Attitude guidance and tracking for spacecraft with two reaction wheels

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This paper addresses the guidance and tracking problem for a rigid-spacecraft using two reaction wheels (RW). The guidance problem is formulated as an optimal control problem on the Special Orthogonal Group SO(3). The optimal motion is solved analytically as a function of time and is used to reduce the original guidance problem to one of computing the minimum of a nonlinear function. A tracking control using two RWs is developed that extends previous singular quaternion stabilization controls to tracking controls on the rotation group. The controller is proved to locally asymptotically track the generated reference motions using Lyapunov's direct method. Simulations of a 3U CubeSat demonstrate that this tracking control is robust to initial rotation errors and angular velocity errors in the controlled axis. For initial angular velocity errors in the uncontrolled axis and under significant disturbances the control fails to track. However, the singular tracking control is combined with a nano-magnetic torquer which simply damps the angular velocity in the uncontrolled axis and is shown to provide a practical control method for tracking in the presence of disturbances and initial condition errors.

I. Introduction

This paper is motivated by a trend towards the miniturisation of spacecraft to reduce launch and operational costs of certain missions. In particular, there is currently a significant interest in the use of nano-spacecraft (1-10kg) for undertaking Earth observation and space science missions. One aspect to enhance the possibility of further miniturization capability is to reduce the mass, power and volume requirements of the sub-systems, for example, by using less actuators. Moreover, a

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nano-sized reaction wheel (RW) can weigh around $0.2 - 0.3$ kg thus accounting for up to 30% of a nano-spacecraft's total mass. Thus, reducing the number of RWs without significantly reducing control performance could help to optimize system design. In addition, nano-spacecraft carry a much higher risk of failure than conventional spacecraft and thus contingency control algorithms must be designed to cope with actuator failure. This paper addresses the problem of optimally re-pointing a spacecraft underactuated in attitude control and thus could be useful in the event of RW failure.

Three-axis stabilization and tracking for spacecraft has been an extensive field of study which has recently focussed on the problem of compensating for external disturbances and parameter uncertainties in the control design.[1] In this paper the problem of three-axis tracking is addressed in the case that the spacecraft is underactuated in control. This problem is particularly challenging as such systems are uncontrollable using continuous, time-invariant controls.[2] However, the attitude of spacecraft is known to be controllable with two thrusters for a symmetric spacecraft[8] and various underactuated attitude stabilization problems with thrusters, including spin-stabilization and those with bounded inputs, have been solved using a complex parameterization of the attitude kinematics to develop a singular control approach[3–5] and using time-varying feedback control.[6] Horri. et al.[7] proposed a singular control approach for three-axis stabilization using RWs, rather than thrusters, while also using a quaternion representation that avoids the singularities of the parameterization.[3–5] Krishnan et. al.[8] developed two discontinuous quaternion stabilization laws for spacecraft with two RWs based on nonholonomic control theory. In this paper the stabilization problem for spacecraft with two RWs is extended in the following ways: (i) to include an optimal guidance method formulated on the Special Orthogonal Group SO(3) which generates kinematically and dynamically feasible motions, (ii) the derivation of a tracking control law that can asymptotically track these time-dependent motions on SO(3) with initial condition errors in the controlled axis (iii) the development of a practical control which combines two RWs with a single magnetic torquer which can provide good tracking performance in the presence of initial condition errors and with disturbance torques.

Firstly, the proposed optimal guidance method is solved using a non-canonical formulation of the Maximum Principle.[16] The implementation is then reduced to the problem of matching the boundary conditions which is achieved through parameter optimization. This approach has been used before to tackle motion planning problems for fully-actuated spacecraft,[10] spin-stabilized spacecraft[11] and spacecraft with path and actuator constraints.[13,14] It has also been used to develop controls for underactuated spacecraft with two thrusters.[15] The guidance method proposed in this paper is specialised to the case of two RWs and is formulated as a left-invariant optimal control problem[16] which, in contrast to general optimal motion planning problems on SO(3), can be solved analytically in closed-form. The closed-form solution is then used to reduce the guidance problem to a parameter optimization problem of a single unconstrained nonlinear function which can be solved using a plethora of well known numerical methods.[17] The analytic formulation also allows the computation of the corresponding (open-loop) torque required to perform the motion. Given the analytic form of the torque time-parameterization[13] can be used to ensure that the physical limits of theRW are not exceeded.

The second main contribution of the paper is a tracking control on SO(3) that is able to track the generated reference motions using only two reaction wheels. Using Lyapunov's direct method the tracking control is shown to track a reference where the rotation error asymptotically converges to zero. The singular tracking control presented here extends that of previous singular stabilization controls to tracking on SO(3) avoiding problems related to unwinding.[18] The tracking control is shown to work effectively when there is zero angular velocity in the uncontrolled axis. However, extending this control law with two RWs to include a magnetic torquer (MT) to damp the angular velocity in the uncontrolled axis, demonstrates good tracking performance with initial condition errors and disturbances for a nano-spacecraft in Low Earth Orbit (LEO). Such a control could be useful to reduce the mass and volume requirements of an attitude control system operating in LEO, for example, tracking could be acheived with two RWs and a MT where in the case of nano-spacecraft the MT can be embedded in the solar panels.

II. Equations of motion and problem formulation

The dynamic equations of motion of a spacecraft with reaction wheels and external disturbance torques $\boldsymbol{d}$ is given by[7]

$$J\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times (J\boldsymbol{\omega} + \boldsymbol{h}) - \dot{\boldsymbol{h}} + \boldsymbol{d}, \tag{1}$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$ is the angular velocity vector, J is the symmetric, inertia tensor (assumed here to contain only the principal moments of inertia terms I_1, I_2, I_3) and $\boldsymbol{h}$ the angular momentum of the RWs. Assuming, without loss of generality, that there are two reaction wheels along the X and Y axis such that $\boldsymbol{h} = [h_1 \quad h_2 \quad 0]^T$. Expressing the attitude kinematics of the spacecraft on the Special Orthogonal Group $R(t) \in$ SO(3):

$$SO(3) \triangleq \left\{ R \in \mathbb{R}^{3\times3} : \;\; R^T R = I_{3\times3}, \;\; \det R = 1 \right\} \tag{2}$$

and with the kinematics expressed as:

$$\dot{R}(t) = R(t)\hat{\omega}, \tag{3}$$

where $\hat{\omega}$ is a skew-symmetric matrix defined by $\hat{\omega} = \omega_1 A_1 + \omega_2 A_2 + \omega_3 A_3$ where A_1, A_2, A_3 are the basis elements of the Lie algebra $\mathfrak{so}(3)$, the space of 3×3 skew-symmetric matrices with the additional structure of a Lie bracket defined by $[X, Y] = XY - YX$ where $X, Y \in \mathfrak{so}(3)$ where $A_1, A_2, A_3 \in \mathfrak{so}(3)$ is

