler problem (1.3).

Steady state solutions of problem (1.4) are found solving the system

$$(2.1) \quad \begin{cases} F(u, v) := u(1 - u - v) = 0 \\ G(u, v) := \left( -\gamma \frac{u}{1 + \tau v} + \delta u - \beta \right) v = 0. \end{cases}$$

In particular, coexistence equilibria of problem (1.4) are found solving the system

$$(2.2) \quad \begin{cases} u = 1 - v \\ \gamma \frac{u}{1 + \tau v} - \delta u + \beta = 0. \end{cases}$$

Hereafter we set  $F_u \equiv F_u(\bar{U})$ ,  $F_v \equiv F_v(\bar{U})$ ,  $G_u \equiv G_u(\bar{U})$ ,  $G_v \equiv G_v(\bar{U})$ . There holds

$$(2.3) \quad F_u = F_v = -\bar{u},$$

$$(2.4) \quad G_u = \left( \delta - \frac{\gamma}{1 + \tau \bar{v}} \right) \bar{v},$$

$$(2.5) \quad G_v = \frac{\gamma \tau}{(1 + \tau \bar{v})^2} \bar{u} \bar{v}.$$

Denote by

$$(2.6) \quad J \equiv J(\bar{u}, \bar{v}) := \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}$$

the linearized operator of the right-hand side on the solution  $\bar{U}$ . By standard results,  $\bar{U}$  is asymptotically stable with respect to the ODE problem (1.4) if

$$(A_0) \quad F_u + G_v < 0, \quad G_u > G_v;$$

in fact, the above conditions ensure that

$$\text{Tr } J = F_u + G_v < 0, \quad \text{Det } J = F_u G_v - F_v G_u > 0.$$

Let  $\bar{U}$  be a solution of (2.2). We wonder whether the Turing destabilization of  $\bar{U}$ , regarded as a spatially homogeneous equilibrium of (1.3), occurs for some values of the parameters  $\lambda$  and  $\chi$ . It turns out that this can only happen if

$$(A_1) \quad 0 \leq \chi < \chi_0 := \frac{G_v}{\bar{v}|F_v|} = \frac{\gamma \tau}{(1 + \tau \bar{v})^2},$$

and

$$(A_2) \quad 0 < \lambda < \lambda_0 := \frac{1}{|F_u|} (2G_u - G_v + \chi \bar{v} F_v - 2\sqrt{(G_u + \chi \bar{v} F_v)(G_u - G_v)})$$

(observe that  $\lambda_0$  is well defined and positive by assumptions  $(A_0)$ – $(A_1)$ ). More precisely, we have the following result.

**THEOREM 2.2.** *Let  $\bar{U} \equiv (\bar{u}, \bar{v})$  be a stationary solution of problem (1.4) such that  $0 < \bar{u} < 1$ ,  $0 < \bar{v} < 1$ , and let assumption  $(A_0)$  be satisfied. Then the homogeneous steady state  $\bar{U}$  is unstable with respect to problem (1.3) if and only if:*

- (i) *the chemotaxis coefficient  $\chi$  satisfies condition  $(A_1)$ , and the diffusion coefficient  $\lambda$  of amoebae satisfies condition  $(A_2)$ ;*
- (ii) *there exists  $n \in \mathbb{N}$  such that*

$$(2.7) \quad k_-(\lambda, \chi) < k_n := n^2 \pi^2 < k_+(\lambda, \chi),$$

where

$$(2.8) \quad k_{\pm}(\lambda, \chi) := \frac{1}{2\lambda} \{F_u \lambda + G_v + \chi \bar{v} F_v \pm \sqrt{(F_u \lambda + G_v + \chi \bar{v} F_v)^2 + 4\lambda F_u (G_u - G_v)}\}.$$

Observe that the functions  $k_{\pm}$  defined in (2.7) are the roots of the equation

$$(2.9) \quad \psi(\lambda, \chi, k) := \lambda k^2 - (F_u \lambda + G_v + \chi \bar{v} F_v)k - F_u (G_u - G_v) = 0.$$

By assumption  $(A_0)$  and equality (2.3) there holds

$$(2.10) \quad \psi(\lambda, \chi, 0) = F_u (G_v - G_u) = |F_u| (G_u - G_v) > 0,$$

thus positive roots of equation (2.9) need not exist. Existence prevails, if assumptions  $(A_1)$ – $(A_2)$  are satisfied; in fact, in this case there holds  $0 < k_-(\lambda, \chi) < k_+(\lambda, \chi)$  (see Section 3).

In the following of this subsection we assume  $\chi \in [0, \chi_0)$  to be fixed. Accordingly, for any fixed  $\chi \in [0, \chi_0)$  we set  $\psi(\lambda, k) \equiv \psi(\lambda, \chi, k)$  and  $k_{\pm}(\lambda) \equiv k_{\pm}(\lambda, \chi)$ . An elementary analysis shows that (see Figure 2.1):

- (a)  $k_-$  is increasing,  $k_+$  decreasing with  $\lambda \in (0, \lambda_0)$  and

$$(2.11) \quad k_{\pm}(\lambda_0) = \frac{F_u \lambda_0 + G_v + \chi \bar{v} F_v}{2\lambda_0};$$

- (b) there holds

$$(2.12) \quad \lim_{\lambda \rightarrow 0^+} k_-(\lambda) = \frac{|F_u|(G_u - G_v)}{G_v + \chi \bar{v} F_v} > 0, \quad \lim_{\lambda \rightarrow 0^+} k_+(\lambda) = \infty.$$

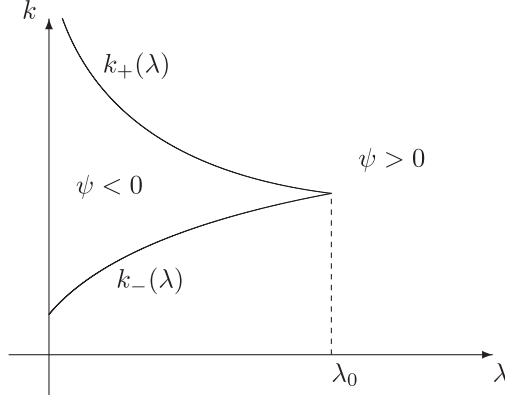


Figure 2.1

The proof of Theorem 2.2 relies on a linearized stability analysis of problem (1.3). The Fréchet derivative of the system in (1.3) at  $\bar{U} \equiv (\bar{u}, \bar{v})$  is the operator-valued matrix

$$\begin{pmatrix} \frac{d^2}{dx^2} + F_u & F_v \\ -\chi \bar{v} \frac{d^2}{dx^2} + G_u & \lambda \frac{d^2}{dx^2} + G_v \end{pmatrix}$$

(see (3.4)), supplemented with homogeneous Neumann boundary conditions. Its spectrum consists of eigenvalues  $\zeta_n \in \mathbb{C}$ , which are the roots of the equation

$$(2.13) \quad \zeta^2 + \phi(\lambda, k_n)\zeta + \psi(\lambda, k_n) = 0$$

where  $k_n := n^2\pi^2$  ( $n \in \mathbb{N} \cup \{0\}$ ) and

$$(2.14) \quad \phi(\lambda, k) := (1 + \lambda)k - (F_u + G_v).$$

Clearly, we are interested in eigenvalues  $\zeta_n$  with positive real part, which turn out to be real positive solutions of equation (2.13). The proof of Theorem 2.2 shows that such solutions exist if and only if the conditions (i)–(ii) of the theorem are satisfied.

Now suppose that assumptions  $(A_0)$ – $(A_2)$  are satisfied. By (2.12) inequality (2.7) is satisfied for any  $n \in \mathbb{N}$  sufficiently large, thus by Theorem 2.2  $\bar{U}$  is unstable for any  $\lambda > 0$  sufficiently small. Then it is natural to conjecture that a pattern of problem (1.3) bifurcates from  $\bar{U}$  at some value  $\lambda \in (0, \lambda_0]$ .

