Object-oriented modelling of a flexible beam including geometric nonlinearities

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Abstract

In this paper, an efficient approach for the modelling and simulation of slender beams subject to heavy inertial loads is presented. The limitations imposed by a linear formulation of elasticity are overcome by a second order expansion of the displacement field, based on a geometrical exact beam model. In light of this, the nonlinearities of the elastic terms are shifted as inertial contributions, which yields an expression of the equations of motion in closed form. Thanks to the formulation in closed form, the proposed model is implemented in Modelica, with particular care to the suitability of the model with respect to the Modelica Multibody library. After describing the model formulation and implementation, the paper presents some simulation results, in order to validate the model with respect to benchmarks, widely adopted in literature. In the context of modern multi-domain modelling, the modular and object oriented approaches are the state-of-the-art paradigms upon which complex models are built. In this respect, multibody dynamics is frequently only one of the domains involved, nevertheless several real-world applications can be found where multibody modelling plays a crucial role in the design of systems, analysis and model-based control architectures. In this framework, modelling techniques and tools have evolved towards the insertion of flexible bodies into the models (MSC Software Corporation, 2017; Claytex Services Ltd; Dymore Solutions, 2016; Spacar, 2016; Heckmann et al., 2006; Ferretti et al., 2014).

Flexible multibody systems can be divided in two main branches according to the *linear* or *nonlinear* constitutive laws employed to model flexible elements. In the first case, the strain-displacement relationships are assumed to be linear and strain components to remain small. Nevertheless, several occurrences can be found where elastic bodies may undergo large overall motion while strains are kept small. Traditionally, linear elasticity has been accounted for using the so called *floating frame of reference approach* (short, FFR), which is natural way to include flexibility in the rigid multibody framework(Shabana, 1998). Indeed, the displacement field is decomposed as the sum of an arbitrary large motion of a suitably selected frame, superposed to an elastic displacement field which is assumed to be small with respect to the overall motion. Thus the elastic displacement field may be computed accurately through a modal expansion, which is extremely efficient from a computational point of view. Furthermore, component mode synthesis techniques, like the well-known Craig-Bampton method (Craig and Bampton, 1968), has been widely adopted in multibody simulation tools and specifically in the context of Modelica, both in commercial (Claytex Services Ltd; Heckmann et al., 2006) and open-source (Ferretti et al., 2014; Bascetta et al., 2015) libraries. By means of this technique, complex geometry can be included in the analysis even tough an external finite element modeling tool is required. Although the concept of floating frame is simple, in practice there are several issues to be handled. The selection of the floating frame is not unique and accuracy of the results is strongly affected by the choice of the modal basis (Schwertassek et al., 1999a), (Heckmann, 2010). Furthermore, the use of linear elasticity may lead to erroneous results when the inertial contribution of the floating frame motion is large enough to produce high loads in bodies with high stiffness. A well-known example is the rotating beam around a fixed axis: the coupling among the axial, flexural and torsional motion, neglected by the linearized theory, is crucial in order to correctly predict the behavior of the structure (Berzeri and Shabana, 2002; Sharf, 1995; Lugrís et al., 2008; Absy and Shabana, 1997). On the other hand, the flaws of this theory pushed the multibody community toward the development of new approaches, closely related to the nonlinear finite element method (Géradin and Cardona, 2001). In this case the domain is divided in sub-domains or finite elements which are connected at nodes to ensure elements compatibility, the nodal displacements and rotations are referred to a common frame, which is selected as the inertial frame. From a theoretical point of view this approach is challenging when considering structural elements such as beams, shells, plates and in particular when large displacements and rotations are considered. *Geometrically exact models* for these elements have been developed in the last decades (Pai, 2007) and are the state-of-the-art to deal with large displacements small strains problems. Nonetheless special care is required for the computation of elastic terms and for the interpolation scheme of spatial rotations in a finite elements framework (Jelenić and Crisfield, 1998). Moreover, a large number of degrees of freedom is required to obtain accurate solutions and the expression of the elastic contribution is highly nonlinear even in the case of small strains. The use of such approach within the Modelica multibody library is difficult because the closed form expression of the discretized equations of motion is not manageable even for simple elements like beams.

Within the FFR method, several approaches have been proposed in the reference literature that overcome the shortcomings of the standard linear approach by accounting for geometrically nonlinear effects (Wallrapp and Wiedemann, 2003; Bremer, 2008; Banerjee, 2016). As already mentioned, the classic linear approach may provide erroneous results due to *a-priori* linearization of the kinematics and of the elastic energy, even if the strains are small. This problem has been deeply studied (Absy and Shabana, 1997) and different techniques have been developed to consistently linearize the equations of motion, so that all the relevant terms are retained while keeping the standard linear elastic terms.

In this work, a general framework to include geometrically nonlinear effects within the FFR approach is presented. The approach is based on a second order expansion of the displacement field, which can be derived from geometrically exact models of simple structural elements. Then, the displacement field is written in terms of a set of generalized deformation variables for which the elastic terms are linear. Thanks to this substitution, the standard linear elastic theory can be exploited and the nonlinearities are expressed as inertial contribution, which can be computed in closed form. Hence, the existing FFR formulation can be employed with minor changes, which is particularly efficient when small elastic deformations are expected: few degrees of freedom are usually required.

The proposed formulation has been applied to slender structural elements which are usually modeled as beams. In the standard approach, a linearized model is adopted to describe the deformation field in the floating frame, such as the Euler-Bernoulli or the Timoshenko beam model. This approach greatly simplifies the computation of the elastic terms but limits the correctness of the results by neglecting nonlinear effects, *e.g.* the geometric stiffening induced by the centrifugal acceleration for fast rotating beams. Within the Modelica framework, the standard linear approach has been implemented in (Schiavo et al., 2006), while in (Heckmann et al., 2006) a second order apporximation of the deformation field is presented together with a Ritz-Galerkin discretization (Ritz, 1909). With respect to (Heckmann et al., 2006), in this work the deformation field is expanded starting from a geometrically exact description of the beam kinematics ((Schwertassek and Wallrapp, 1999)) and the model is discretized according to a finite element approach. Finally, the Craig-Bampton reduction is applied to obtain a computationally efficient set of equations. The model is implemented in Modelica following an approach similar to (Ferretti et al., 2014) and two validation benchmarks taken from literature are reproduced, showing trustworthy agreement between simulation results and literature data.