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

where physically A_1, A_2, A_3 define the infinitesimal rotations in the roll, pitch and yaw directions respectively. An isomorphism between a vector $\boldsymbol{\omega} \in \mathbb{R}^3$ and $\hat{\omega} \in \mathfrak{so}(3)$ is given by the hat map $\hat{\cdot} : \boldsymbol{\omega} \to \hat{\omega}$ where $\boldsymbol{\omega} \times \boldsymbol{v} = \hat{\omega}\boldsymbol{v}$ for any $\boldsymbol{v} \in \mathbb{R}^3$. In addition the inverse of the hat map is denoted by

$^\vee : \hat{x} \to \boldsymbol{x}$ and $\hat{\boldsymbol{\omega}}^\vee = \boldsymbol{\omega}$, explicitly we have

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{5}$$

For the development of the guidance method we initially assume $\boldsymbol{d} = \vec{0}$ and the zero-total angular momentum condition $\boldsymbol{h} = -J\boldsymbol{\omega}$. Note that the disturbances are re-introduced in the tracking problem in Section VI. With these assumptions the dynamics (1) and kinematics (3) reduce to:

$$\begin{aligned} \dot{\boldsymbol{h}} &= \boldsymbol{T}, \\ \dot{R}(t) &= -R(t)(\tfrac{h_1}{I_1}A_1 + \tfrac{h_2}{I_2}A_2), \end{aligned} \tag{6}$$

where $\boldsymbol{T} = [T_1, T_2, 0]^T$ is the torque applied to the two RW. The system (6) is controllable between any two configurations $R(0)$ and $R(T)$ as the Lie bracket of the controlled basis elements A_1, A_2 generate an infinitesimal motion in the third axis i.e. $[A_1, A_2] = A_3$.[16] The guidance problem is then formulated as the following optimal control problem of computing $R(t)$ subject to the following constraints, cost function and boundary conditions:

$$\begin{cases} \dot{\boldsymbol{h}} = \boldsymbol{T}, & |\boldsymbol{T}| < \delta \\ \dot{R}(t) & = -R(t)(\tfrac{h_1}{I_1}A_1 + \tfrac{h_2}{I_2}A_2) \\ \min & J = \tfrac{1}{2}\int_0^T h_1^2 + c h_2^2 dt, \\ \text{where} & R(0) = R_0, R(T) = R_d \end{cases} \tag{7}$$

where $c > 0$ is a scalar weight of the cost function and δ is an upper-bound on the magnitude of the torque that can be applied to the reaction wheels. In the following section we analytically compute the necessary conditions for optimality and the form of the optimal motion. In Section IV we present a method to match the boundary conditions and ensure that the (open-loop) torque required to perform the motion is within the physical bounds of the actuator.

III. Optimal geometric attitude guidance

In this section we derive an the analytic form of the optimal motion and the required (open-loop) torques. We make use of Pontryagin's Maximum Principle applied to left-invariant systems on SO(3)[16] which is most naturally expressed using a non-canonical formulation exploiting the geometry of the cotangent bundle T^*SO(3) of SO(3). The cotangent bundle T^*SO(3) can be trivialized (from the left) such that T^*SO(3)= SO(3) $\times\mathfrak{so}(3)^*$, where $\mathfrak{so}(3)^*$ is the dual space of the Lie algebra $\mathfrak{so}(3)$. The dual space $\mathfrak{so}(3)^*$ has a natural Poisson structure, called the minus Lie-Poisson structure, given by

$$\{F, H\}(p) = -p([dF(p), dH(p)])$$

for $p \in \mathfrak{so}(3)^*$ and $F, H \in C^\infty(\mathfrak{so}(3)^*)$ in canonical form as:

$$H = \langle \boldsymbol{\lambda}, f(\boldsymbol{x}, \boldsymbol{u}, t)\rangle - L(\boldsymbol{x}, \boldsymbol{u}, t) \tag{8}$$

where $(\boldsymbol{\lambda}, \boldsymbol{x}) \in \mathbb{R}^{2n}$ and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$, where $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, t)$ is a differential constraint, $\boldsymbol{\lambda}$ is a vector of co-states and $L(\boldsymbol{x}, \boldsymbol{u}, t)$ the Lagrangian associated with the cost-function $\int L(\boldsymbol{x}, \boldsymbol{u}, t)dt$ to be minimized. However, in this case it is convenient to use a non-canonical formulation as we can express the dynamic constraint on the Lie algebra as $\hat{f}(h_1, h_2) = R(t)^T \dot{R}(t) = -(\frac{h_1}{I_1}A_1 + \frac{h_2}{I_2}A_2) \in \mathfrak{so}(3)$ with $\hat{\lambda} = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3$. Also noting that minimising the function J in (7) is equivalent to minimising the function

$$J_1 = \frac{1}{2}\int_0^T \frac{h_1^2}{I_1^2} + k\frac{h_2^2}{I_2^2}dt, \tag{9}$$

we can write the Hamiltonian (independently of $R(t) \in$ SO(3)) as

$$H = \left\langle \hat{\lambda}, \hat{f}(h_1, h_2)\right\rangle - \frac{1}{2}(\frac{h_1^2}{I_1^2} + k\frac{h_2^2}{I_2^2}) \tag{10}$$

where $\left\langle \hat{A}, \hat{B}\right\rangle = -\frac{1}{2}trace(\hat{A}\hat{B})$ is the inner product on the Lie algebra $\mathfrak{so}(3)$ of the Lie group SO(3).[16]

From the Maximum Principle and given that H is a concave function in h_1, h_2 the necessary conditions for optimality are:

$$\begin{aligned} \tfrac{\partial H}{\partial h_1} &= 0 \Rightarrow \lambda_1(t) = -\tfrac{h_1}{I_1} \\ \tfrac{\partial H}{\partial h_2} &= 0 \Rightarrow \lambda_2(t) = -k\tfrac{h_2}{I_2}. \end{aligned} \tag{11}$$

Substituting (11) into (10) we obtain the optimal Hamiltonian as a function of the co-states,

$$H = \frac{1}{2}\left(\lambda_1{}^2(t) + \frac{\lambda_2{}^2(t)}{k}\right). \tag{12}$$

It is well known that the Hamiltonian vectors fields can be constructed using the Poisson bracket for arbitrary co-ordinates $z_i \in G \times T^*G$ where $G \times T^*G$ is the co-tangent bundle where G is the manifold and T^*G is the co-tangent space such that:

$$\dot{z}_i(t) = \{z_i(t), H\}, \tag{13}$$

where the general equation for the Poisson bracket of two functions H, G in arbitrary co-ordinates can be stated as:

$$\{F, H\} = \frac{\partial F}{\partial z_i}\{z_i, z_j\}\frac{\partial H}{\partial z_j}, \tag{14}$$