The above question can be addressed by standard methods of bifurcation theory (*e.g.*, see [1, 14]). In fact, let there exist  $n_0 \in \mathbb{N}$  and  $\hat{\lambda}_0 \in (0, \lambda_0]$  such that  $k_{n_0} = k_-(\hat{\lambda}_0)$ . Then  $\psi(\hat{\lambda}_0, k_{n_0}) = \psi(\hat{\lambda}_0, k_-(\hat{\lambda}_0)) = 0$ , and  $\zeta(k_{n_0}) = 0$  is an eigenvalue of the operator  $A_{\hat{\lambda}_0}$ . To avoid technicalities, we only consider the case

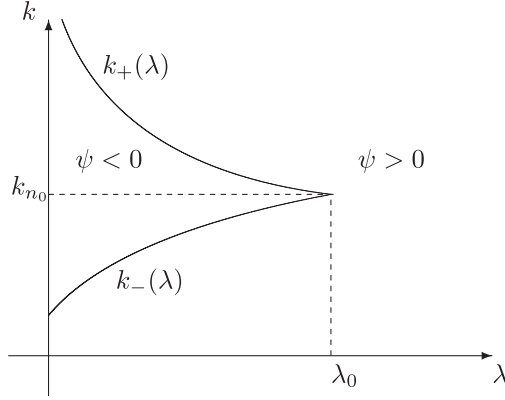


Figure 2.2: Condition (2.15)

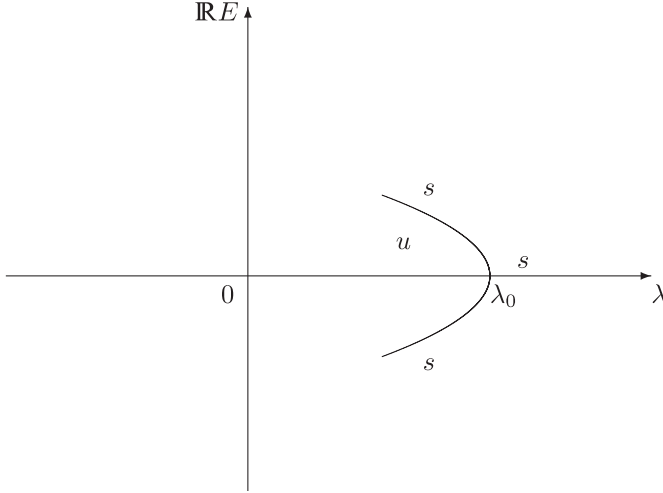


Figure 2.3: Theorem 2.3

when this eigenvalue is simple. This is certainly the case if  $\tilde{\lambda}_0 = \lambda_0$ , since for any  $n \in \mathbb{N} \setminus \{n_0\}$  there holds  $\psi(\lambda_0, k_n) > 0$ , thus the real part of  $\zeta(k_n)$  is negative (see Figure 2.2). Then we have the following result (see Figure 2.3), where the labels  $s$  and  $u$  stand for “stable” and “unstable”, respectively, and  $E$  denotes the eigenvector (3.9).

**THEOREM 2.3.** *Let  $\bar{U}$  be the homogeneous steady state considered in Theorem 2.2. Let assumptions  $(A_0)$ – $(A_2)$  be satisfied. Moreover, suppose that*

$$(2.15) \quad \text{there exists } n_0 \in \mathbb{N} \text{ such that } k_{n_0} = k_{\pm}(\lambda_0).$$

Then:

- (i)  $(\lambda_0, \bar{U})$  is a bifurcation point of stationary solutions of problem (1.3);
- (ii) the bifurcating stationary solutions are nonconstant, and exist in some neighbourhood of the bifurcation point  $(\lambda_0, \bar{U})$ ;
- (iii) the bifurcation is subcritical, and the bifurcating nonconstant stationary solutions are asymptotically stable.

REMARK 2.1. Concerning statement (ii) of Theorem 2.3, the set of bifurcating solutions can be described as follows (see [1, Proposition 26.13]). Denote by  $Y$  the Banach space

$$(2.16) \quad Y := \{U \equiv (u, v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega}) \mid u'(0) = v'(0) = u'(1) = v'(1) = 0\}$$

with norm

$$\|U\|_Y := \sum_{k=0}^2 \{\|u^{(k)}\|_{\infty} + \|v^{(k)}\|_{\infty}\}$$

for any  $U \equiv (u, v) \in Y$ . Then there exist  $\varepsilon > 0$  and a smooth map  $U : (-\varepsilon, \varepsilon) \rightarrow Y$  such that for any  $s \in (-\varepsilon, \varepsilon)$  and  $x \in \bar{\Omega}$  the bifurcating stationary solutions are given by the equality

$$(2.17) \quad U(s, x) = \bar{U} + s \left[ \left( \cos(\sqrt{k_{n_0}}x), \frac{k_{n_0} - F_u}{F_v} \cos(\sqrt{k_{n_0}}x) \right) + y(s, x) \right],$$

where the map  $s \rightarrow y(s, \cdot)$  belongs to  $C^1((-\varepsilon, \varepsilon); N_c)$  for some closed subspace  $N_c \subseteq Y$ , and  $y(0, \cdot) = 0$ . Moreover, there exists a smooth map  $\lambda : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}_+$  such that  $\lambda = \lambda(s)$  for any  $s \in (-\varepsilon, \varepsilon)$ ,  $\lambda \in \mathbb{R}_+$  being the parameter in problem (1.3) and  $\lambda_0 = \lambda(0)$ .

In view of Theorem 2.3 and Remark 2.1, there exists  $\varepsilon > 0$  such that for any  $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$  there exist patterns of problem (1.3). Observe that, under the assumptions of Theorem 2.3, the steady state  $\bar{U}$  is unstable with respect to problem (1.3) for any  $\lambda \in (0, \lambda_0)$ , whereas it is asymptotically stable for any  $\lambda > \lambda_0$  (see Theorem 2.2 and Figure 2.3).

REMARK 2.2. Conclusions similar to those of Theorem 2.3 hold in more general situations. In fact, let there exist  $n_0 \in \mathbb{N}$  such that

$$(A_3)(i) \quad k_-(0) < k_{n_0} < k_{\pm}(\lambda_0).$$

Then there exists a unique  $\tilde{\lambda}_0 \in (0, \lambda_0)$  such that  $k_{n_0} = k_-(\tilde{\lambda}_0)$  (see Figure 2.4a). Suppose that

$$(A_3)(ii) \quad k_{n_0+1} > k_+(\tilde{\lambda}_0).$$

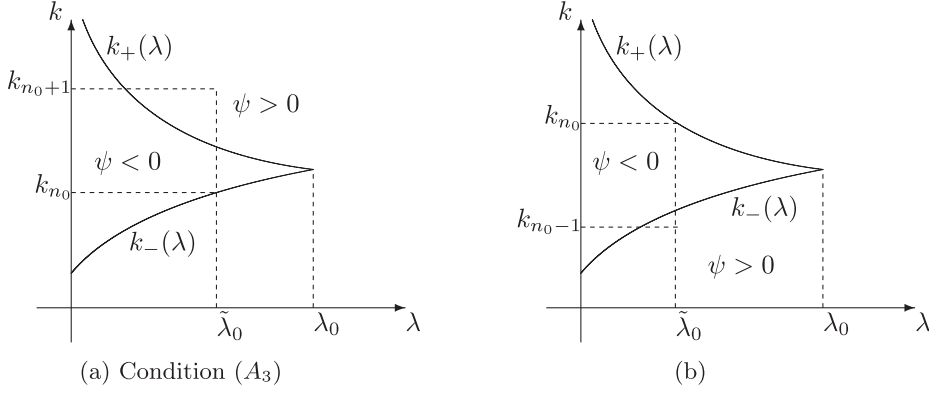


Figure 2.4

Since the function  $\psi$  is increasing in  $(k_{\pm}(\lambda_0), \infty)$ , this implies that  $\psi(k_n) > 0$  for every  $n \geq n_0 + 1$ . Plainly, it follows that the real part of  $\zeta(k_n)$  is negative for any  $n \in \mathbb{N} \setminus \{n_0\}$ . Then the same conclusions of Theorem 2.3 hold with  $\lambda_0$  replaced by  $\tilde{\lambda}_0$ . Similar remarks hold in analogous situations (*e.g.*, see Figure 2.4b); we leave their formulation to the reader.

**2.3. Existence of patterns: nonlocal interaction.** Let us now regard the coexistence steady state  $\bar{U}$  as a spatially homogeneous equilibrium of problem (1.1). It will be seen below that in this case the functions  $k_{\pm}$  of the previous analysis (see (2.8)) are replaced by

$$(2.18) \quad \tilde{k}_{\pm}(\lambda, \chi) := \frac{1}{2\lambda} \{F_u \lambda + G_v + \chi \bar{v} F_v \pm \sqrt{(F_u \lambda + G_v + \chi \bar{v} F_v)^2 - 4\lambda F_u (G_v - G_u + \delta \bar{v})}\},$$

which are the roots of the equation

$$(2.19) \quad \tilde{\psi}(\lambda, \chi, k) = 0;$$

here

$$(2.20) \quad \begin{aligned} \tilde{\psi}(\lambda, \chi, k) &:= \psi(\lambda, \chi, k) + \delta \bar{v} F_v \\ &= \lambda k^2 - (F_u \lambda + G_v + \chi \bar{v} F_v)k + F_u (G_v - G_u + \delta \bar{v}) \end{aligned}$$

Observe that, at variance from the previous case (see (2.10)), there holds

$$(2.21) \quad \tilde{\psi}(\lambda, \chi, 0) = F_u (G_v - G_u + \delta \bar{v}) = -\gamma \frac{\bar{u} \bar{v}}{(1 + \tau \bar{v})^2} [1 + \tau(\bar{u} + \bar{v})] < 0.$$