The paper is organized as follows: In section 1 the modelling framework is described for the generic flexible body and the equations of motion are formulated. Section 2 goes into details of the beams by describing the mathematical formulation pointed out in the previous section. In section 3 the implementation and the simulation results are described. In particular, the results are compared with two well known literature benchmarks. Section 4 concludes the paper.

1 Equations of motion of a flexible boby

Within the FFR approach, the absolute position $\mathbf{p}$ of a generic point of the flexible body is composed by the sum of three contributions

$$\mathbf{p} = \mathbf{r} + \mathbf{u}_0 + \mathbf{u}_f, \tag{1}$$

where $\mathbf{r}$ is the vector describing the position of the reference frame $\{\mathbf{O}_i, x_i, y_i, z_i\}$ with respect to the inertial frame $\{\mathbf{O}_w, x_w, y_w, z_w\}$, $\mathbf{u}_0$ is the undeformed position of the point with respect to the local reference frame and $\mathbf{u}_f$ is the deformation field, as shown by Figure 1. The components of $\mathbf{u}_0$ resolved in $\{\mathbf{O}_i, x_i, y_i, z_i\}$ are named *material coordinates*.

In order to obtain the equations of motion, the *principle of virtual work* is exploited, i.e.:

$$\delta W_e = \delta W_i \tag{2}$$

where δW_e and δW_i are the external and internal virtual works. According to the FFR approach, the virtual displacement related to (1) can be computed as follows:

$$\delta\mathbf{p} = \delta\mathbf{r} + \delta\phi \times \left(\mathbf{u}_0 + \mathbf{u}_f\right) + \delta\mathbf{u}_f \tag{3}$$

where $\delta\phi$ is the virtual rotations vector of $\{\mathbf{O}_i, x_i, y_i, z_i\}$ while $\delta\mathbf{u}_f$ is the virtual variation of the deformation field with respect to the local reference frame of the body.

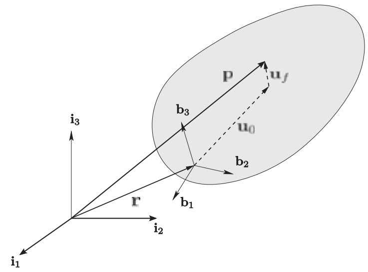

Figure 1. Flexible body reference frames

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The external and internal virtual works for a generic flexible body can be defined as:

$$\delta W_e = \int_V \delta \mathbf{p} \cdot (-\rho \mathbf{a} + \mathbf{f}) \, \mathrm{d}V + \int_{A_N} \delta \mathbf{p} \cdot \mathbf{t} \, \mathrm{d}A \quad (4)$$

$$\delta W_i = \int_V \delta \mathbf{B} : \mathbf{J} \, \mathrm{d}V \quad (5)$$

where ρ is the density of the body, $\mathbf{B} \in \mathbb{R}^{3x3}$ and $\mathbf{J} \in \mathbb{R}^{3x3}$ are the Jaumann strain and stress tensor respectively (see (Pai, 2007) for more details). By using the indicial notation, the double inner tensor product ":" is defined by:

$$\int_V \delta \mathbf{B} : \mathbf{J} \, \mathrm{d}V = \int_V \sum_{i=1}^{3} \sum_{j=1}^{3} \delta B_{ij} J_{ij} \, \mathrm{d}V. \quad (6)$$

Moreover, $\mathbf{f}$ is the body force per unit volume and $\mathbf{t}$ the surface traction per unit area, V is the volume of the body and A is the unconstrained portion of the body surface.

It must be pointed out that the Jaumann strain tensor is related to the deformation gradient $\mathbf{F} \in \mathbb{R}^{3x3}$ as follows (see (Hodges, 2006)):

$$\mathbf{B} = \mathbf{U} - \mathbf{I} \qquad \mathbf{U}^2 = \mathbf{F}^T \cdot \mathbf{F} \quad (7)$$

where $\mathbf{I}$ is the identity tensor and $\mathbf{U}$ is the right stretch tensor.

It must be pointed out that the Jaumann strains are an objective strain measure suitable for large displacements-small strains analysis since they are co-rotated engineering strains. As a consequence, in a linear elastic framework the reduced material stiffness matrix can be derived from standard experiments (Pai, 2007). The term of δW_e relative to inertial virtual work can be expanded by substituting (3) in (4), thus obtaining:

$$- \int_V \delta p \cdot \rho \left(\dot{\mathbf{v}} + \omega \times \mathbf{v} + \dot{\omega} \times \left(\mathbf{u}_0 + u_f \right) + \right. \quad (8)$$
$$\left. + \omega \times \left(\omega \times \left(\mathbf{u}_0 + u_f \right) \right) + \ddot{\mathbf{u}}_f + 2\omega \times \dot{\mathbf{u}}_f \right) \mathrm{d}V$$

where $\mathbf{v}$ and ω are the body translational and angular velocities of the FFR.