Note that if z_i is canonical with $z_i = (p_i, q_i)$ ($\{z_i, z_j\} = \{q_i, p_i\} = \partial_{ij}$ and $\{p_i, q_i\} = -\partial_{ij}$ where ∂_{ij} is the Kronecker delta function) then we retrieve the well-known canonical Poisson bracket and the Hamiltonian vector fields (13) yield Hamilton's classic canonical equations. In the case of the Hamiltonian (12) which is defined on the dual of the Lie algebra $\mathfrak{so}(3)^*$ then the Poission bracket is defined in terms of the Lie bracket

$$\{\lambda_i(t), \lambda_j(t)\} = \{p(A_i), p(A_j)\} = -p([A_i, A_j]) \tag{15}$$

this allows us to compute the Hamiltonian vector fields corresponding to H using the formula:

$$\dot{\lambda}_i(t) = \{\lambda_i(t), H\} = \{\lambda_i(t), \lambda_j(t)\}\frac{\partial H}{\partial \lambda_j(t)} \tag{16}$$

Particularly, in the case of the optimal control problem (7),

Proposition 1. *The equations for the optimal co-states are given by*

$$\begin{aligned} \dot{\lambda}_1(t) &= -\tfrac{\lambda_2(t)\lambda_3(t)}{k} \\ \dot{\lambda}_2(t) &= \lambda_1(t)\lambda_3(t) \\ \dot{\lambda}_3(t) &= \tfrac{1-k}{k}\lambda_1(t)\lambda_2(t). \end{aligned} \tag{17}$$

where the optimal angular velocities of the spacecraft are $\lambda_1(t) = \omega_1$ *and* $\lambda_2(t) = k\omega_2$.

Proof. Trivially following (16) for the left-invariant Hamiltonian (12) with (11) yields (17). □

From these necessary conditions for optimality we can construct the form of the optimal torques of the RWs.

A. Open-loop control for a spacecraft with two RWs

In this section we develop an analytic form of the attitude motions that satisfy the necessary conditions for optimality (17) in terms of the Jacobi Elliptic Functions $\mathrm{sn}(\cdot,\cdot)$, $\mathrm{cn}(\cdot,\cdot)$ and $\mathrm{dn}(\cdot,\cdot)$.

Theorem 1. *The optimal open-loop control laws, under the condition of zero-total angular momentum, that satisfy the necessary conditions for optimaility of the optimal control problem* (7) *are of the form:*

$$\begin{aligned} T_1 &= \mathrm{sgn}(\lambda_2)\mathrm{sgn}(\lambda_3)\sqrt{2H}I_1\alpha\,\mathrm{cn}(\alpha t+\beta, m)\,\mathrm{dn}(\alpha t+\beta, m) \\ T_2 &= \mathrm{sgn}(\lambda_2)k\sqrt{2Hk}I_2\alpha\,\mathrm{dn}(\alpha t+\beta, m)\,\mathrm{sn}(\alpha t+\beta, m) \end{aligned} \tag{18}$$

where sgn *is the sign function and the constants are defined as*

$$\begin{cases} \alpha &= \sqrt{(M-2Hk)/k} \\ \beta &= \mathrm{cn}^{-1}\left(\frac{\mathrm{sgn}(\lambda_2)\lambda_2}{\sqrt{2Hk}}, m\right), \\ m &= \sqrt{\frac{2H(k-1)}{2Hk-M}}, \end{cases} \tag{19}$$

where $H = \frac{1}{2}(\lambda_1^2 + \frac{\lambda_2^2}{k})$ *and* $M = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. $\lambda_1, \lambda_2, \lambda_3$ *are the initial co-states that must be chosen to satisfy the boundary conditions* $R(0) = R_0$ *and* $R(T) = R_d$.

Proof. To solve (17) observe the Hamiltonian (12) implicitly defines an elliptic cylinder and therefore it can be parameterized by Jacobi Elliptic Functions:

$$\lambda_1(t) = \alpha_1 \text{sn}(\theta, m), \lambda_2(t) = \alpha_2 \text{cn}(\theta, m) \tag{20}$$

substituting these into (12) gives $\alpha_1^2 = 2H, \alpha_2^2 = 2Hk$. Substituting (26) into (17) gives:

$$\lambda_3(t) = \text{sgn}(\lambda_3(0))\sqrt{k}\dot{\theta}\text{dn}(\theta, m) \tag{21}$$

where $\theta = \sqrt{\left(\frac{1-k}{k}\right)\frac{2H}{m^2}}t + \beta$. To compute m we observe that the function

$$M = \lambda_1^2(t) + \lambda_2^2(t) + \lambda_3^2(t), \tag{22}$$

is a constant of motion ($\{M, H\} = 0$) and substituting in (26) and (21) yields:

$$m = \sqrt{\frac{2H(k-1)}{2Hk - M}}. \tag{23}$$

The signs of α_2 can be determined by inspection of $\lambda_2(t)$ and the sign of α_1 by the equations (17) such that $\text{sgn}(\alpha_2) = \text{sgn}(\lambda_2(0))$ and $\text{sgn}(\alpha_1) = -\text{sgn}(\lambda_2(0))\text{sgn}(\lambda_3(0))$. This yields

$$\begin{aligned} \omega_1 &= -\text{sgn}(\lambda_2)\text{sgn}(\lambda_3)\sqrt{2H}\text{sn}\left(\alpha t + \beta, m\right) \\ \omega_2 &= \text{sgn}(\lambda_2)\sqrt{2Hk}\text{cn}\left(\alpha t + \beta, m\right) \end{aligned} \tag{24}$$

then differentiating (24) and substituting into (11) and (6) gives (18). For simplicity we write $\lambda_i(0) = \lambda_i$ □

Note that when $k = 1$ the open-loop optimal torque (18) applied to the reaction wheels simplify to the form:

$$\begin{aligned} T_1 &= \text{sgn}(\lambda_2)\text{sgn}(\lambda_3)\sqrt{2H}I_1\alpha\cos(\alpha t + \beta) \\ T_2 &= \text{sgn}(\lambda_2)\sqrt{2H}I_2\alpha\sin(\alpha t + \beta) \end{aligned} \tag{25}$$

where $\alpha = \sqrt{(M-2H)}$ and $\beta = \tan^{-1}(-\mathrm{sgn}(\lambda_3)\frac{\lambda_1}{\lambda_2})$.