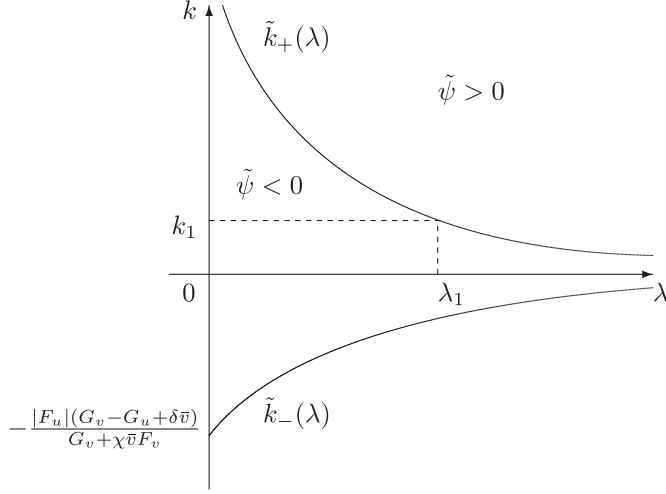


Figure 2.5

Hence for any  $\lambda > 0$  and  $\chi \geq 0$

$$(2.22) \quad \tilde{k}_-(\lambda, \chi) < 0 < \tilde{k}(\lambda, \chi)$$

(see Figure 2.5, where  $\chi \geq 0$  is fixed). The root  $\tilde{k}_-$  has no role in the subsequent analysis since it is always negative, thus we set  $\tilde{k} \equiv \tilde{k}_+$  hereafter. Observe that assumptions  $(A_1)$ – $(A_2)$  have no counterpart in the present case. However, it is worth mentioning that

$$\lim_{\lambda \rightarrow 0^+} \tilde{k}_-(\lambda) = \begin{cases} \frac{F_u(G_v - G_u + \delta \bar{v})}{G_v + \chi \bar{v} F_v} < 0 & \text{if } (A_1) \text{ holds} \\ -\infty & \text{otherwise.} \end{cases}$$

Let  $\chi \geq 0$  be fixed, and set  $\tilde{\psi}(\lambda, k) \equiv \tilde{\psi}(\lambda, \chi, k)$ ,  $\tilde{k}(\lambda) \equiv \tilde{k}(\lambda, \chi)$ . It is easily checked that  $\tilde{k}$  is decreasing in  $(0, \infty)$ , and

$$\lim_{\lambda \rightarrow 0^+} \tilde{k}(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \infty} \tilde{k}(\lambda) = 0.$$

Denote by  $\lambda_1 \in (0, \infty)$  the unique root of the equation  $\tilde{k}(\lambda) = k_1$ , namely

$$(2.23) \quad \lambda = \lambda_1 \Leftrightarrow \tilde{k}(\lambda) = k_1$$

(recall that  $k_1 := \pi^2$ ). Arguing as in Subsection 2.2, we obtain the following result.

**THEOREM 2.4.** *Let  $\bar{U} \equiv (\bar{u}, \bar{v})$  be a stationary solution of problem (1.4) such that  $0 < \bar{u} < 1$ ,  $0 < \bar{v} < 1$ , and let assumption  $(A_0)$  be satisfied. Let  $\lambda_1 \in (0, \infty)$  be the*

unique root of the equation  $\tilde{k}(\lambda) = k_1$ . Then the homogeneous steady state  $\bar{U}$  is unstable with respect to problem (1.1) if and only if  $\lambda \in (0, \lambda_1)$ .

As in the case of local interaction, the proof of Theorem 2.4 is based on a linearized stability analysis of problem (1.1). The Fréchet derivative of the system in (1.1) at  $\bar{U} \equiv (\bar{u}, \bar{v})$  is the operator-valued matrix

$$(2.24) \quad \begin{pmatrix} \frac{d^2}{dx^2} + F_u & F_v \\ -\chi \bar{v} \frac{d^2}{dx^2} + G_u + \delta \bar{v}[\langle 1, \cdot \rangle - 1] & \lambda \frac{d^2}{dx^2} + G_v \end{pmatrix}$$

(see Section 4), supplemented with homogeneous Neumann boundary conditions; here the linear functional  $\langle 1, \cdot \rangle$  is defined in (1.2),  $F_u \equiv F_u(\bar{u}, \bar{v})$ , and so on. By analogy with the situation encountered for the case of local interaction, it is natural to conjecture that  $(\bar{U}, \lambda_1)$  be a bifurcation point of patterns of problem (1.1). The affirmative answer is the content of the following theorem.

**THEOREM 2.5.** *Let  $\bar{U}$  be the homogeneous steady state considered in Theorem 2.2, and let assumption  $(A_0)$  be satisfied. Let  $\lambda_1 \in (0, \infty)$  be the unique root of the equation  $\tilde{k}(\lambda) = k_1$ . Then the conclusions of Theorem 2.3 hold true, with  $\lambda_0$  replaced by  $\lambda_1$ . Moreover, the nonconstant bifurcating stationary solutions are of the form (2.17) with  $k_{n_0}$  replaced by  $k_1$ .*

### 3. LOCAL INTERACTION: PROOFS

Consider the Banach space  $X := C(\bar{\Omega}) \times C(\bar{\Omega})$  endowed with the norm

$$\|U\|_X := \|u\|_\infty + \|v\|_\infty \quad (U \equiv (u, v) \in X).$$

Define a bounded nonlinear operator  $\mathcal{F} : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \times Y \rightarrow X$ , with  $Y$  as in (2.16)), by setting

$$(3.1) \quad \mathcal{F}(\lambda, \chi, U) := \begin{pmatrix} u'' + F(u, v) \\ \lambda v'' - \chi(u'v)' + G(u, v) \end{pmatrix}$$

for any  $\lambda > 0$ ,  $\chi \geq 0$  and  $U \equiv (u, v) \in Y$ . Then problem (1.3) reads as the abstract Cauchy problem

$$(3.2) \quad \begin{cases} U_t = \mathcal{F}(\lambda, \chi, U) & \text{in } \mathbb{R}_+ \\ U(0) = U_0 := (u_0, v_0). \end{cases}$$

**PROOF OF THEOREM 2.1.** For every  $\lambda > 0$ ,  $\chi \geq 0$  the map  $\mathcal{F}(\lambda, \chi, \cdot) : Y \rightarrow X$  is locally Lipschitz continuous, thus for each  $U_0 \in X$  a unique local solution exists. The solution is global by elementary a priori estimates. The claim concerning nonnegativity follows by the maximum principle.

To prove Theorem 2.2 we need a linearized stability analysis of problem (1.3), which is conveniently thought of in the abstract form (3.2). If so, stationary solutions of problem (1.3) satisfy

$$(3.3) \quad \mathcal{F}(\lambda, \chi, U) = 0.$$

Clearly, for any  $U \equiv (u_1, v_1) \in Y$

$$\mathcal{F}_U(\lambda, \chi, U)U_1 = \begin{pmatrix} u_1'' + F_u(u, v)u_1 + F_v(u, v)v_1 \\ \lambda v_1'' - \chi[(u'v_1)' + (u_1'v)'] + G_u(u, v)u_1 + G_v(u, v)v_1 \end{pmatrix};$$

hereafter, by  $\mathcal{F}_U, \mathcal{F}_{\lambda U}, \mathcal{F}_{UU}, \mathcal{F}_{UUU}$  we denote the Fréchet partial derivatives of  $\mathcal{F}$  with respect to its arguments. Observe that  $\mathcal{F}_U(\lambda, \chi, U) \in \mathcal{L}(Y, X)$  ( $\mathcal{L}(W, Z)$ ) denoting the space of bounded linear operators from the Banach space  $W$  to the Banach space  $Z$ .

Let  $\bar{U} \equiv (\bar{u}, \bar{v})$  be a stationary solution of problem (1.4). By the above equality, the linearized operator at  $\bar{U}$  of the right-hand side of (3.2) is

$$(3.4) \quad A_{\lambda, \chi} \equiv \mathcal{F}_U(\lambda, \chi, \bar{U}) = \begin{pmatrix} \frac{d^2}{dx^2} + F_u & F_v \\ -\chi \bar{v} \frac{d^2}{dx^2} + G_u & \lambda \frac{d^2}{dx^2} + G_v \end{pmatrix},$$

where  $F_u \equiv F_u(\bar{u}, \bar{v})$ , and so on. It is easily seen that the linearized operator  $A_{\lambda, \chi}$  has compact resolvent, thus purely point spectrum. Its eigenvalues are the roots  $\zeta_n \in \mathbb{C}$  of the equation

$$(3.5) \quad \begin{vmatrix} \zeta + k_n - F_u & -F_v \\ -\chi \bar{v} k_n - G_u & \zeta + \lambda k_n - G_v \end{vmatrix} = 0 \quad \Leftrightarrow \quad \zeta^2 + \phi(\lambda, k_n)\zeta + \psi(\lambda, \chi, k_n) = 0,$$

where  $k_n := n^2\pi^2$  ( $n \in \mathbb{N} \cup \{0\}$ ) and the functions  $\phi, \psi$  are defined by (2.14), respectively (2.9) (here use of equalities (2.3) has been made). The corresponding eigenfunctions are

$$(3.6) \quad \Phi_n \equiv (\varphi_1^n, \varphi_2^n) = (a \cos(\sqrt{k_n}x), b \cos(\sqrt{k_n}x)),$$

with  $a, b \in \mathbb{R}$  to be chosen. By the completeness of the trigonometric system it is easily seen that no other eigenfunctions and eigenvalues exist.