In the classic linear approach, the deformation field measured in the FFR is assumed to be infinitesimal such that the computation of the internal virtual work can be approximated with the standard linear theory:

$$\int_V \delta \mathbf{B} : \mathbf{J} \, dV \approx \int_V \sum_{i=1}^{3} \sum_{j=1}^{3} \delta \varepsilon_{ij} \sigma_{ij} \, \mathrm{d}V \quad (9)$$

where $\varepsilon_{ij} = \frac{1}{2} (F_{ij} + F_{ji}) - \delta_{ij}$ is the infinitesimal deformation tensor, and σ_{ij} the conjugated stress tensor, both resolved in the undeformed basis $\{\mathbf{O}_i, x_i, y_i, z_i\}$. The corresponding components of the deformation gradient are:

$$F_{ij} = \delta_{ij} + \frac{\partial u_{f\,i}}{\partial u_{0\,j}} \quad (10)$$

where δ_{ij} are the components of the identity tensor:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} . \quad (11)$$

As already mentioned in the introduction, the classic linear approach may lead to erroneous results since several terms are *a-priori* neglected. Instead, a consistent approximation of (5) is based on the decomposition of the deformation gradient such that

$$B_{ij} \approx \frac{1}{2} (F_{ij} + F_{ji}) - \delta_{ij} \quad (12)$$

where $F_{ij} = \sum_{k=1}^{3} R_{ik} F_{kj}$ and R_{ik} is the rotation matrix describing the orientation of a suitable frame with respect to the local reference frame. Under the small strains assumption, the aforementioned frame can be selected such that $\hat{F}_{ij} \approx \delta_{ij}$, even if displacements are large (see (Bauchau et al., 2016) for more details).

Within this framework, a second order approximation of the deformation field is considered in this paper. In particular, it is possible to operate a change of variables such that elastic forces can be derived from standard linear theory whereas geometrical effects are computed as an inertial contribution. In order to perform such change, the following assumptions are introduced:

- the deformation field $\mathbf{u}_f$ depends on a finite set of *physical* deformation functions $\mathbf{d}_p = \mathbf{d}_p(\mathbf{u}_0)$, *i.e.*, $\mathbf{u}_f = \mathbf{u}_f(\mathbf{d}_p, \mathbf{u}_0)$. Physical deformations depend only on a reduced set of material coordinates $\mathbf{u}_0$, *e.g.*, in beam models, the reference axis displacements and the cross-section rotation angles are functions of the reference axis coordinate alone;

- the physical deformations $\mathbf{d}_p$ can be expressed in terms of generalized deformation functions $\mathbf{d}_g$, *i.e.*, $\mathbf{d}_p = \mathbf{d}_p(\mathbf{d}_g, \mathbf{u}_0)$, such that the deformation gradient $\mathbf{F}$ (and thus elastic forces) are linear in $\mathbf{d}_g$ when strains (but not displacements) are small;

- the nonlinear relation $\mathbf{u}_f = \mathbf{u}_f(\mathbf{d}_p(\mathbf{d}_g), \mathbf{u}_0)$ is expanded up to the second order in terms of the generalized deformations.

It is worth remarking that these assumptions are not restrictive as they hold true for geometrically nonlinear models of beams, plates and shells. On the other hand, it is not trivial to obtain the relationship among physical and generalized deformation variables (Schwertassek et al., 1999b).

Assuming that the generalized deformations are expanded by means of a Ritz-Galerking approach (Ritz, 1909), i.e.,

$$\mathbf{d}_g = \Phi(\mathbf{u}_0) \, \mathbf{q}(t), \quad (13)$$

the displacement field, up to the second order, reads:

$$\mathbf{u}_f = \mathbf{S}\mathbf{q} + \mathbf{G}(\mathbf{q})\mathbf{q}, \quad (14)$$

where $\Phi(\mathbf{u}_0)$ are spatial mode functions, $\mathbf{q}$ is a vector of generalized coordinates and $\mathbf{S}(\mathbf{u}_0)$ is the standard matrix of shape functions obtained with linear models. The matrix $\mathbf{G}(\mathbf{q})$ linearly depends on $\mathbf{q}$ and allows to account for geometrically nonlinear terms:

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} \mathbf{q}^T\mathbf{G}_1 \\ \mathbf{q}^T\mathbf{G}_2 \\ \mathbf{q}^T\mathbf{G}_3 \end{bmatrix} \tag{15}$$

where $\mathbf{G}_i = \mathbf{G}_i(\mathbf{u}_0)$ is a symmetric matrix depending only on the material coordinates. Adopting the approximation (14) in the definition of the internal and external virtual works (4,5) leads to the derivation of the equations of motion in the following form:

$$m(\dot{\mathbf{v}} - \mathbf{g}) + m\widetilde{\mathbf{d}}_C^T\dot{\omega} + \mathbf{C}_t^T\ddot{\mathbf{q}} + \widetilde{\omega}m\widetilde{\mathbf{d}}_C^T\omega + 2\widetilde{\omega}\mathbf{C}_t^T\dot{\mathbf{q}} + \widetilde{\omega}m\mathbf{v}_1 = \mathbf{h}_e^r \tag{16}$$

$$m\widetilde{\mathbf{d}}_C(\dot{\mathbf{v}} - \mathbf{g}) + \mathbf{J}\dot{\omega} + \mathbf{C}_r^T\ddot{\mathbf{q}} + \widetilde{\omega}\mathbf{J}\omega + m\widetilde{\mathbf{d}}_C\widetilde{\omega}\mathbf{v} + 2\widetilde{\omega}\mathbf{C}_r^T\dot{\mathbf{q}} = \mathbf{h}_e^\theta \tag{17}$$

$$\mathbf{C}_t^g(\dot{\mathbf{v}} - \mathbf{g}) + \mathbf{C}_r^g\dot{\omega} + \mathbf{M}_e\ddot{\mathbf{q}} + \mathbf{C}_t\widetilde{\omega}\mathbf{v} + (\mathbf{D}_e + \mathbf{D}_{cr})\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{h}_e^f - \mathbf{h}_e^{ct} \tag{18}$$

where the terms of the inertia and stiffness matrices include additional terms with respect to the classical linear Newton-Euler approach(Shabana, 1998). In particular, the generalized stiffness matrix is expressed as follows:

$$\mathbf{K} = \mathbf{K}_e + \mathbf{K}_{ct} + \mathbf{K}_g^1 + \mathbf{K}_g^2 + \mathbf{K}_g^r \tag{19}$$

and contains additional contributions relative the motion induced stiffness ($\mathbf{K}_g^1, \mathbf{K}_g^2$) and the external action which account for geometrically nonlinear effects. The motion induced stiffness ($\mathbf{K}_g^r$) contribution depends on the reference frame motion and accounts for the loss of stiffness induced by the centrifugal acceleration ($\mathbf{K}_{ct}$). The aforementioned terms appear in the equations of motion as a consequence of the presence of the first order term in the virtual variation of the generalized coordinates formulation:

$$\delta\mathbf{u}_f = \mathbf{S}\delta\mathbf{q} + \mathbf{G}(\mathbf{q})\delta\mathbf{q} \tag{20}$$

which yields a contribution in the external virtual work formulation

$$\delta W_e = \delta W_e^c + \delta W_g \tag{21}$$

where the standard terms of the external virtual work are contained in δW_e^c and the geometric contribution is described by δW_g. The complete derivation of the terms is not reported here for the sake of brevity, it must be however pointed out that all the terms of the geometric contribution can be computed as function of invariants. The formulation described above enhances the classic linear FFR approach through the addition of further terms, providing a simple solution for including geometric nonlinearities in the equations of motion. This can be considered as a relevant advantage of the proposed formulation. The nonlinearities are isolated in the inertial terms, hence, a closed form expression for the geometrical stiffening effects is derived, which is essential for the application of the proposed approach in the context of the Modelica framework. The definition of the standard terms $\left(m, \mathbf{d}_C, \mathbf{C}_t, \mathbf{J}, \mathbf{C}_r, \mathbf{D}_e, \mathbf{K}_e, \mathbf{M}_e, \mathbf{h}_e^r, \mathbf{h}_e^\theta, \mathbf{h}_e^f\right)$ and the computation of the corresponding invariants can be found in (Bascetta et al., 2015).

2 Geometrically exact modeling of slender beams

The method developed in Section 1 is applied to derive the equations of motion of slender beams, for which the cross-section plane is assumed to remain normal to the reference axis during the deformation. The motion of the flexible beam can be described in terms of three reference frames as shown in Figure 2, frame $\{\mathbf{O}_w, x_w, y_w, z_w\}$ is the world reference frame, while the undeformed beam geometry is represented by frame $\{\mathbf{O}_r, x_r, y_r, z_r\}$ where $\mathbf{b_1}$ is the direction of the undeformed beam axis, $\mathbf{b_2}$ and $\mathbf{b_3}$ define the cross section principal axis. Finally, frame $\{\mathbf{O}_d, x_d, y_d, z_d\}$ describes the motion of the beam cross-section. The out-of-plane displacement are assumed to be negligible.

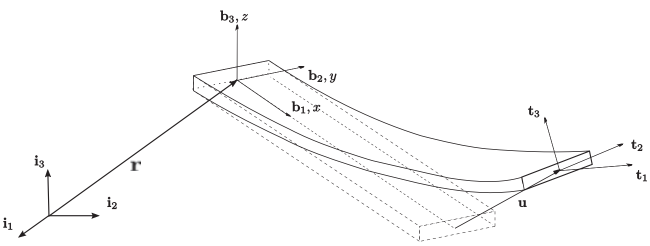

Figure 2. Beam reference frames

According to the reference frames described above, the position of a generic point on the deformed beam, with respect to the inertial frame, (see eq.1) is given by:

$$\mathbf{p} = \mathbf{r} + x\mathbf{b_1} + \mathbf{u} + \mathbf{R}\cdot\xi \tag{22}$$

where $\mathbf{u} = u(x)\mathbf{b_1} + v(x)\mathbf{b_2} + w(x)\mathbf{b_3}$ represents the vector of the cross-section translation dofs, $\xi = y\mathbf{b_2} + z\mathbf{b_3}$

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and $\mathbf{R}(x)$ is the rigid rotation tensor of the cross-section which can be parametrized by means of Euler angles $(\theta(x), \phi(x), \psi(x))$. The deformation field, defined as $\mathbf{u}_f = \mathbf{u} + \mathbf{R} \cdot \xi$, is a nonlinear function of a set of physical displacements $\mathbf{d}_p = (u, v, w, \theta, \phi, \psi)$, as required by the first assumption in Section 1.

As already mentioned, the Jaumann strain tensor $\mathbf{B}$ is chosen as the strain measure. According to the small deformations assumption, $\mathbf{B}$ can be consistently approximated as:

$$\begin{aligned} B_{11} &= e + zk_2 - yk_3 \\ 2B_{12} &= 2B_{21} = -zk_1 \\ 2B_{13} &= 2B_{31} = yk_1 \\ B_{22}&, B_{33}, B_{23}, B_{32} = 0 \end{aligned} \tag{23}$$

where e represents the axial stretch, k_1 the twisting curvature and k_2, k_3 the bending curvatures. These quantities are called generalized strains and are nonlinear functions of the physical displacements and their derivatives, see (Hodges, 2006) for further details. The internal virtual work is expressed in terms of the Jaumann strains B_{ij} in (23) and their work-conjugate stresses J_{ij} as follows:

$$\delta W_i = \int_V (\delta B_{11} J_{11} + 2\delta B_{12} J_{12} + 2\delta B_{13} J_{13})\, \mathrm{d}V. \tag{24}$$

After substituting (23), the internal virtual work can be compactly written by introducing the generalized axial force F_1 and moments M_1, M_2, M_3 as follows

$$\delta W_i = \int_0^\ell (\delta e F_1 + \delta k_1 M_1 + \delta k_2 M_2 + \delta k_3 M_3)\, \mathrm{d}x. \tag{25}$$

Assuming a linear elastic constitutive law for an isotropic material, the generalized force and moment are related to the corresponding strains as:

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EJ_{yy} & 0 \\ 0 & 0 & 0 & EJ_{zz} \end{bmatrix} \begin{Bmatrix} e \\ k_1 \\ k_2 \\ k_3 \end{Bmatrix}$$

where EA, GJ, EJ_{yy}, EJ_{zz} are the axial, torsional, and bending stiffness, respectively.

Thus, by substituting the constitutive law in the internal virtual work expression:

$$\int_0^\ell (\delta e EA e + \delta k_1 GJ k_1 + \delta k_1 GJ k_1 + \delta k_2 EJ_{yy} k_2 + \delta k_3 EJ_{zz} k_3)\, \mathrm{d}x. \tag{26}$$

It is clear that the above expression is linear in generalized strains and has the same mathematical form of the classic linear approach. Nonetheless, it is valid also in the the case of large displacements and small strains. In order to adopt the described approach, a suitable change of variables is introduced, such that the elastic forces are linear in these new variables, according to the second assumption in Section 1.