Theorem 2. The general form $R(t) \in$ SO(3) for the optimal rotation corresponding to the necessary conditions for optimality

$$\begin{aligned}\lambda_1(t) &= -\mathrm{sgn}(\lambda_2)\mathrm{sgn}(\lambda_3)\sqrt{2H}\mathrm{sn}(\alpha t+\beta, m),\\ \lambda_2(t) &= \mathrm{sgn}(\lambda_2)\sqrt{2Hk}\mathrm{cn}(\alpha t+\beta, m),\\ \lambda_3(t) &= \mathrm{sgn}(\lambda_3(0))\sqrt{k}\alpha\mathrm{dn}(\alpha t+\beta, m)\end{aligned} \tag{26}$$

where α, β, m are defined in (19) are described by the equation:

$$R(t) = R(0)R_p(0)^T R_p(t) \tag{27}$$

where

$$R_p(t) = \exp(\phi_1 A_3)\exp(\phi_2 A_1)\exp(\phi_3 A_3), \tag{28}$$

with

$$\begin{aligned}\cos\phi_2 &= \tfrac{\lambda_3}{\sqrt{M}}, \sin\phi_2 = \tfrac{\sqrt{M-\lambda_3^2}}{\sqrt{M}}\\ \cos\phi_3 &= \tfrac{\lambda_1}{\sqrt{M-\lambda_3^2}}, \sin\phi_3 = \tfrac{\lambda_2}{\sqrt{M-\lambda_3^2}},\end{aligned} \tag{29}$$

and

$$\dot{\phi}_1 = \sqrt{M}\frac{\lambda_1^2 + (\lambda_2^2/k)}{\lambda_1^2+\lambda_2^2}. \tag{30}$$

Proof. It is well known that for left-invariant control systems on SO(3) with quadratic cost functions of the form $J = \frac{1}{2}\int_0^T \boldsymbol{\omega}^T Q\boldsymbol{\omega} dt$ can be lifted to Hamiltonian vector fields described by the Lax Pair equations $\dot{\hat{\lambda}} = [\hat{\lambda}, \nabla H]$[16] where $\hat{\lambda}, \nabla H \in \mathfrak{so}(3)$ with general solution:

$$\hat{\lambda}(t) = R(t)^T\hat{\lambda}(0)R(t). \tag{31}$$

then analagous to the integration procedure used to solve the optimal control problem for a spin-stabilized spacecraft on the quaternions[11] and the natural dynamics[12] we can choose an initial

$R_p(0)$ such that $R_p(0)\hat{\lambda}(0)R_p(0)^T = \sqrt{M}A_3$ then we have

$$R_p(t)\hat{\lambda}(t)R_p(t)^T = \sqrt{M}A_3, \tag{32}$$

then substituting in (28) (noting that $\exp(-\phi_1 A_3)A_3\exp(\phi_1 A_3) = A_3$) and solving gives (29). Then substituting (28) into $\dot{R} = R(-\frac{h_1}{I_1}A_1 - \frac{h_2}{I_2}A_2)$ and simplifying gives

$$\begin{cases} \lambda_1\dot{\phi}_1 + \dot{\phi}_2\cos\phi_3 &= \lambda_1 \\ \lambda_2\dot{\phi}_1 - \dot{\phi}_2\sin\phi_3 &= \frac{\lambda_2}{k} \\ \dot{\phi}_1\lambda_3 + \dot{\phi}_3 &= 0. \end{cases} \tag{33}$$

finally substituting (29) into (33) yields (30) □

Lemma 1. *For the case of $k = 1$ a convenient closed-form solution can be obtained where $R_p(t) = (\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z})$ where the orthonormal vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are defined by:*

$$\boldsymbol{x} = \begin{bmatrix} \cos\phi_1\cos\theta - C_1\sin\phi_1\sin\theta \\ \sin\phi_1\cos\theta + C_1\cos\phi_1\sin\theta \\ C_2\sin\theta \end{bmatrix} \tag{34}$$

$$\boldsymbol{y} = \begin{bmatrix} -\cos\phi_1\sin\theta - C_1\sin\phi_1\cos\theta \\ -\sin\phi_1\sin\theta + C_1\cos\phi_1\cos\theta \\ C_2\cos\theta \end{bmatrix} \tag{35}$$

$$\boldsymbol{z} = \begin{bmatrix} C_2\sin\phi_1 \\ -C_2\cos\phi_1 \\ C_1 \end{bmatrix} \tag{36}$$

where

$$
\begin{aligned}
\phi_1 &= \sqrt{M}t, \\
\theta &= -\text{sgn}(\lambda_3)(\sqrt{M-2H}t+\beta), \\
C_1 &= \text{sgn}(\lambda_3)\sqrt{1-(2H/M)}, \\
C_2 &= \text{sgn}(\lambda_2)\sqrt{2H/M}.
\end{aligned} \tag{37}
$$

Proof. This follows directly from Theorem 2 evaluated explicitly at $k = 1$. □

IV. Open-loop control (guidance) algorithm implementation

In this section the analytic form of the optimal attitude motion and the torque derived in the previous section is used to develop a guidance method. Moreover, the parameters $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]$ must be selected so that the boundary conditions on the rotation are matched. In the first stage a kinematically feasible motion is computed to match the boundary conditions on the rotation on a virtual domain $t \in [0, 1]$. Note that in the second stage, due to the semi-analytical nature of the method, the time can be parameterized to ensure that the angular velocities and torques are feasible.[13] Matching the boundary conditions reduces to the problem of minimizing an appropriate metric of the rotation error which in this case is taken to be:

$$J_2 = \min_{\boldsymbol{\lambda}} \|R_e(1)\| \tag{38}$$

where

$$\|R_e(t)\| = \text{tr}[I_{3\times3} - R_e(t)]. \tag{39}$$

Following this the dynamic feasibility of the motion is considered by converting the problem on the virtual domain $t \in [0, 1]$ to one on a real time domain $\tau \in [0, T_f]$, a procedure often used in robotic motion planning,[13] where $t = \tau/T_f$ such that

$$T_i = -\frac{I_i}{T_f^2}\dot{\omega}(t/T_f) \tag{40}$$

which for $k = 1$ is:

$$\begin{aligned} T_1 &= \text{sgn}(\lambda_2)\text{sgn}(\lambda_3)\tfrac{I_1\sqrt{2H(M-2H)}}{T_f^2}\cos\left(\tfrac{\sqrt{M-2H}}{T_f}t+\beta\right) \\ T_2 &= \text{sgn}(\lambda_2)\tfrac{I_2\sqrt{2H(M-2H)}}{T_f^2}\sin\left(\tfrac{\sqrt{M-2H}}{T_f}t+\beta\right) \end{aligned} \tag{41}$$

therefore given a maximum torque capability of the RW we can select T_f using the following formula:

$$\frac{I_i\sqrt{2H(M-2H)}}{\max\{T_i\}} \leq T_f^2 \tag{42}$$

To demonstrate this we set the boundary conditions to $R(0) = I_{3\times 3}$ and a randomly selected final rotation:

$$R_d = \begin{bmatrix} 0 & 1 & 0 \\ -0.623 & 0 & 0.782 \\ 0.782 & 0 & 0.623 \end{bmatrix}. \tag{43}$$