In the following we suppose that assumption  $(A_0)$  is satisfied. Let us seek conditions on the parameter  $\lambda$  ensuring that some eigenvalue  $\zeta_n$  of the linearized operator  $A_\lambda$  has positive real part, so that the steady state  $(\bar{u}, \bar{v})$  becomes unstable with respect to solutions of the PDE problem (1.3). In fact, this amounts to prove Theorem 2.2.

**PROOF OF THEOREM 2.2.** Every complex root  $\zeta = \zeta_1 + i\zeta_2$  of the equation

$$(3.7) \quad \zeta^2 + \phi(\lambda, k)\zeta + \psi(\lambda, \chi, k) = 0 \quad (k \geq 0)$$

satisfies the system

$$\begin{cases} \zeta_1^2 - \zeta_2^2 + \phi(\lambda, k)\zeta_1 + \psi(\lambda, \chi, k) = 0 \\ [2\zeta_1 + \phi(\lambda, k)]\zeta_2 = 0. \end{cases}$$

Since we seek solutions with  $\zeta_1 > 0$ , and there holds  $\phi(\lambda, k) > 0$  for any  $\lambda, k \geq 0$  (see (2.14) and recall that  $F_u + G_v < 0$  by assumption  $(A_0)$ ), the second equation gives  $\zeta_2 = 0$ . Hence solutions of the above system with  $\zeta_1 > 0$  exist, if and only if there exist real positive solutions of equation (3.7). Since  $\phi(\lambda, k) > 0$  for any  $\lambda, k \geq 0$ , this happens if and only if  $\psi(\lambda, \chi, k) < 0$  for some  $\lambda > 0$ ,  $\chi \geq 0$  and  $k > 0$ .

By equalities (2.8)–(2.9) and (2.10) there holds

$$\psi(\lambda, \chi, k) < 0 \quad \text{for some } k > 0 \quad \Leftrightarrow \quad \begin{cases} F_u\lambda + G_v + \chi\bar{v}F_v > 0 \\ (F_u\lambda + G_v + \chi\bar{v}F_v)^2 + 4F_u(G_u - G_v)\lambda > 0. \end{cases}$$

The second inequality of the above system is satisfied if either  $\lambda < \lambda_0$ , or

$$\lambda > \lambda^{(2)} := \frac{1}{|F_u|} (2G_u - G_v + \chi\bar{v}F_v + 2\sqrt{(G_u + \chi\bar{v}F_v)(G_u - G_v)}),$$

whereas the first inequality yields

$$\lambda < \lambda^{(1)} := \frac{G_v + \chi\bar{v}F_v}{|F_u|}.$$

By assumptions  $(A_0)$ – $(A_1)$  there holds  $0 < \lambda_0 < \lambda^{(1)} < \lambda^{(2)}$ , thus condition  $(A_2)$  ensures that the system of inequalities above is satisfied.

Therefore, if  $\chi$  and  $\lambda$  satisfy conditions  $(A_1)$  and  $(A_2)$  respectively, there exists  $k > 0$  such that  $\psi(\lambda, \chi, k) < 0$ . On the other hand, there holds  $\psi(\lambda, \chi, 0) > 0$  (see (2.10)) and  $\psi(\lambda, \chi, k) \rightarrow \infty$  as  $k \rightarrow \infty$ , since by assumption  $\lambda > 0$ . Then, by continuity, for every  $\chi \in [0, \chi_0)$  and  $\lambda \in (0, \lambda_0)$  there exist  $0 < k_-(\lambda, \chi) < k_+(\lambda, \chi)$  such that  $\psi(\lambda, \chi, k_\pm(\lambda, \chi)) = 0$  and  $\psi(\lambda, \chi, k) < 0$  for any  $k \in (k_-(\lambda, \chi), k_+(\lambda, \chi))$ . Therefore, an eigenvalue of the linearized operator  $A_\lambda$  with positive real part (namely, a real positive solution of equation (3.7) with  $k = k_n$ ) exists if and only if inequality (2.7) is satisfied for some  $n \in \mathbb{N}$ . This completes the proof.

In the remaining part of this section we suppose that  $\chi \in [0, \chi_0)$  is fixed, thus we denote by  $\mathcal{F}(\lambda, \bar{U}) \equiv \mathcal{F}(\lambda, \chi, \bar{U})$  the operator defined in (3.1).

If condition (2.15) is satisfied, the roots of equation (2.13) are  $\underline{\zeta}_+(k_{n_0}) = 0$ ,  $\underline{\zeta}_-(k_{n_0}) = -\phi(\lambda, k_{n_0}) < 0$ . Then the linearized operator  $A_{\lambda_0} \equiv \mathcal{F}_U(\lambda_0, \bar{U})$  is not invertible, since it has an eigenvalue equal to zero. Moreover, if assumptions  $(A_0)$ – $(A_2)$  are satisfied,  $\bar{U}$  is unstable with respect to problem (1.3) for any  $\lambda \in (0, \lambda_0)$ . This suggests that the point  $(\lambda_0, \bar{U})$  is a bifurcation point of equation (3.3), as in fact the following proposition shows.

**PROPOSITION 3.1.** *Let the assumptions of Theorem 2.3 be satisfied. Then the statements (i)–(ii) of the same theorem hold true.*

To prove Proposition 3.1 we need some preliminary remarks. Set  $\Phi \equiv (\varphi_1, \varphi_2) \in Y$ . Then the eigenvalue equation  $A_{\lambda_0}\Phi = \zeta\Phi$  reads (see (3.4))

$$(3.8) \quad \begin{cases} \varphi_1'' + F_u\varphi_1 + F_v\varphi_2 = \zeta\varphi_1 \\ \lambda_0\varphi_2'' - \chi\bar{v}\varphi_1'' + G_u\varphi_1 + G_v\varphi_2 = \zeta\varphi_2 \quad \text{in } \Omega \\ \varphi_1'(0) = \varphi_2'(0) = \varphi_1'(1) = \varphi_2'(1) = 0. \end{cases}$$

It is immediately checked that any vector  $E \in Y$ ,

$$(3.9) \quad E \equiv (e_1, e_2) := (a \cos(\sqrt{k_{n_0}}x), b \cos(\sqrt{k_{n_0}}x)) \quad (x \in \bar{\Omega})$$

(see (3.6)) is an eigenvector of the linearized operator  $A_{\lambda_0}$  with eigenvalue 0, if  $k_{n_0}$  and  $\lambda_0$  are related by equality (2.15) and

$$(3.10) \quad a \in \mathbb{R} \setminus \{0\}, \quad b := \frac{k_{n_0} - F_u}{F_v} a.$$

Observe that in this case the first equality in (3.5) reads

$$(3.11) \quad \begin{vmatrix} k_{n_0} - F_u & -F_v \\ -\chi\bar{v}k_{n_0} - G_u & k_{n_0}\lambda_0 - G_v \end{vmatrix} = 0.$$

It has been already observed that the eigenvalue 0 is simple, thus the kernel  $\mathcal{N}(A_{\lambda_0}) \subseteq Y$  of the operator  $A_{\lambda_0}$  coincides with the linear span of the vector  $E$ .

Consider also any vector  $E^* \in Y$ ,

$$(3.12) \quad E^* \equiv (e_1^*, e_2^*) := (a^* \cos(\sqrt{k_{n_0}}x), b^* \cos(\sqrt{k_{n_0}}x)) \quad (x \in \bar{\Omega}),$$

with  $k_{n_0}$  and  $\lambda_0$  related by equality (2.15) and

$$(3.13) \quad a^* \in \mathbb{R} \setminus \{0\} \quad b^* := \frac{F_v}{k_{n_0}\lambda_0 - G_v} a^*.$$

It is easily checked that  $E^*$  is an eigenvector with eigenvalue 0 of the formal adjoint  $A_{\lambda_0}^*$  of  $A_{\lambda_0}$ ,

$$A_{\lambda_0}^* := \begin{pmatrix} \frac{d^2}{dx^2} + F_u & -\chi\bar{v}\frac{d^2}{dx^2} + G_u \\ F_v & \lambda_0\frac{d^2}{dx^2} + G_v \end{pmatrix}.$$

It is also easily seen that

$$(3.14) \quad ((E^*, Z)) = 0 \quad \text{for any } Z \in \mathcal{R}(A_{\lambda_0}),$$

where  $\mathcal{R}(A_{\lambda_0}) \subseteq X$  denotes the range of the operator  $A_{\lambda_0}$  and  $((\cdot, \cdot))$  the scalar product in  $L^2(\Omega) \times L^2(\Omega)$ , namely

$$((F, G)) := \int_0^1 \{f_1 g_1 + f_2 g_2\} dx$$

for any  $F \equiv (f_1, f_2)$ ,  $G \equiv (g_1, g_2) \in L^2(\Omega) \times L^2(\Omega)$ . In fact, let  $Z \equiv (z_1, z_2) \in \mathcal{R}(A_{\lambda_0})$ . Then there exists  $W \equiv (w_1, w_2) \in Y$  such that  $Z = A_{\lambda_0} W$ , namely

$$\begin{cases} w_1'' + F_u w_1 + F_v w_2 = z_1 \\ \lambda_0 w_2'' - \chi \bar{v} w_1'' + G_u w_1 + G_v w_2 = z_2 \quad \text{in } \Omega. \end{cases}$$