The generalized strain components are written in terms of a set of generalized deformation functions $\mathbf{d}_g = \big(\bar{u}(x), \bar{v}(x), \bar{w}(x), \bar{\phi}(x)\big)$ by defining:

$$e = \frac{\partial \bar{u}}{\partial x}, \qquad k_1 = \frac{\partial \bar{\phi}}{\partial x}, \qquad k_2 = -\frac{\partial^2 \bar{w}}{\partial x^2}, \qquad k_3 = \frac{\partial^2 \bar{v}}{\partial x^2}. \tag{27}$$

As shown in (Schwertassek and Wallrapp, 1999), by describing the physical variables in terms of generalized strains (27) and expanding up to the second order the corresponding relation, one can compute the components of the deformation field $\mathbf{u}_f$ as:

$$\begin{aligned} u_{f1} &= \bar{u} - y\frac{\partial \bar{v}}{\partial x} - z\frac{\partial \bar{w}}{\partial x} - \frac{1}{2}\int_0^x \left[\left(\frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{w}}{\partial x}\right)^2\right] dx + \\ &\quad -y\left[\int_0^x \left(\bar{\phi}\frac{\partial^2 \bar{w}}{\partial x^2} - \frac{\partial \bar{u}}{\partial x}\frac{\partial^2 \bar{v}}{\partial x^2}\right) dx + \bar{\phi}\frac{\partial \bar{w}}{\partial x}\right] - z\left[\int_0^x \left(\frac{\partial \bar{u}}{\partial x}\frac{\partial^2 \bar{w}}{\partial x^2} + \bar{\phi}\frac{\partial^2 \bar{v}}{\partial x^2}\right) dx - \bar{\phi}\frac{\partial \bar{v}}{\partial x}\right] \end{aligned} \tag{28}$$

$$\begin{aligned} u_{f2} &= \bar{v} - z\bar{\phi} + \int_0^x \left[\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{v}}{\partial x} + \int_0^x \left(\frac{\partial \bar{u}}{\partial x}\frac{\partial^2 \bar{v}}{\partial x^2} - \bar{\phi}\frac{\partial^2 \bar{w}}{\partial x^2}\right) dx\right] dx - \frac{1}{2}y\left[\left(\frac{\partial \bar{v}}{\partial x}\right)^2 + \bar{\phi}^2\right] + \\ &\quad -z\left[\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{w}}{\partial x} + \int_0^x \left(\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{\phi}}{\partial x} - \frac{\partial \bar{w}}{\partial x}\frac{\partial^2 \bar{v}}{\partial x^2}\right) dx\right] \end{aligned} \tag{29}$$

$$\begin{aligned} u_{f3} &= \bar{w} + y\bar{\phi} + \int_0^x \left[\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{w}}{\partial x} dx + \int_0^x \left(\frac{\partial \bar{u}}{\partial x}\frac{\partial^2 \bar{w}}{\partial x^2} + \bar{\phi}\frac{\partial^2 \bar{v}}{\partial x^2}\right) dx\right] dx - \frac{1}{2}z\left[\left(\frac{\partial \bar{w}}{\partial x}\right)^2 + \bar{\phi}^2\right] + \\ &\quad +y\left[\int_0^x \left(\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{\phi}}{\partial x} - \frac{\partial \bar{w}}{\partial x}\frac{\partial^2 \bar{v}}{\partial x^2}\right) dx\right]. \end{aligned} \tag{30}$$

The direction cosine matrix of the cross-section, expanded up to the second order, can be defined accordingly. A full treatise can be found in (Schwertassek et al., 1999b).
Finally, the generalized deformations, namely: $\bar{u}(x)$, $\bar{v}(x)$, $\bar{w}(x)$ and $\bar{\phi}(x)$ can be approximated as a linear combination of shape functions in terms of the generalized coordinates $\mathbf{q}$ by means of the classical Ritz-Galerkin method (Ritz, 1909), in particular:

$$\bar{u} = \Phi_1 \mathbf{q} \qquad \bar{v} = \Phi_2 \mathbf{q} \tag{31}$$

$$\bar{w} = \Phi_3 \mathbf{q} \qquad \bar{\phi} = \Phi_4 \mathbf{q} \tag{32}$$

where Φ are rows of admissible shape functions. It is worth to remark that the boundary conditions of the generalized displacements are the same of physical displacements. In this work, a finite element approach is used to discretize the beam domain. After the assembly procedure, the Craig-Bampton (Craig and Bampton, 1968) reduction procedure is applied by projecting the equations on the corresponding modal basis, which include constraint as well as normal modes. This is particularly efficient from a computational point of view since the final model includes only a few degrees of freedom while providing satisfactory results. The deformation field (14) can be written as:

$$\begin{Bmatrix} u_{f1} \\ u_{f2} \\ u_{f3} \end{Bmatrix} = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \mathbf{S}_3 \end{bmatrix} \mathbf{q} + \begin{bmatrix} \mathbf{q}^T \mathbf{G}_1 \\ \mathbf{q}^T \mathbf{G}_2 \\ \mathbf{q}^T \mathbf{G}_3 \end{bmatrix} \mathbf{q}$$

where

$$\mathbf{S}_1 = \Phi_1 - y\frac{\partial \Phi_2}{\partial x} - z\frac{\partial \Phi_3}{\partial x}$$
$$\mathbf{S}_2 = \Phi_2 - z\Phi_4$$
$$\mathbf{S}_3 = \Phi_3 + y\Phi_4$$