The error (39) is a nonlinear function of $\boldsymbol{\lambda}$ and can be solved numerically by formulating a parameter optimization problem. This is a similar optimization problem to that found in[13] where a comparison of different optimizers was undertaken in Mathematica. Essentially, the numerical optimizer needs to be robust to problems with multiple local minima such as the Random Search (RS) method that is used in Mathematica and gives the following values for the parameters $\lambda_1 = 2.80745, \lambda_2 = -1.73597, \lambda_3 = -3.60479$. Although this provides suitable results on a PC for practical implementation on-board a spacecraft the computational efficiency of the algorithm would be critical when selecting the optimizer. The computed parameters yield the following open-loop controls that will match the boundary conditions on $t \in [0, T_f]$:

$$\begin{aligned} T_1 &= 0.129696\cos(-3.60479t/T_f + 1.017)/(T_f^2) \\ T_2 &= 0.599696\sin(-3.60479t/T_f + 1.017)/(T_f^2) \end{aligned} \tag{44}$$

As an example we assume the maximum torque for each reaction wheel is 10 mNm which the open loop controls (44) violate. By using the equation (41) and choosing an appropriate value for T_f such as $T_f = 100$ secs the open-loop controls fall within the feasible limits. Note that for the zero-angular momentum condition we must have $\omega(0) = [0.0280745 \quad -0.0173597 \quad 0]^T$. However, if there is an error in these initial conditions the open-loop controls will not respect the boundary condition $R(T) = R_T$ as is demonstrated in Fig. 1. which shows the open-loop control under the zero-angular momentum condition (continuous line) and with initial condition errors (dashed line) where $\omega(0) = [0.025 \quad -0.017 \quad 0]^T$.

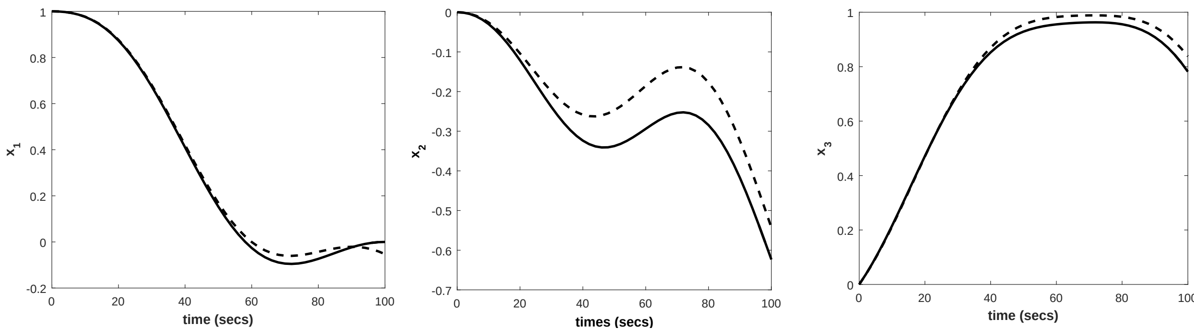

Figure 1. The pointing vector of the attitude motion (the first column vector of $R(t)$) (i) with the zero-angular momentum condition (continuous line) and (ii) with initial condition errors (dashed line)

As the open-loop control is not robust to errors in the initial conditions it is essential for practical implementation that the motion generated under perfect conditions is tracked with a closed-loop control. The development of a closed-loop feedback tracking control using only two RWs is addressed in the following section.

V. Attitude tracking with two RWs

In this section a tracking controller is developed that is able to track the generated attitude motions in the presence of initial condition errors and disturbances. This method extends the singular stabilization controls for two RWs in[7] to the tracking problem. In the first instance we develop an inner-loop for the controller which defines a virtual angular velocity $\boldsymbol{\omega}^*$ that will be tracked using an outer-loop controller. Defining the rotation error $R_e = R_d^T R$ where R_d is a feasible reference motion (such as those generated in the previous section) that satisfies the kinematic equation $\dot{R}_d = R_d \hat{\omega}_d$ where $\boldsymbol{\omega}_d$ is the angular velocity of the reference motion in the body-fixed frame and where the error dynamics are

$$\dot{R}_e = R_e \hat{\omega}_e \tag{45}$$

with $\boldsymbol{\omega}_e = \boldsymbol{\omega}^* - R_e^T \boldsymbol{\omega}_d$ with the virtual angular velocity $\boldsymbol{\omega}^* = [\omega_1^* \quad \omega_2^* \quad \omega_3^*]^T$ equal to

$$\begin{aligned}
\omega_1^* &= (R_e^T \boldsymbol{\omega}_d - k_1(R_e - {R_e}^T)^\vee) \cdot \boldsymbol{e_1} - k_2(R_e^{(1,2)} - R_e^{(1,1)}\omega_{e3})/R_e^{(1,3)} \\
\omega_2^* &= (R_e^T \boldsymbol{\omega}_d - k_1(R_e - {R_e}^T)^\vee) \cdot \boldsymbol{e_2} + k_2(R_e^{(2,1)} - R_e^{(2,2)}\omega_{e3})/R_e^{(2,3)} \\
\omega_3^* &= 0
\end{aligned} \tag{46}$$

where $k_1, k_2 > 0$ are scalar gains, $\boldsymbol{e_1} = [1 \quad 0 \quad 0]^T$, $\boldsymbol{e_2} = [0 \quad 1 \quad 0]^T$ and $R_e^{(i,j)}$ are the i th row and j th column component of the error rotation R_e. Intuitively, we can see from the error dynamics (45) the component of the virtual angular velocity (46), $-k_1(R_e - {R_e}^T)^\vee \cdot \boldsymbol{e_i}$ where $i = 1, 2$ will drive the errors to zero in the first two controlled axis such that $R_e^{(3,2)}, R_e^{(2,3)}, R_e^{(1,3)}, R_e^{(3,1)} \to 0$ as $t \to \infty$. The components $-k_2(R_e^{(1,2)} - R_e^{(1,1)}\omega_{e3})/R_e^{(1,3)}$ and $k_2(R_e^{(2,1)} - R_e^{(2,2)}\omega_{e3})/R_e^{(2,3)}$ have the effect of ensuring that $R_e^{(2,1)}, R_e^{(1,2)} \to 0$ as $t \to \infty$. To see this substitute $\omega_1^* = -k_2(R_e^{(1,2)} - R_e^{(1,1)}\omega_{e3})/R_e^{(1,3)}$ and $\omega_2^* = k_2(R_e^{(2,1)} - R_e^{(2,2)}\omega_{e3})/R_e^{(2,3)}$ into (45) with $\boldsymbol{\omega}_e = \boldsymbol{\omega}^* - R_e^T \boldsymbol{\omega}_d$ which yields $\dot{R}_e^{(2,1)} = -k_2 R_e^{(2,1)}$ and $\dot{R}_e^{(1,2)} = -k_2 R_e^{(1,2)}$. Therefore, the cumulative effect of the control is that $R_e \to Id$ as $t \to \infty$.