Since  $w_1'(0) = w_2'(0) = w_1'(1) = w_2'(1) = 0$ , by the definition of  $b$  (see (3.10)) and equality (3.11) there holds

$$\begin{aligned} ((E^*, Z)) &= \frac{1}{2} \{a^* [(F_u - k_{n_0})a + F_v b] \\ &\quad + b^* [(\chi \bar{v} k_{n_0} + G_u)a + (G_v - k_{n_0} \lambda_0)b]\} = 0. \end{aligned}$$

Further, observe that

$$\begin{aligned} ((E^*, E)) &= (aa^* + bb^*) \int_0^1 \cos^2(\sqrt{k_{n_0}} x) dx \\ &= \frac{aa^*}{2} \left[ 1 + \frac{k_{n_0} - F_u}{k_{n_0} \lambda_0 - G_v} \right]. \end{aligned}$$

Then choosing

$$(3.15) \quad a^* := \frac{2}{a} \frac{k_{n_0} \lambda_0 - G_v}{k_{n_0} (\lambda_0 + 1) - (F_u + G_v)}$$

we have

$$(3.16) \quad ((E^*, E)) = 1.$$

Without loss of generality, hereafter we suppose

$$(3.17) \quad a > 0$$

and  $b, b^*, a^*$  chosen as in (3.10), (3.13) and (3.15), respectively. Observe that by equalities (2.3), (2.11) and (2.15)

$$(3.18) \quad k_{n_0} \lambda_0 - G_v = -\frac{|F_u| \lambda_0 + G_v + \chi \bar{v} |F_v|}{2} < 0.$$

Then by assumption  $(A_0)$ , equality (2.3) and inequality (3.18) there holds

$$(3.19) \quad b < 0, \quad a^* < 0, \quad b^* < 0.$$

Now we can prove Proposition 3.1.

**PROOF OF PROPOSITION 3.1.** Consider the second Fréchet derivative

$$\mathcal{F}_{\lambda U}(\lambda, \bar{U}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}$$

(observe that  $\mathcal{F}_{\lambda U}(\lambda, \bar{U}) \in \mathcal{L}(\mathbb{R}, \mathcal{L}(Y, X)) \simeq \mathcal{L}(Y, X)$ ). By the Lyapunov-Schmidt theorem (*e.g.*, see [1, Theorem 26.13]), the result will follow if we prove that

$$(3.20) \quad \mathcal{F}_{\lambda U}(\lambda, \bar{U})[\mathcal{N}(A_{\lambda_0})] \not\subseteq \mathcal{R}(A_{\lambda_0}),$$

where  $\mathcal{N}(A_{\lambda_0}) \subseteq Y$  denotes the kernel of the operator  $A_{\lambda_0}$  and  $\mathcal{F}_{\lambda U}(\lambda, \bar{U})[\mathcal{N}(A_{\lambda_0})]$  its image under the operator  $\mathcal{F}_{\lambda U}(\lambda, \bar{U})$ .

Let  $E, E^*$  be the vectors defined in (3.9) and (3.12). Clearly, there holds

$$\mathcal{F}_{\lambda U}(\lambda, \bar{U})E = (0, -bk_{n_0} \cos(\sqrt{k_{n_0}}x)),$$

whence by (3.19)

$$(3.21) \quad ((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E)) = -bb^*k_{n_0} \int_0^1 \cos^2(\sqrt{k_{n_0}}x) dx = -\frac{bb^*k_{n_0}}{2} < 0.$$

Since  $\mathcal{N}(A_{\lambda_0})$  coincides with the linear span of the vector  $E$ , by equality (3.14) and inequality (3.21) we obtain (3.20). Then the conclusion follows.

It is easily seen that for any  $U_1 \equiv (u_1, v_1)$  and  $U_2 \equiv (u_2, v_2) \in Y$

$$\begin{aligned} & \mathcal{F}_{UU}(\lambda, \bar{U})U_1U_2 \\ &= \begin{pmatrix} F_{uu}u_1u_2 + F_{uv}(u_1v_2 + u_2v_1) + F_{vv}v_1v_2 \\ -\chi[(u'_2v_1)' + (u'_1v_2)'] + G_{uu}u_1u_2 + G_{uv}(u_1v_2 + u_2v_1) + G_{vv}v_1v_2 \end{pmatrix}, \end{aligned}$$

where  $F_u \equiv F_u(\bar{u}, \bar{v})$  and so on, whereas

$$\begin{aligned} & \mathcal{F}_{UUU}(\lambda, \bar{U})U_1U_2U_3 \\ &= \begin{pmatrix} F_{uuu}u_1u_2u_3 + F_{uuv}\alpha_1(U_1, U_2, U_3) + F_{uvv}\alpha_2(U_1, U_2, U_3) + F_{vvv}v_1v_2v_3 \\ G_{uuu}u_1u_2u_3 + G_{uuv}\alpha_1(U_1, U_2, U_3) + G_{uvv}\alpha_2(U_1, U_2, U_3) + G_{vvv}v_1v_2v_3 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha_1(U_1, U_2, U_3) &= u_1u_2v_3 + u_1v_2u_3 + v_1u_2u_3, \\ \alpha_2(U_1, U_2, U_3) &= u_1v_2v_3 + v_1u_2v_3 + v_1v_2u_3 \end{aligned}$$

for any  $U_1 \equiv (u_1, v_1)$ ,  $U_2 \equiv (u_2, v_2)$  and  $U_3 \equiv (u_3, v_3) \in Y$ . Observe that  $\mathcal{F}_{UU}(\lambda, \bar{U}) \in \mathcal{L}(Y, \mathcal{L}(Y, X)) \simeq \mathcal{L}_2(Y \times Y, X)$  and  $\mathcal{F}_{UUU}(\lambda, \bar{U}) \in \mathcal{L}(Y, \mathcal{L}_2(Y \times Y, X)) \simeq \mathcal{L}_3(Y \times Y \times Y, X)$  ( $\mathcal{L}_n(W, Z)$ ,  $n \in \mathbb{N}$ , denoting the space of bounded multilinear operators from the Banach space  $W := \underbrace{Y \times \cdots \times Y}_{n \text{ times}}$  to the Banach space  $X$ ). If  $U_1 = U_2 = U_3$  we write  $\mathcal{F}_{UU}(\lambda, \bar{U})U_1^2$

and  $\mathcal{F}_{UUU}(\lambda, \bar{U})U_1^3$ , with obvious meaning of the symbols.

Now we can complete the proof of Theorem 2.3.

**PROOF OF THEOREM 2.3.** In view of Proposition 3.1, we only have to prove statement (iii).

Let us first prove that the bifurcation is subcritical. Let  $\lambda : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}_+$  ( $\varepsilon > 0$ ),  $\lambda_0 = \lambda(0)$  be the smooth map which appears in the parametrization of the bifurcation curve  $(\lambda(s), U(s)) \subseteq \mathbb{R}_+ \times Y$  (see Remark 2.1). By [1, Remark 27.6] and the proof of [1, Proposition 27.7], we have:

$$(3.22) \quad \lambda'(0) = -\frac{1}{2} \frac{((E^*, \mathcal{F}_{UU}(\lambda_0, \bar{U})E^2))}{((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E))},$$

$$(3.23) \quad \lambda''(0) = -\frac{1}{3} \frac{((E^*, \mathcal{F}_{UUU}(\lambda_0, \bar{U})E^3))}{((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E))}.$$

Then by [1, Proposition 27.7] and (3.20) the claim will follow, if we prove that

$$(3.24) \quad \lambda'(0) = 0, \quad \lambda''(0) < 0.$$

Recalling equality (3.9) and the definition of the functions  $F, G$  (see (2.1)), from the above expressions of  $\mathcal{F}_{UU}(\lambda, \bar{U})U_1U_2$  and  $\mathcal{F}_{UUU}(\lambda, \bar{U})U_1U_2U_3$  we obtain

$$(3.25) \quad \mathcal{F}_{UU}(\lambda_0, \bar{U})E^2 = \begin{pmatrix} F_{uu}e_1^2 + 2F_{uv}e_1e_2 + F_{vv}e_1^2 \\ -2\chi(e_1'e_2)' + G_{uu}e_1^2 + 2G_{uv}e_1e_2 + G_{vv}e_2^2 \end{pmatrix} \\ = 2 \begin{pmatrix} -e_1^2 - e_1e_2 \\ -\chi(e_1'e_2)' + \left[ \delta - \frac{\gamma}{(1 + \tau\bar{v})^2} \right] e_1e_2 + \frac{\gamma\tau\bar{u}}{(1 + \tau\bar{v})^3} e_2^2 \end{pmatrix},$$

respectively

$$\mathcal{F}_{UUU}(\lambda_0, \bar{U})E^3 = \begin{pmatrix} F_{uuu}e_1^3 + 3F_{uuv}e_1^2e_2 + 3F_{uvv}e_1e_2^2 + F_{vvv}e_2^3 \\ G_{uuu}e_1^3 + 3G_{uuv}e_1^2e_2 + 3G_{uvv}e_1e_2^2 + G_{vvv}e_2^3 \end{pmatrix} \\ = \begin{pmatrix} 0 \\ \frac{6\gamma\tau}{(1 + \tau\bar{v})^3} e_1e_2^2 - \frac{6\gamma\tau^2\bar{u}}{(1 + \tau\bar{v})^4} e_2^3 \end{pmatrix}.$$