$$\begin{aligned} \mathbf{G}_1 = & -\frac{1}{2}\int_0^x \left(\frac{\partial \Phi_2}{\partial x}^T \frac{\partial \Phi_2}{\partial x} + \frac{\partial \Phi_3}{\partial x}^T \frac{\partial \Phi_3}{\partial x} \right) dx - y \left[\int_0^x \left(\Phi_4^T \frac{\partial^2 \Phi_3}{\partial x^2} - \frac{\partial \Phi_1}{\partial x}^T \frac{\partial^2 \Phi_2}{\partial x^2} \right) dx + \Phi_4^T \frac{\partial \Phi_3}{\partial x} \right] + \\ & -z \left[\int_0^x \left(\Phi_4^T \frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial \Phi_1}{\partial x}^T \frac{\partial^2 \Phi_3}{\partial x^2} \right) dx - \Phi_4^T \frac{\partial \Phi_2}{\partial x} \right] \end{aligned} \tag{33}$$

$$\begin{aligned} \mathbf{G}_2 = & \int_0^x \left[\frac{\partial \Phi_1}{\partial x}^T \frac{\partial \Phi_2}{\partial x} + \int_0^x \left(\frac{\partial \Phi_1}{\partial x}^T \frac{\partial^2 \Phi_2}{\partial x^2} - \Phi_4^T \frac{\partial^2 \Phi_3}{\partial x^2} \right) dx \right] dx - \frac{1}{2} y \left(\frac{\partial \Phi_2}{\partial x}^T \frac{\partial \Phi_2}{\partial x} + \Phi_4^T \Phi_4 \right) + \\ & -z \left[\frac{\partial \Phi_2}{\partial x}^T \frac{\partial \Phi_3}{\partial x} + \int_0^x \left(\frac{\partial \Phi_1}{\partial x}^T \frac{\partial \Phi_4}{\partial x} - \frac{\partial \Phi_3}{\partial x}^T \frac{\partial^2 \Phi_2}{\partial x^2} \right) dx \right] \end{aligned} \tag{34}$$

$$\begin{aligned} \mathbf{G}_3 = & \int_0^x \left[\frac{\partial \Phi_1}{\partial x}^T \frac{\partial \Phi_3}{\partial x} + \int_0^x \left(\frac{\partial \Phi_1}{\partial x}^T \frac{\partial^2 \Phi_3}{\partial x^2} + \Phi_4^T \frac{\partial^2 \Phi_2}{\partial x^2} \right) dx \right] dx + \\ & +y \int_0^x \left(\frac{\partial \Phi_1}{\partial x}^T \frac{\partial \Phi_4}{\partial x} - \frac{\partial \Phi_3}{\partial x}^T \frac{\partial^2 \Phi_2}{\partial x^2} \right) dx - \frac{1}{2} z \left(\frac{\partial \Phi_3}{\partial x}^T \frac{\partial \Phi_3}{\partial x} + \Phi_4^T \Phi_4 \right). \end{aligned} \tag{35}$$

The terms of the direction cosines matrix are not reported here for brevity, the procedure for the computation is similar.

3 Model implementation and validation

3.1 Model implementation

The implementation of the model is similar to (Ferretti et al., 2014), a component fully compatible with the standard Modelica multibody library has been developed. The shape functions and the invariants which assemble the terms of eq.(18) are collected in a Modelica `record` defined as `replaceable`, in order to exploit the object-oriented approach of the language and possibly instantiate multiple flexible beams in the same model. The aforementioned record has been calculated offline by means of an external script written in Matlab (Mathworks, 2014) starting from the geometric and material parameters of the beam. The symbolic computation toolbox has been used to solve eqs.(28,29,30).

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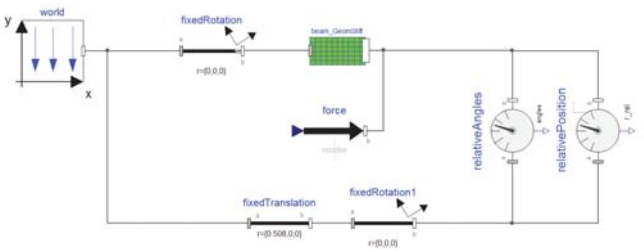

Figure 3. Diagram level scheme of the beam in a simple model

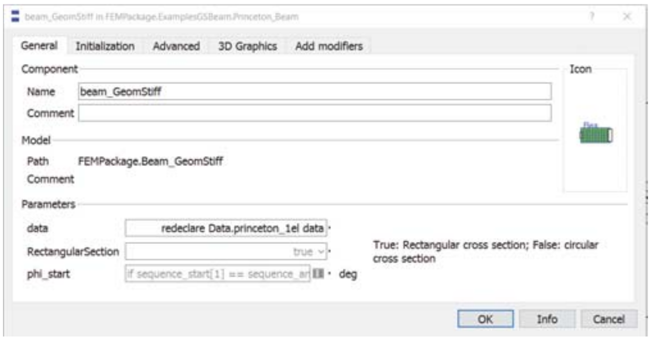

Figure 4. User interface of the beam model

The connectors are placed at the ends of the beam and the number is limited to two. As an example of model usage, fig. 3 shows a simple implementation containing the beam, while fig. 4 shows the GUI of the beam model, where the replaceable data record can be modified.

In the rigid `body` components of the multibody library, the body coordinates are used as state variables when the component is floating, this is carried out by selecting the component as *root* of one connection tree (see (The Modelica Association, 2009; Otter et al., 2003) for details). Conversely, if the body is connected to a *root*, a `branch` statement is declared and the kinematic is computed from the states of the joints connecting the component to the root tree. This mechanism is reproduced in the implementation described here, the FFR (placed in the `FrameA` connector) is assigned as root of the connection tree if the body is floating, if the body is part of a kinematic chain a `branch` is declared between `FrameA` and `FrameB`. Finally, the 3D visualization of the component is provided by means of the `Advanced.Shape` visualizer of the standard multibody library.

3.2 Model validation

The model has been thoroughly validated by means of two simulation scenarios. Initially, the well-known Princeton beam experiment (Bauchau et al., 2016) has been reproduced in order to validate the quasi-static behaviour of the model. The beam is subject to a lateral load in different root orientations ranging from 0 to 90 degrees. The setup is briefly shown in fig. 5 while the geometric and material parameters of the beam are shown in tab. 1. This experiment is particularly effective in order to validate the static deflection of the beam as well as the coupled bending/torsional behaviour, due to the different relative orientation of the load and beam.