The tracking control law for 2 RWs, under the assumption of zero-total angular momentum, can then be stated as:

$$\boldsymbol{T} = k_3(\boldsymbol{\omega} - \boldsymbol{\omega^*}) - J\frac{d\boldsymbol{\omega}^*}{dt} \tag{47}$$

where $\boldsymbol{\omega}^* = [\omega_1^* \quad \omega_2^* \quad \omega_3^*]^T$ is defined in (46).

This control is shown to locally asymptotically stabilize the closed-loop system by considering the Lyapunov function:

$$V = \frac{1}{2}\left\langle \boldsymbol{\omega} - \boldsymbol{\omega}_e - R_e^T \boldsymbol{\omega}_d, \quad J(\boldsymbol{\omega} - \boldsymbol{\omega}_e - R_e^T \boldsymbol{\omega}_d) \right\rangle \tag{48}$$

when $R = R_d$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_d$ it can be seen from (46) and (48) that $V = 0$. Simplifying (48) we write

$$V = \frac{1}{2}\left\langle \boldsymbol{\omega} - \boldsymbol{\omega^*}, \quad J(\boldsymbol{\omega} - \boldsymbol{\omega^*}) \right\rangle \tag{49}$$

and differentiating with respect to time gives

$$\dot{V} = \left\langle \ \boldsymbol{\omega} - \boldsymbol{\omega}^{\boldsymbol{*}}, \quad -\boldsymbol{T} - J\frac{d\boldsymbol{\omega}^{*}}{dt} \ \right\rangle \tag{50}$$

then setting the control to (47) gives

$$\dot{V} = -k_3 \left\langle \ \boldsymbol{\omega} - \boldsymbol{\omega}^{\boldsymbol{*}}, \quad \boldsymbol{\omega} - \boldsymbol{\omega}^{\boldsymbol{*}} \ \right\rangle \tag{51}$$

then locally about the equilibrium point $R_e = Id, \boldsymbol{\omega}_e = 0$ we see that $\dot{V} < 0$ and thus the desired state is locally asymptotically stable. The simulations are undertaken using the full equations for the attitude dynamics with RWs (1) with the principal moments of inertia equal to that of the 3U CubeSat UKube-1[12] with the initial rotation equal to

$$R(0) = \begin{bmatrix} 0.997377 & -0.0524115 & 0.0499167 \\ 0.0499167 & 0.997502 & 0.0499792 \\ -0.0524115 & -0.0473564 & 0.997502 \end{bmatrix}. \tag{52}$$

$\boldsymbol{\omega}(0) = [0 \quad 0 \quad 0]^T$ to induce an initial rotation and angular velocity error. From Fig 2. it can be

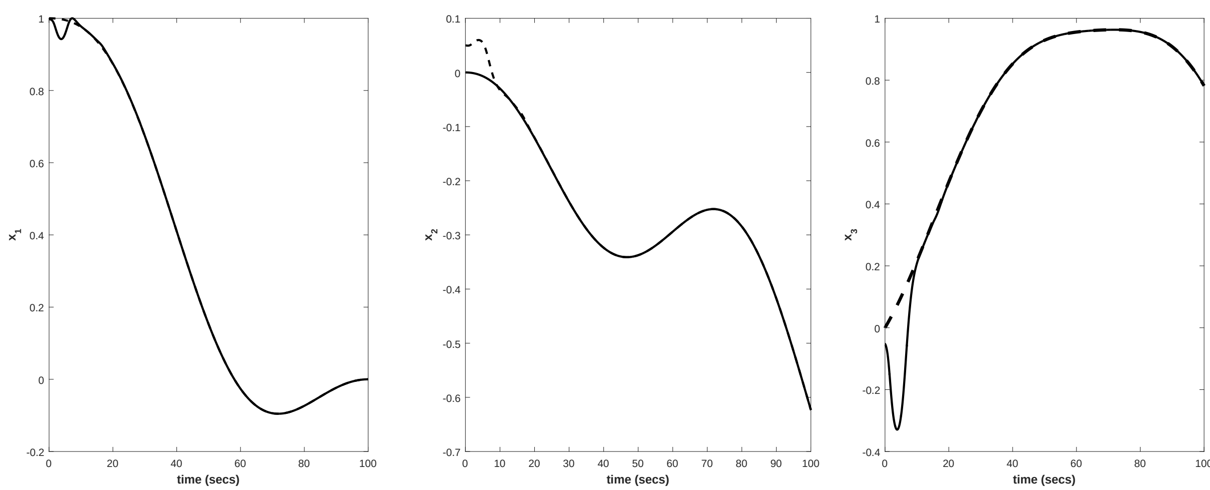

Figure 2. The pointing axis (first column of the orthonormal frame $R(t)$) where the dashed line is the reference trajectory generated by the guidance method and the solid line the actual trajectory with initial conditions errors in the angular velocity

seen that after approximately 10 seconds the trajectory of the pointing direction of the spacecraft converges closely to the reference trajectory. In the case with no initial condition errors the motion exactly replicates the motion induced by the open-loop control under perfect conditions. In Fig. 3

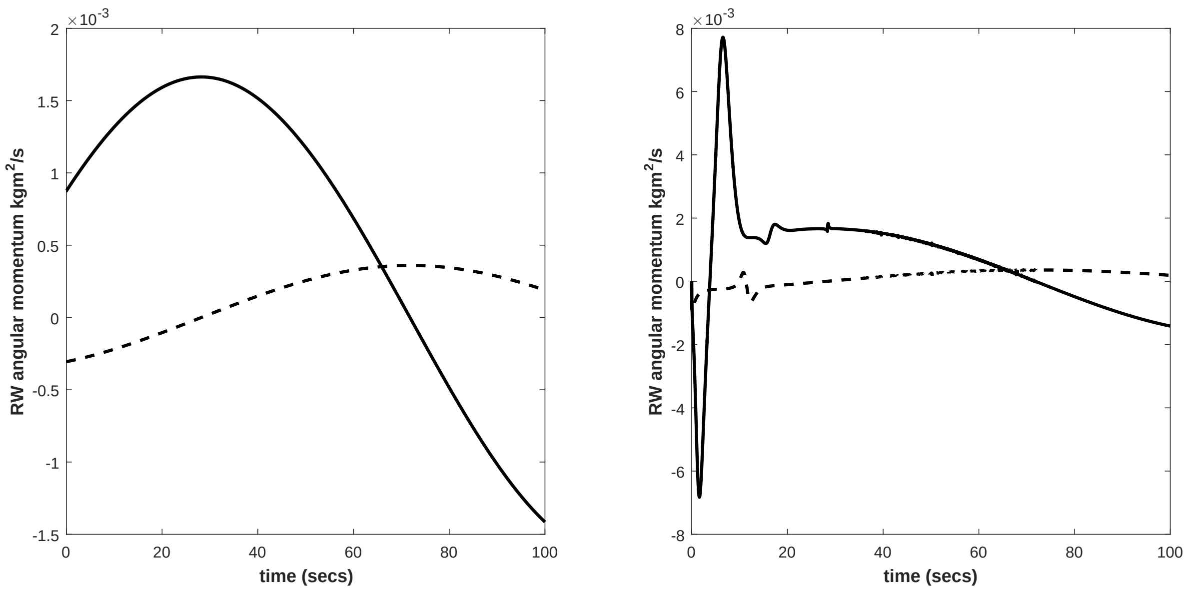

Figure 3. Angular momentum of the RW for (i) no initial state error (ii) errors in the initial angular velocity and rotation