It is easily checked that

$$\begin{aligned}
((E^*, \mathcal{F}_{UU}(\lambda_0, \bar{U})E^2)) &= 2 \left\{ -a^2 a^* - aa^* b + \left[ \delta - \frac{\gamma}{(1 + \tau\bar{v})^2} \right] abb^* \right. \\
&\quad \left. + \frac{\gamma\tau\bar{u}}{(1 + \tau\bar{v})^3} b^2 b^* \right\} \int_0^1 \cos^3(\sqrt{k_{n_0}}x) dx \\
&\quad - 2\chi abb^* k_{n_0} \int_0^1 \sin^2(\sqrt{k_{n_0}}x) \cos(\sqrt{k_{n_0}}x) dx = 0,
\end{aligned}$$

whence  $\lambda'(0) = 0$  by equality (3.22). Moreover, there holds

$$((E^*, \mathcal{F}_{UUU}(\lambda_0, \bar{U})E^3)) = \frac{6\gamma\tau b^2 b^*}{(1 + \tau\bar{v})^3} \left( a - \frac{\tau\bar{u}}{1 + \tau\bar{v}} b \right) \int_0^1 \cos^4(\sqrt{k_{n_0}}x) dx < 0$$

(here use of (3.17) and (3.19) has been made). Then by (3.21) and (3.23) we obtain that  $\lambda''(0) < 0$ . This proves (3.24), whence the claim follows.

Let us now prove that the stationary bifurcating solutions  $U(s) \equiv (u(s), v(s))$  (see (2.17)) are asymptotically stable. By [1, Proposition 26.24] there exists a unique continuation  $\kappa(s) \in \sigma(\mathcal{F}_U(\lambda(s), U(s)))$  of the zero eigenvalue of  $A_{\lambda_0} \equiv \mathcal{F}_U(\lambda_0, \bar{U})$  along the curve  $\{U(s) \mid s \in (-\varepsilon, \varepsilon)\}$  of bifurcating solutions—namely, there exists a smooth function  $(\kappa, \tilde{E}) : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R} \times Y$ , with  $\kappa(0) = 0$  and  $\tilde{E}(0) = E$ , such that

$$\mathcal{F}_U(\lambda(s), U(s))\tilde{E}(s) = \kappa(s)\tilde{E}(s) \quad \text{for any } s \in (-\varepsilon, \varepsilon).$$

By [1, Theorem 27.2] there exists (possibly for some smaller  $\varepsilon$ ) a function  $\alpha \in C((-\varepsilon, \varepsilon), \mathbb{R})$  such that

$$(3.26) \quad \kappa(s) = \alpha(s)s\lambda'(s) \quad \text{for any } s \in (-\varepsilon, \varepsilon);$$

moreover,

$$(3.27) \quad \alpha(0) = -((E^*, \mathcal{F}_{\lambda U}(\lambda_0, \bar{U})E)).$$

Since  $\lambda'(0) = 0$ , by (3.26) we have

$$(3.28) \quad \kappa(s) = \alpha(s)[s^2\lambda''(0) + o(s^2)] \quad \text{as } s \rightarrow 0,$$

where  $o(s^2)$  denotes a term of higher order with respect to  $s^2$ . On the other hand, by (3.21) and (3.27) there holds  $\alpha(0) > 0$ . Then by continuity of the  $\alpha(\cdot)$  and the inequality in (3.24), from (3.28) we obtain that  $\kappa(s) < 0$  for any  $|s| \in (0, \varepsilon)$  sufficiently small. Hence the conclusion follows.

#### 4. NONLOCAL INTERACTION: PROOFS

Consider the open subset of the space  $Y$

$$B := \left\{ U \equiv (u, v) \in Y \mid \int_0^1 v(x) dx \neq 0 \right\},$$

and consider the map  $\tilde{\mathcal{F}} : \mathbb{R}_+ \times \bar{\mathbb{R}}_+ \times B \rightarrow X$ ,

$$\tilde{\mathcal{F}}(\lambda, \chi, U) := \begin{pmatrix} u'' + F(u, v) \\ \lambda v'' - \chi(u'v)' + \tilde{G}(u, v) \end{pmatrix},$$

for any  $\lambda > 0$ ,  $\chi \geq 0$  and  $U \equiv (u, v) \in B$ ; here

$$\begin{aligned} \tilde{G}(u, v) &:= -\gamma \frac{uv}{1 + \tau v} + \delta \frac{\langle u, v \rangle}{\langle 1, v \rangle} v \\ &= G(u, v) + \delta \left[ \frac{\langle u, v \rangle}{\langle 1, v \rangle} - u \right] v. \end{aligned}$$

Then problem (1.1) can be given the abstract form

$$(4.1) \quad \begin{cases} U_t = \tilde{\mathcal{F}}(\lambda, \chi, U) & \text{in } \mathbb{R}_+ \\ U(0) = U_0. \end{cases}$$

Observe that

$$(4.2) \quad \tilde{\mathcal{F}}(\lambda, \chi, U) = \mathcal{F}(\lambda, \chi, U) + \delta \begin{pmatrix} 0 & 0 \\ \mathcal{H}(U) - uv & 0 \end{pmatrix},$$

where

$$\mathcal{H} : \hat{B} := \left\{ U \equiv (u, v) \in X \mid \int_0^1 v(x) dx \neq 0 \right\} \rightarrow C(\bar{\Omega}), \quad U \mapsto \mathcal{H}(U) := \frac{\langle u, v \rangle}{\langle 1, v \rangle} v.$$

By  $\mathcal{H}'(U) \in \mathcal{L}(X, C(\bar{\Omega}))$ ,  $\mathcal{H}''(U) \in \mathcal{L}_2(X, C(\bar{\Omega}))$ ,  $\mathcal{H}'''(U) \in \mathcal{L}_3(X, C(\bar{\Omega}))$  we shall denote the first three derivatives of  $\mathcal{H}(U)$  evaluated at some  $U \in \hat{B}$ . The following technical lemma will be used in the sequel (in particular, to obtain the expression (2.24) of the Fréchet derivative  $\tilde{A}_\lambda \equiv \tilde{\mathcal{F}}_U(\lambda, \chi, \bar{U})$ ).

**LEMMA 4.1.** *Let  $U \equiv (u, v) \in \hat{B}$ . Then:*

(i) *for any  $U_1 \equiv (u_1, v_1) \in X$*

$$\mathcal{H}'(U)U_1 = \frac{\langle u, v_1 \rangle + \langle u_1, v \rangle}{\langle 1, v \rangle} v + \frac{\langle u, v \rangle}{\langle 1, v \rangle} v_1 - \frac{\langle u, v \rangle \langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v;$$

(ii) for any  $U_1 \equiv (u_1, v_1)$  and  $U_2 \equiv (u_2, v_2) \in X$

$$\begin{aligned} \mathcal{H}''(U)U_1U_2 &= \frac{\langle u_2, v_1 \rangle + \langle u_1, v_2 \rangle}{\langle 1, v \rangle} v \\ &\quad + \frac{\langle u, v_1 \rangle + \langle u_1, v \rangle}{\langle 1, v \rangle} v_2 + \frac{\langle u, v_2 \rangle + \langle u_2, v \rangle}{\langle 1, v \rangle} v_1 \\ &\quad - \frac{[\langle u, v_1 \rangle + \langle u_1, v \rangle]\langle 1, v_2 \rangle + [\langle u, v_2 \rangle + \langle u_2, v \rangle]\langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v \\ &\quad - \frac{\langle u, v \rangle \langle 1, v_2 \rangle}{\langle 1, v \rangle^2} v_1 - \frac{\langle u, v \rangle \langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v_2 + \frac{2\langle u, v \rangle \langle 1, v_1 \rangle \langle 1, v_2 \rangle}{\langle 1, v \rangle^3} v; \end{aligned}$$