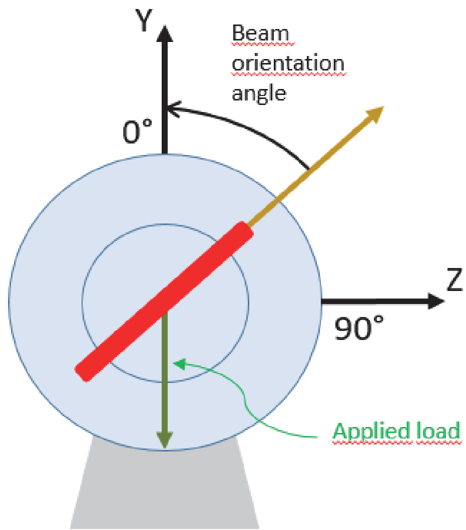

Figure 5. The Princeton beam experiment, reference scheme

Length	$0.508\ m$
Height	$3.2024x10^{-3}\ m$
Width	$12.377x10^{-3}\ m$
Axial S	$2.842x10^{6}\ N$
Shearing K_{22}	$0.6401x10^{6}\ N$
Shearing K_{33}	$0.9039x10^{6}\ N$
Torsional H_{11}	$3.103\ Nm^2$
Bending H_{22}	$36.28\ Nm^2$
Bending H_{33}	$2.429\ Nm^2$

Table 1. Princeton beam parameters

The simulations have been carried out in three different loading conditions, namely $P_1 = 4448N$, $P_2 = 8896N$ and $P_3 = 13345N$ and the results have been compared with simulations performed in the software Dymore(Dymore Solutions, 2016) where a geometrically exact beam theory is implemented. A photogram of the animation is shown in fig. 6, where the green arrow on the tip of the bent beam represents the applied load. Moreover, a substructutred instance has been tested in order to show the difference in performance and accuracy. The beam has been divided in 5 elements (6 dofs each) placed in series by simply connecting five instances of the model. Figs. 7 and 8 show the absolute displacement of the transverse components of the beam with respect to the beam root orientation, while fig. 9 shows the twisting angle in the same circumstances. The continuous lines represent the Dymore solutions while the triangles represent the simulations performed in Dymola (Dynasim AB). As shown in the figures, the results of the single element model are in good accordance with the exact solution in the first loading case (blue), while five substructuring elements are sufficient to correctly reproduce the exact solution in the other loading cases. Indeed, in the second and third case, large displacements are expected and a single element with a second order approximation is not adequate. As expected from a theoretical point of view, the proposed model is slightly stiffer because shear deformations are not included. It is also worth to remark that a null twisting angle (fig. 9) would be predicted by a linearized beam model whilst the coupling effect between bending and twisting is correctly captured by the present formulation even with a single element.

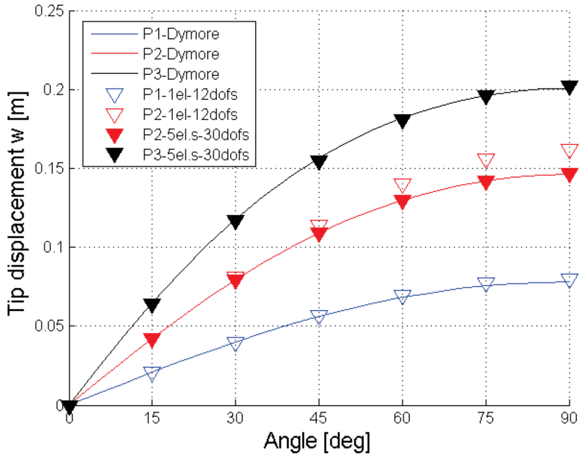

Figure 8. The Princeton beam experiment, transverse tip displacement (w)

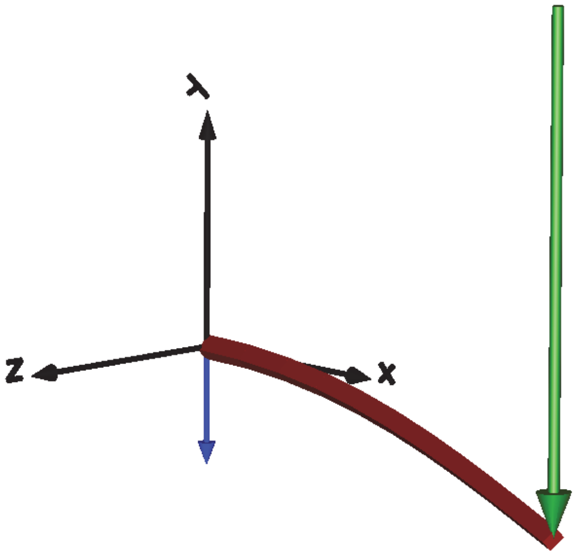

Figure 6. The Princeton beam experiment, simulation visualization

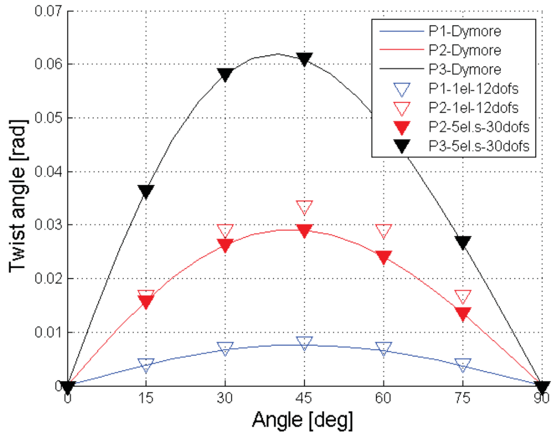

Figure 9. The Princeton beam experiment, twisting angle

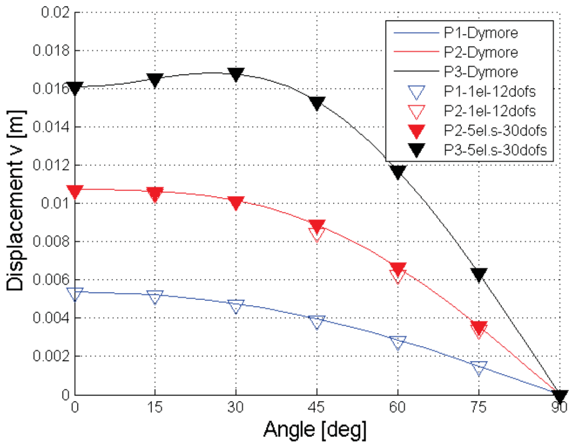

Figure 7. The Princeton beam experiment, transverse tip displacement (v)