the corresponding angular momentums of the RW are presented. In Fig. 3 (i) the angular momentum of the closed-loop controller exactly replicates that given in the case of the open-loop control. This is due to the fact that there are no disturbances and no initial condition errors, so we have perfect tracking. In Fig. 3 (ii) an initial condition error is included such that $\boldsymbol{\omega}(0) = [0 \quad 0 \quad 0]^T$. After an initial transient the angular momentum of the closed-loop controller converges to that of the open-loop controller. However, we note that the presented tracking control is only useful if there are no initial condition errors in the uncontrolled axis. In addition this section has not considered the environmental disturbance torques typical of a nano-spacecraft in LEO. Although the control in its current form is not practical (in the sense that the assumption $\omega_3 = 0$ does not hold in practise even though it maybe small) it can be utilized effectively with other attitude actuators. For example, an additional actuator in the uncontrolled axis would only need to compensate for small disturbances. Therefore, a smaller or less power consuming actuator can be used in the uncontrolled axis to undertake the less torque expensive damping task. For example, magnetic torquers (MT) are cheap and efficient actuators that can be used for damping the angular velocities (de-tumbling) of a spacecraft in LEO. However, on their own MTs do not have the capability to precisely track reference motions but could be utilized with the presented algorithm for two RWs to ensure that the angular velocity in the uncontrolled axis is damped out. This is demonstrated in the following section.

VI. Attitude tracking with external disturbances and initial condition errors

The previous section designed a singular control for attitude tracking using only two RWs. The assumption for controllability of this system is that the total-angular momentum is zero. Thus, in the uncontrolled axis we must have $\omega_3 = 0$ for all time. However, in practise there will be initial condition errors and disturbance torques in all three axis. In this section we demonstrate a practical use of the algorithm alongside a MT which performs the simple task of damping the angular velocity in the z-axis using a proportional control law,

$$D_x = -\frac{k_x}{B_\mathrm{y}}\omega_3 \tag{53}$$

where $k_x > 0$ is a gain parameter. To derive a simple controller for the 2RWs and 1 MT actuator configuration observe that the Magnetic torque $\boldsymbol{\tau}_m = [\tau_{m_1}, \tau_{m_2}, \tau_{m_3}]^T$ is given by the vector product of the dipole moment $\boldsymbol{D} = [D_x, D_y, D_z]^T$ and the magnetic field $\boldsymbol{B}$ which is modelled to 13th order in LEO as described in.[19] Assuming the use of only one MT we set $D_\mathrm{y} = D_\mathrm{z} = 0$ with the control torque in the x-axis and y-axis provided by RWs then the total applied torque $\boldsymbol{M} = [M_\mathrm{x} \quad M_\mathrm{y} \quad M_\mathrm{z}]^T$ is:

$$\begin{bmatrix} M_\mathrm{x} \\ M_\mathrm{y} \\ M_\mathrm{z} \end{bmatrix} = \begin{bmatrix} 0 & B_\mathrm{z} & -B_\mathrm{y} \\ -B_\mathrm{z} & 0 & B_\mathrm{x} \\ B_\mathrm{y} & -B_\mathrm{x} & 0 \end{bmatrix} \begin{bmatrix} D_\mathrm{x} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} T_1 \\ T_2 \\ 0 \end{bmatrix}. \tag{54}$$

Assuming a simple disturbance torque model in the dynamics (1)

$$\boldsymbol{d} = [sin(0.1\pi t), sin(0.2\pi t), sin(0.3\pi t)]^T \times 10^{-4} N \cdot m \tag{55}$$

which is representative of the magnitude of order of the disturbance of a nano-spacecraft in LEO[12] and using the same simulation parameters as in the previous section but with initial angular velocity errors in all three axis set to $\boldsymbol{\omega}(0) = [0.01, 0.01, 0.01]^T rad/s$ and the tuning parameter set to $k_x = 6$, a maximum torque of the nano-RWs set to $u_{\max} = 0.004N \cdot m$ and the maximum dipole moment of the nano-MT set to $D_{max} = 0.2A \cdot m^2$ the simulation results are shown in Fig.4 and Fig. 5. It can be seen in Fig 4. that good tracking performance can be obtained. In Fig. 5 it can be seen that the angular momentum of the RWs required for tracking are feasible and after an initial transient is of the same order of magnitude as in the ideal system without disturbances and initial condition errors. In other words after an initial transient the closed-loop control converges closely to that of the open-loop control computed using the guidance method. In addition Fig. 5 shows the magnetic dipole of the single MT required to perform the tracking and implies that it is feasible with a nano-MT. It can be seen that as the attitude converges to the reference motion the magnetic dipole of the MT converges to a very small bounded region around the origin. In Fig. 4 it can be seen that even in the presence of environmental disturbance torques and initial condition errors in all three axis that the trajectories converge to the reference attitude after an initial transient of about 10 seconds. The tracking error in each case is small and due to the fact that the disturbance torques have not been compensated for in the controller. This tracking error could potentially be improved further by compensating for the disturbances in the control, for example, by extending it to include a disturbance observer.[21] Nevertheless, the simulation shows that it is possible to track a fast attitude reference motion using only two RWs with the addition of an actuator that simply damps the angular velocity in the uncontrolled axis. In Fig. 5 the angular momentum of the reactions wheels and the magnetic dipole of the magnetic torquer are shown. In each case the controls have been saturated so that they they are representative of nano-spacecraft actuators. It is shown that tracking is feasible with current nano-spacecraft actuators.