(iii) for any  $U_1 \equiv (u_1, v_1)$ ,  $U_2 \equiv (u_2, v_2)$  and  $U_3 \equiv (u_3, v_3) \in X$

$$\begin{aligned} \mathcal{H}'''(U)U_1U_2U_3 &= \frac{\langle u_2, v_3 \rangle + \langle u_3, v_2 \rangle}{\langle 1, v \rangle} v_1 + \frac{\langle u_1, v_3 \rangle + \langle u_3, v_1 \rangle}{\langle 1, v \rangle} v_2 \\ &\quad + \frac{\langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle}{\langle 1, v \rangle} v_3 - \frac{[\langle u, v_1 \rangle + \langle u_1, v \rangle]\langle 1, v_3 \rangle}{\langle 1, v \rangle^2} v_2 \\ &\quad - \frac{[\langle u_1, v_3 \rangle + \langle u_3, v_1 \rangle]\langle 1, v_2 \rangle + [\langle u_2, v_3 \rangle + \langle u_3, v_2 \rangle]\langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v \\ &\quad - \frac{[\langle u, v_1 \rangle + \langle u_1, v \rangle]\langle 1, v_2 \rangle + [\langle u, v_2 \rangle + \langle u_2, v \rangle]\langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v_3 \\ &\quad - \frac{[\langle u_2, v_1 \rangle + \langle u_1, v_2 \rangle]\langle 1, v_3 \rangle}{\langle 1, v \rangle^2} v - \frac{[\langle u, v_2 \rangle + \langle u_2, v \rangle]\langle 1, v_3 \rangle}{\langle 1, v \rangle^2} v_1 \\ &\quad - \frac{[\langle u, v_3 \rangle + \langle u_3, v \rangle]\langle 1, v_2 \rangle}{\langle 1, v \rangle^2} v_1 - \frac{[\langle u, v_3 \rangle + \langle u_3, v \rangle]\langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v_2 \\ &\quad + \frac{2\langle u, v \rangle \langle 1, v_2 \rangle \langle 1, v_3 \rangle}{\langle 1, v \rangle^3} v_1 + \frac{2\langle u, v \rangle \langle 1, v_1 \rangle \langle 1, v_3 \rangle}{\langle 1, v \rangle^3} v_2 \\ &\quad + \frac{2\{[\langle u, v_1 \rangle + \langle u_1, v \rangle]\langle 1, v_2 \rangle + [\langle u, v_2 \rangle + \langle u_2, v \rangle]\langle 1, v_1 \rangle\}\langle 1, v_3 \rangle}{\langle 1, v \rangle^3} v \\ &\quad + \frac{2[\langle u, v_3 \rangle + \langle u_3, v \rangle]\langle 1, v_1 \rangle \langle 1, v_2 \rangle}{\langle 1, v \rangle^3} v + \frac{2\langle u, v \rangle \langle 1, v_2 \rangle \langle 1, v_1 \rangle}{\langle 1, v \rangle^3} v_3 \\ &\quad - \frac{6\langle u, v \rangle \langle 1, v_1 \rangle \langle 1, v_2 \rangle \langle 1, v_3 \rangle}{\langle 1, v \rangle^4} v. \end{aligned}$$

PROOF. We only prove claim (i); the lengthy proof of (ii)–(iii) is similar, thus we omit it. For any  $U \equiv (u, v) \in \hat{B}$ ,  $U_1 \equiv (u_1, v_1) \in X$  and  $\varepsilon > 0$  sufficiently small

there holds  $U + \varepsilon U_1 \in \hat{\mathcal{B}}$ . Then, denoting by  $o(\varepsilon)$  any term of higher order with respect to  $\varepsilon$ , we have that

$$\begin{aligned}
\mathcal{H}(U + \varepsilon U_1) &= \frac{\langle u + \varepsilon u_1, v + \varepsilon v_1 \rangle}{\langle 1, v + \varepsilon v_1 \rangle} (v + \varepsilon v_1) \\
&= \frac{\langle u, v \rangle + \varepsilon[\langle u, v_1 \rangle + \langle u_1, v \rangle] + o(\varepsilon)}{\langle 1, v \rangle \left(1 + \varepsilon \frac{\langle 1, v_1 \rangle}{\langle 1, v \rangle}\right)} (v + \varepsilon v_1) \\
&= \frac{\langle u, v \rangle + \varepsilon[\langle u, v_1 \rangle + \langle u_1, v \rangle] + o(\varepsilon)}{\langle 1, v \rangle} \left(1 - \varepsilon \frac{\langle 1, v_1 \rangle}{\langle 1, v \rangle} + o(\varepsilon)\right) (v + \varepsilon v_1) \\
&= \frac{\langle u, v \rangle + \varepsilon[\langle u, v_1 \rangle + \langle u_1, v \rangle] + o(\varepsilon)}{\langle 1, v \rangle} \left(v + \varepsilon v_1 - \varepsilon \frac{\langle 1, v_1 \rangle}{\langle 1, v \rangle} v + o(\varepsilon)\right) \\
&= \mathcal{H}(u, v) + \frac{\varepsilon[\langle u, v_1 \rangle + \langle u_1, v \rangle]}{\langle 1, v \rangle} v + \varepsilon \frac{\langle u, v \rangle}{\langle 1, v \rangle} v_1 \\
&\quad - \varepsilon \frac{\langle u, v \rangle \langle 1, v_1 \rangle}{\langle 1, v \rangle^2} v + o(\varepsilon).
\end{aligned}$$

Hence the claim follows.

Now we can show that the linearized operator  $\tilde{A}_\lambda \equiv \tilde{\mathcal{F}}_U(\lambda, \chi, \bar{U})$  at the constant stationary solution  $\bar{U}$  has the expression given by (2.24). In fact, applying the Fréchet derivative of the operator-valued matrix in equality (4.2) to any  $U_1 \equiv (u_1, v_1) \in X$  we get

$$(4.3) \quad \begin{pmatrix} 0 & 0 \\ \mathcal{H}'(\bar{U})U_1 - \bar{u}v_1 - \bar{v}u_1 & 0 \end{pmatrix}.$$

Since  $\bar{u}$  and  $\bar{v}$  are constant, there holds

$$\langle 1, \bar{u} \rangle = \bar{u}, \quad \langle 1, \bar{v} \rangle = \bar{v}.$$

Then by Lemma 4.1 we plainly obtain

$$\begin{aligned}
\mathcal{H}'(\bar{U})U_1 &= \frac{\langle \bar{u}, v_1 \rangle + \langle u_1, \bar{v} \rangle}{\langle 1, \bar{v} \rangle} \bar{v} + \frac{\langle \bar{u}, \bar{v} \rangle}{\langle 1, \bar{v} \rangle} v_1 - \frac{\langle \bar{u}, \bar{v} \rangle \langle 1, v_1 \rangle}{\langle 1, \bar{v} \rangle^2} \bar{v} \\
&= \bar{u} \langle 1, v_1 \rangle + \bar{v} \langle 1, u_1 \rangle + \bar{u}v_1 - \bar{u} \langle 1, v_1 \rangle \\
&= \bar{v} \langle 1, u_1 \rangle + \bar{u}v_1,
\end{aligned}$$

thus

$$\mathcal{H}'(\bar{U})U_1 - \bar{u}v_1 - \bar{v}u_1 = \bar{v}[\langle 1, u_1 \rangle - u_1].$$

By (4.2) and the above equality we obtain

$$(4.4) \quad \tilde{A}_\lambda = A_\lambda + \delta \begin{pmatrix} 0 & 0 \\ \bar{v}[\langle 1, \cdot \rangle - 1] & 0 \end{pmatrix},$$

whence equality (2.24) follows. In particular, equality (4.4) shows that the operator  $\tilde{A}_\lambda$  has compact resolvent (since this holds for  $A_\lambda$ ), thus its spectrum consists of eigenvalues.

In view of (2.24), the eigenvalue equation  $\tilde{A}_\lambda \Phi = \zeta \Phi$  ( $\Phi \equiv (\varphi_1, \varphi_2) \in Y$ ) reads

$$(4.5) \quad \begin{cases} \varphi_1'' + F_u \varphi_1 + F_v \varphi_2 = \zeta \varphi_1 \\ \lambda \varphi_2'' - \chi \bar{v} \varphi_1'' + G_u \varphi_1 + \delta \bar{v}[\langle 1, \varphi_1 \rangle - \varphi_1] + G_v \varphi_2 = \zeta \varphi_2 \quad \text{in } \Omega \\ \varphi_1'(0) = \varphi_2'(0) = \varphi_1'(1) = \varphi_2'(1) = 0. \end{cases}$$

As for system (3.8), choose

$$\Phi_n \equiv (\varphi_1^n, \varphi_2^n) = (a \cos(\sqrt{k_n}x), b \cos(\sqrt{k_n}x)),$$

with  $k_n := n^2 \pi^2$  ( $n \in \mathbb{N} \cup \{0\}$ ) and  $a, b \in \mathbb{R}$  to be fixed, as trial functions. Observe that

$$(4.6) \quad \begin{aligned} \langle 1, \varphi_1^0 \rangle = a &\Rightarrow [\langle 1, \varphi_1^0 \rangle - \varphi_1^0] = 0, \\ \langle 1, \varphi_1^n \rangle = 0 &\Rightarrow [\langle 1, \varphi_1^n \rangle - \varphi_1^n] = -\varphi_1^n \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

Hence  $\Phi_0 \equiv (a, b)$  is an eigenfunction of  $\tilde{A}_\lambda$  if and only if  $\zeta$  is a root of the equation

$$(4.7) \quad \begin{vmatrix} \zeta - F_u & -F_v \\ -G_u & \zeta - G_v \end{vmatrix} = 0,$$

namely an eigenvalue of the linearized operator (2.6) without space dependence. By assumption  $(A_0)$  both eigenvalues of this operator have negative real part. Therefore, to have Turing destabilization of the stationary solution  $\bar{U}$  we must consider eigenvalues of system (4.5) with  $n \in \mathbb{N}$ . By (4.6), these are the roots of the equation

$$(4.8) \quad \begin{vmatrix} \zeta + k_n - F_u & -F_v \\ -\chi \bar{v} k_n - G_u + \delta \bar{v} & \zeta + \lambda k_n - G_v \end{vmatrix} = 0 \\ \Leftrightarrow \zeta^2 + \phi(\lambda, k_n) \zeta + \tilde{\psi}(\lambda, \chi, k_n) = 0,$$

where  $\phi(\lambda, k)$  and  $\tilde{\psi}(\lambda, \chi, k)$  are defined by (2.14) and (2.20), respectively.