In order to validate the dynamic behaviour of the model, the classical planar spin-up manoeuver benchmark has been considered (Berzeri and Shabana, 2002; Valembois et al., 1997; Shi et al., 2001; Wu and Haug, 1988). A flexible beam rotates about one end with a prescribed angular law, a diagram of the mechanism is reported in fig.10. The law describing the time evolution of the angle θ is the following:

$$\theta(t) = \begin{cases} \frac{\Omega}{T}\left[\frac{t^2}{2} + (\frac{T}{2\pi})^2(\cos(\frac{2\pi t}{T}) - 1)\right], & t < T \\ \Omega(t - T/2), & t \geq T \end{cases} \tag{36}$$

thus, the spin-up starts at $t = 0$ and ends at $t = T$, when a constant angular velocity is reached, where it has been assumed $T = 15s$ in the considered experiment. This benchmark is widely used in literature in order to demonstrate the effectiveness of the substructuring technique as well as the robustness of nonlinear formulations. The following geometrical data were assumed for the beam: length $L = 8m$, cross sectional area $A = 7.3 \cdot 10^{-5} m^2$, modulus of elasticity $E = 1.3359 \cdot 10^{10} N/M^2$, second moment of inertia $I = 8.218 \cdot 10^{-9} m^4$ and density $\rho = 2766 Kg/m^3$.

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The tip transverse deflection for a 20 seconds simulation is here compared with the results obtained with a completely different approach, thoroughly described in (Boer et al., 2011). A single element beam has been used by retaining two additional normal modes in order to increase accuracy. Thus the number of active dofs is five, considering only the planar components of the end node deformation. Fig. 11 shows satisfactory results in terms of accordance between the two approaches. Moreover, the dynamics shown here is in good accordance with the other results in literature (see (Boer et al., 2011; Wu and Haug, 1988)), and the maximum tip deflection is similar to other numerical results as shown in tab. 2. The proposed formulation captures correctly the geometric stiffening effect induced by the rotation and overcomes the shortcomings of the standard linear approach, while keeping low the computational effort.

Model	Number of elements	Max. deflection [m]
Present formulation	1 (5 dofs)	0.536
SPACAR, nonlinear beam	4 (8 dofs)	0.5388
SPACAR, superelement	4 superelements (8 dofs)	0.5375
Wu and Haung (Wu and Haug, 1988)	6 substructure	0.543

Table 2. Maximum tip deflection, comparison with other simulation results

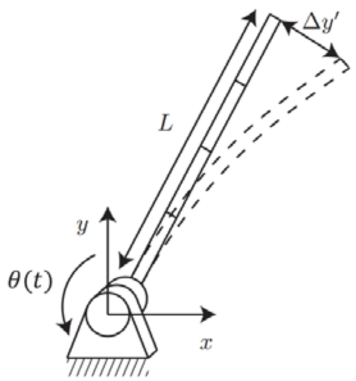

Figure 10. Planar spin-up, scheme of the setup

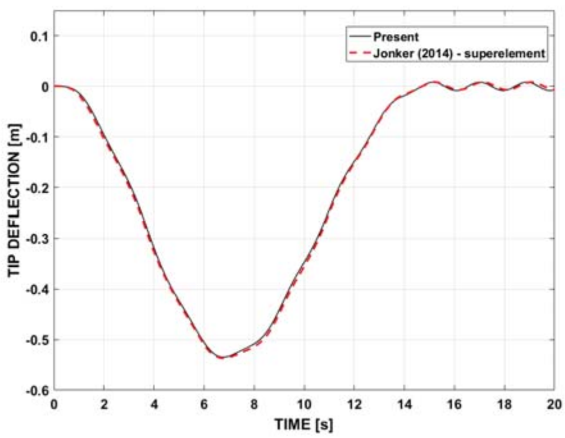

Figure 11. Planar spin-up, tip transverse deflection

4 Conclusion

In this paper, an approximated dynamic model for flexible beams is presented, including geometrically nonlinear phenomena. The equations of motion for a generic flexible body are developed starting from the approximation of the Jaumann strain tensor under the small strain hypothesis. Assuming that the deformation field of the continuum model can be expanded up to the second order, a closed form expression of the equations of motion is presented including only a few additional terms with respect to the standard floating frame of reference approach. Subsequently, the proposed formulation is applied to slender structures. A second order model is consistently derived from a geometrically exact beam model. The finite element approach is adopted in order to discretize the beam, finally the Craig-Bampton method is applied to reduce the number of degrees of freedom. The theoretical model is implemented in the Modelica framework by adopting an efficient approach where the invariant terms are computed offline and the resulting model is fully integrated in the Modelica multibody library. The model is finally validated by means of comparison with well known literature benchmarks, the numerical results are compared with simulations obtained by means of completely different approaches. The model is suitable to perform small strains, moderate displacements analysis and can be employed in large displacements cases by means of substructuring, which is naturally managed in Modelica. The proposed model will allow to consider the realistic behaviour of slender beams subject to high angular velocities, as well as to correctly consider the geometrical nonlinear phenomena in slender beams. The development of this model constitutes a step forward in the state of the art of the flexible multibody Modelica models, leading to more efficient models for real-world applications. The beam model, as well as the other flexible multibody models developed by the authors are freely available upon request, hence, the author would encourage possible users to contact them.

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