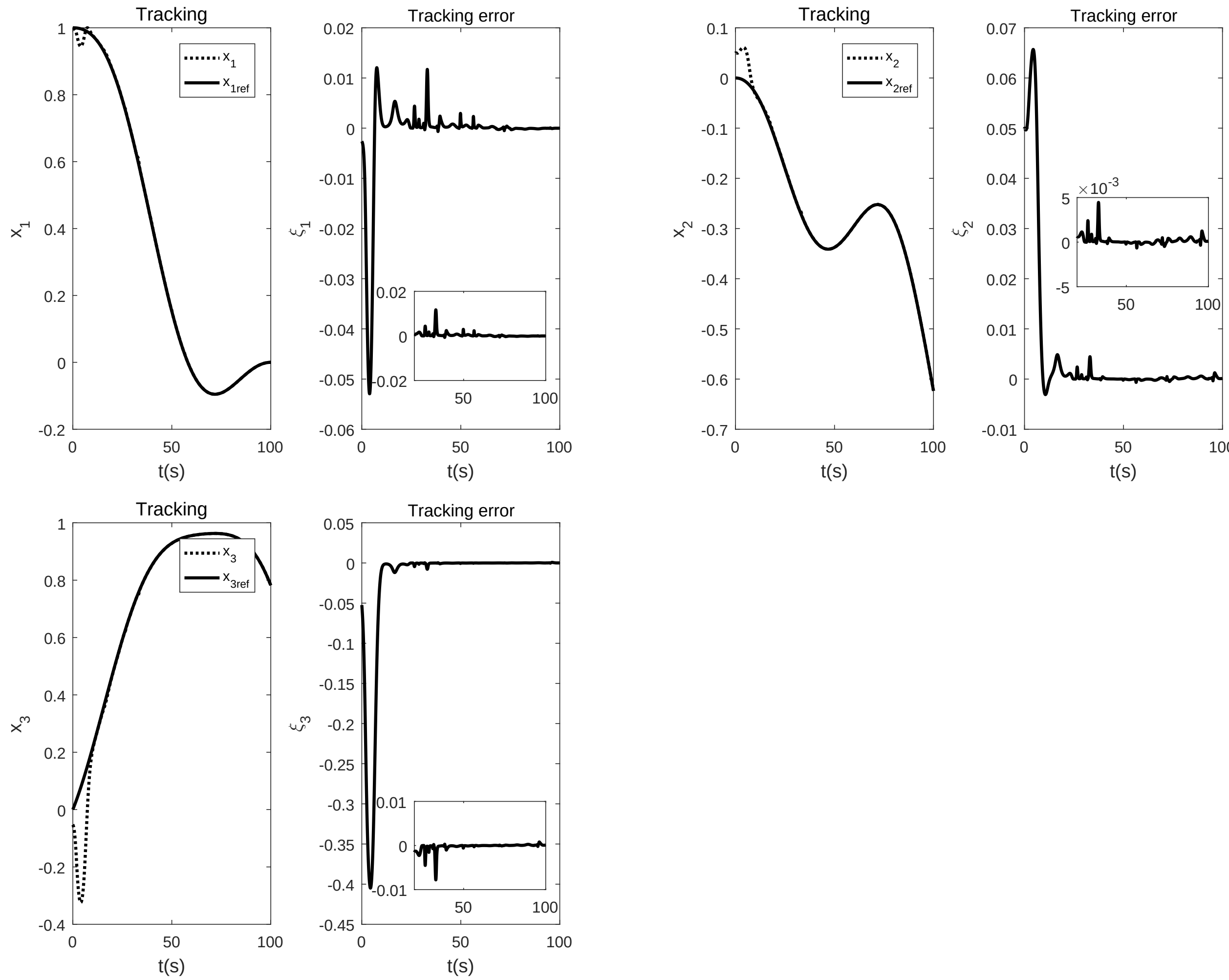

Figure 4. The pointing vector (the first column of the rotation matrix) and the tracking error in each axis ξ_i over time with the initial angular velocity $\omega(0) = [0.01 \quad 0.01 \quad 0.01]^T$.

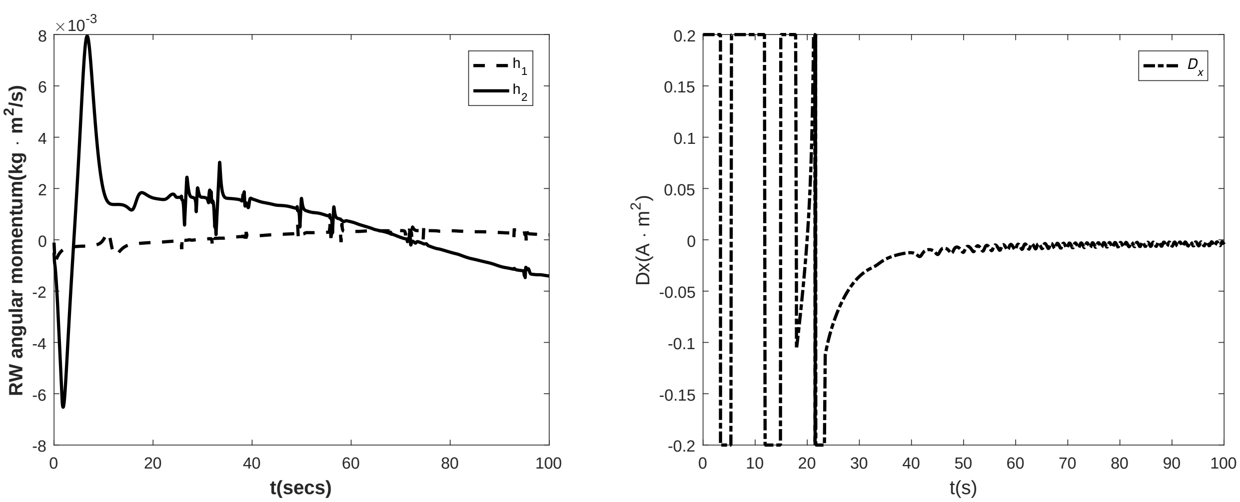

Figure 5. Angular momentum of each RW and the magnetic dipole of the magnetic torquer over time

VII. Conclusion

This paper has developed an optimal attitude guidance method for a spacecraft with two reaction wheels on the Special Orthogonal Group SO(3). The approach uses a non-canonical formulation of the Maximum Principle and solves the extremal curves, optimal angular velocities and rotation analytically. This result is used to construct a semi-analytical method to generate feasible motions with respect to both the kinematic and dynamic constraints of an underactuated spacecraft. A tracking control algorithm that can track the generated reference motions, when there are initial condition uncertainties in the controlled axis, is presented and proved to locally asymptotically track the references using Lyapunov's direct method. However, for practical implementation of this method it must also be robust to disturbance torques and initial condition errors in all three axis. It is shown that combining the tracking control method with a single magnetic torquer that the tracking control is robust to disturbances and initial condition uncertainties. As most nano-spacecraft in Low Earth Orbit are equipped with magnetic torquers for de-tumbling and reaction wheels for precision pointing the presented algorithms could be useful when a reaction wheel fails. Furthermore, a nano-spacecraft could be designed with attitude tracking capability using only two reaction wheels and a magnetic torquer for damping. In nano-spacecraft it is possible to embed the magnetic torquers in the body fixed solar panels and thus significant mass and volume savings could be made. Other possibilities for utilizing this control would be in the event of partial failure of a reaction wheel whereby two reaction wheels are fully functional and a third loses control authority through fault. In this case the faulty reaction wheel may be able to perform the simpler task of damping angular velocity while the two working reaction wheels perform the tracking.

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