Let us now prove Theorem 2.4.

**PROOF OF THEOREM 2.4.** As in the proof of Theorem 2.2, by (4.8) a necessary condition for the Turing destabilization of  $\bar{U}$  is the existence of real positive solutions of the equation

$$(4.9) \quad \zeta^2 + \phi(\lambda, k) \zeta + \tilde{\psi}(\lambda, \chi, k) = 0 \quad (k > 0).$$

Since  $\phi(\lambda, k) > 0$ , such solutions exist if and only if  $\tilde{\psi}(\lambda, \chi, k) < 0$ , namely if and only if  $0 < k < \tilde{k}(\lambda, \chi)$  (see Figure 2.5). Therefore, a positive eigenvalue of system (4.5) exists if and only if  $0 < k_n < \tilde{k}(\lambda, \chi)$  for some  $n \in \mathbb{N}$ . Since  $0 < k_1 < \dots < k_n < \dots$ , for any  $\chi \geq 0$  this happens if and only if

$$0 < k_1 < \tilde{k}(\lambda, \chi) \Leftrightarrow \lambda \in (0, \lambda_1).$$

Then the conclusion follows.

The following analogue of Proposition 3.1 holds true.

**PROPOSITION 4.2.** *Let the assumptions of Theorem 2.5 be satisfied. Then  $(\lambda_1, \bar{U})$  is a bifurcation point of nonconstant stationary solutions of problem (1.1).*

The proof of the above proposition is almost verbatim the same of Proposition 3.1 (observe that  $\mathcal{F}_{\lambda U}(\lambda, \bar{U}) = \tilde{\mathcal{F}}_{\lambda U}(\lambda, \bar{U})$  by equality (4.2)), thus we omit it. Let us only mention for future reference that in the present case the vectors  $E, E^*$  are replaced by the vectors  $D, D^* \in Y$ ,

$$(4.10) \quad D \equiv (d_1, d_2) := (c \cos(\sqrt{k_1}x), d \cos(\sqrt{k_1}x)) \quad (x \in \bar{\Omega})$$

with

$$(4.11) \quad c \in \mathbb{R} \setminus \{0\}, \quad d := \frac{k_1 - F_u}{F_v} c,$$

and

$$(4.12) \quad D^* \equiv (d_1^*, d_2^*) := (c^* \cos(\sqrt{k_1}x), d^* \cos(\sqrt{k_1}x)) \quad (x \in \bar{\Omega}),$$

with

$$(4.13) \quad c^* := \frac{2}{c} \frac{k_1 \lambda_1 - G_v}{k_1(\lambda_1 + 1) - (F_u + G_v)} \quad d^* := \frac{F_v}{k_1 \lambda_1 - G_v} c^*.$$

By the above choice in (4.11) and (4.13), there holds  $((D^*, D)) = 1$ . Without loss of generality, we assume that

$$(4.14) \quad c > 0.$$

Then, by equalities (2.3) and assumption  $(A_0)$ , from (4.11) and (4.13) we get

$$(4.15) \quad d < 0, \quad d^* = \frac{2}{c} \frac{F_v}{k_1(\lambda_1 + 1) - (F_u + G_v)} < 0.$$

Now we can prove Theorem 2.5.

PROOF OF THEOREM 2.5. By Proposition 4.2, we only have to prove that the bifurcation is subcritical, and the bifurcating nonconstant stationary solutions are asymptotically stable.

By the same notations used in the proof of Theorem 2.3, now we have (see (3.22)–(3.23))

$$(4.16) \quad \lambda'(0) = -\frac{1}{2} \frac{((D^*, \tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2))}{((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D))},$$

$$(4.17) \quad \lambda''(0) = -\frac{1}{3} \frac{((D^*, \tilde{\mathcal{F}}_{UUU}(\lambda_1, \bar{U})D^3))}{((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D))}.$$

By equality (4.2) there holds

$$(4.18) \quad \tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2 = \mathcal{F}_{UU}(\lambda_1, \bar{U})D^2 + \delta \begin{pmatrix} 0 & 0 \\ \mathcal{H}''(\bar{U})D^2 - 2d_1d_2 & 0 \end{pmatrix}.$$

Since  $\bar{U}$  is constant and

$$\langle 1, d_1 \rangle = c \int_0^1 \cos(\sqrt{k_1}x) dx = 0 = \langle 1, d_2 \rangle$$

(see (4.10)), from Lemma 4.1-(ii) plainly we get

$$\mathcal{H}''(\bar{U})D^2 = 2 \int_0^1 d_1(x)d_2(x) dx.$$

Then from (4.18) and the above equality we obtain

$$\tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2 = 2 \begin{pmatrix} -d_1^2 - d_1d_2 \\ -\chi(d_2d_1)' + \delta \int_0^1 d_1d_2 dx - \frac{\gamma}{(1+\tau\bar{v})^2}d_1d_2 + \frac{\gamma\tau\bar{u}}{(1+\tau\bar{v})^3}d_2^2 \end{pmatrix}$$

(see (3.25)). Then there holds

$$\begin{aligned} & ((D^*, \tilde{\mathcal{F}}_{UU}(\lambda_1, \bar{U})D^2)) \\ &= 2 \left\{ -c^2c^* - cc^*d - \frac{\gamma}{(1+\tau\bar{v})^2}cdd^* + \frac{\gamma\tau\bar{u}}{(1+\tau\bar{v})^3}d^2d^* \right\} \int_0^1 \cos^3(\sqrt{k_1}x) dx \\ & \quad + 2\delta cdd^* \int_0^1 \cos^2(\sqrt{k_1}x) dx \int_0^1 \cos(\sqrt{k_1}x) dx \\ & \quad - 2k_1\chi cdd^* \int_0^1 \sin^2(\sqrt{k_1}x) \cos(\sqrt{k_1}x) dx = 0, \end{aligned}$$

thus  $\lambda'(0) = 0$  by (4.16).

On the other hand, from Lemma 4.1-(iii) we easily obtain

$$\mathcal{H}'''(\bar{U})D^3 = \frac{6}{\bar{v}}cd^2\left(\int_0^1 d_1(x)d_2(x)dx\right)d_2.$$

Then by (4.18) and the above equality we have

$$\tilde{\mathcal{F}}_{UUU}(\lambda_1, \bar{U})D^3 = \left( \frac{6\gamma\tau}{(1+\tau\bar{v})^3}d_1d_2^2 - \frac{6\gamma\tau^2\bar{u}}{(1+\tau\bar{v})^4}d_2^3 + \frac{6\delta}{\bar{v}}cd^2\left(\int_0^1 d_1d_2dx\right)d_2 \right),$$

whence

$$\begin{aligned} ((D^*, \tilde{\mathcal{F}}_{UUU}(\lambda_1, \bar{U})D^3)) &= \frac{6\gamma\tau d^2 d^*}{(1+\tau\bar{v})^3} \left( c - \frac{\tau\bar{u}}{1+\tau\bar{v}}d \right) \int_0^1 \cos^4(\sqrt{k_1}x) \\ &\quad + \frac{6\delta}{\bar{v}}cd^2d^* \left( \int_0^1 \cos^2(\sqrt{k_1}x) \right)^2 < 0 \end{aligned}$$

(here use of (4.14)–(4.15) has been made). Moreover, arguing as for (3.21) we have

$$(4.19) \quad ((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D)) = -dd^*k_1 \int_0^1 \cos^2(\sqrt{k_1}x)dx < 0,$$

thus  $\lambda''(0) < 0$  by equality (4.17). Then the same argument used in the proof of Proposition 3.1 proves that the bifurcation is subcritical.

Finally, replacing equality (3.27) by

$$(4.20) \quad \alpha(0) = -((D^*, \tilde{\mathcal{F}}_{\lambda U}(\lambda_1, \bar{U})D))$$

and inequality (3.21) by (4.19), the above calculations and the same arguments used in the proof of Theorem 2.3 prove that the stationary bifurcating solutions are asymptotically stable. This completes the proof.

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