Stochastic H_∞ Filtering for Neural Networks with Leakage Delay and Mixed Time-Varying Delays

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Abstract: This paper deals with the problem of H_∞ filtering for stochastic neural networks (SNNs) with a mixed of time-varying interval delays, time-varying distributed delays, and leakage delays. A novel quintuple integral Lyapunov–Krasovskii functional (LKF) is constructed to improve the performance of the SNN. Sufficient criteria can be obtained by applying the linear matrix inequality (LMI) approach and developing a new mathematical analysis, which ensures the filtering error system is asymptotically stable in the mean square. Finally, simulation results are provided to show the superiority and usefulness of the proposed method.

Keywords: H_∞ filtering, Leakage delay, Linear matrix inequality, Stochastic neural networks, Time-varying delay

1. Introduction

In recent years, the dynamic behaviors of neural networks (NNs) have been effectively investigated due to their great potential in many applications, such as pattern recognition [8], smart antenna arrays [31], and associative memory [42]. The success of these applications depends on the dynamic behaviors of the concerned NNs; thus, several articles analyzing the dynamic behaviors of NNs have been published in literature, see for instance [1–5,10,11,18,22,29,32,38,43]. Numerous practical applications require the equilibrium point of a designed NN to be unique and asymptotically stable. Stability

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analysis of NNs have been conducted, and different types of stability conditions have been proposed in [24, 36]. It should be noted that synaptic transmission is a difficult process due to the random variations in neurotransmitters and the various causes in real nerve systems. The establishments in [26, 44] showed that with some stochastic inputs, NNs can be either stabilized or destabilized. Thus, this has real-life importance for the study of stochastic effects on the dynamic behaviors of delayed NNs [28, 40]. Furthermore, the problem of stability analysis for stochastic neural networks (SNNs) is usually harder than the stability analysis of deterministic NNs. Thus, it is necessary to examine the stability of NNs with stochasticity [7, 39].

As is common knowledge, time delays are the main source of divergence, oscillation, and instability of the model. The time delays that appear in the system can be constant or various [14–16]. Generally, NNs have spatial areas due to the appearance of parallel pathways including different axon lengths and sizes. That is, the conduction velocities are distributed together with the above pathways. Therefore, distribution can occur in an extension of delays. Distributed time-delayed signals reflect the distributed signal propagation in neurons during a time span in the above pathways. Moreover, the lower bound of the delay is not always zero. Thus, different types of time-varying delays, the so-called time-varying interval delays that appear in some engineering systems, are commonly found in networked control systems. Thus, we assume that the time-varying delay belongs to a given interval. Recently, the stability criteria for NNs that includes both interval and distributed time-varying delays have been presented in [6, 17].

Leakage delay has a wide range of effects on the dynamic behavior of NNs. In fact, the characteristic of the leakage term may destabilize the dynamic behaviors of the NNs, and leakage delays are generally difficult to handle. Therefore, it is important to analyze the impact of leakage delays when investigating the state estimation and filtering design problems of NNs; many researchers intend to study leakage delay problems. In [41], the authors conducted a passivity analysis of SNNs with leakage and time-varying delays. Leakage delays have a propensity to destabilize NNs, and they are usually difficult to handle. Therefore, it is necessary to study the effects of leakage-delayed signals when analyzing the stability of NNs. Note that the existence of leakage-delayed signals in NNs has led to some new research topics in recent years [30, 37, 45]. Gan in [9] considered the exponential synchronization problem for SNNs with reaction-diffusion terms and leakage delay. Furthermore, the authors in [28] studied the synchronization problems of coupled SNNs with leakage and time-varying delays.

A general NN is a strongly interconnected model that contains a great number of neurons. NNs generally require substantial connections between neurons to handle complicated nonlinear problems. It is highly expensive or complex to get the complete state information of all neurons in an NNs in most engineering systems due to their tangled structure. Thus, evaluating the states of the neurons through available measurements to make useful NN applications is a major undertaking. Hence, it becomes important to examine the state estimation problems of delayed NNs [13, 20, 33]. The H_∞ state estimation results of static NNs were reported in [25, 34]. Saravanakumar et al. investigated the H_∞ state estimation of SNNs in [34]. H_∞ filtering for SNNs with leakage, time-varying interval, and time-varying distributed delay signals have not been completely studied in the existing literature to the authors' best knowledge, which is one of the motivations for this work.

The problem of H_∞ filtering is studied for SNNs with mixed delays in this paper. The stability criteria for delayed SNNs are obtained based on the *Itô* differential formula and theory of stochastic stability to ensure mean-square asymptotic stability with a prescribed H_∞ performance. The main contributions of this work are twofold: 1) An efficient approach is proposed to study the H_∞ filtering problem for SNNs with leakage delay and mixed time-varying delays by introducing a new Lyapunov–Krasovskii functional (LKF) including quintuple integral terms; and 2) The quintuple integral terms $\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{\alpha}^{0}\int_{t+\sigma}^{t} m^T(s)S_7m(s)dsd\sigma d\alpha d\mu d\delta$ and $\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{\alpha}^{0}\int_{t+\sigma}^{t} m^T(s)S_8m(s)dsd\sigma d\alpha d\mu d\delta$ that include the information of the activation function are utilized when obtaining the time derivative of the constructed LKF.

Notation: $\mathcal{R}^n$ and $\mathcal{R}^{n\times n}$ respectively represent the n-dimensional Euclidean space and the set of all $n \times n$ real matrices. The notation $*$ denotes the entries induced by symmetry. A^T denotes the matrix transpose of A. $X > 0$ ($X \geq 0$) denotes that X is a real symmetric positive definite (positive semi-definite) matrix. $diag\{a, b, ..., z\}$ denotes the block-diagonal matrix with elements a, b, ..., z in the diagonal entries. I is the identity matrix with appropriate dimensions.

2. System Formulation

Consider the following SNNs with mixed delays including time-varying interval delays, time-varying distributed delays, and leakage delays:

$$du(t) = \Big[- \mathcal{B}u(t-h) + W_0 g(u(t)) + W_1 g(u(t-\vartheta(t))) + W_2 \int_{t-b(t)}^{t} g(u(s))ds + B_1 v(t)\Big] dt$$

$$+\Big[Eu(t)+E_1u(t-\vartheta(t))+H_0g(u(t))+H_1g(u(t-\vartheta(t)))+H_2\int_{t-b(t)}^{t}g(u(s))ds\Big]d\omega(t), \quad (1)$$

$$x(t)=Cu(t-h)+Du(t-\vartheta(t))+B_2v(t),$$

$$\overline{\beta}(t)=Ju(t),$$

$$u(t)=\rho(t),\ t\in[-max(\vartheta_2,d,h),0],$$

where $u(t)=[u_1(t),u_2(t),...,u_n(t)]^T\in\mathcal{R}^n$ is the neuron state vector signal. $g(u(t))=[g(u_1(t)),...,g(u_n(t))]^T\in\mathcal{R}^n$ represents the activation function vector of neurons. The measurement signal $x(t)\in\mathcal{R}^m$ is the network output, $\overline{\beta}(t)\in\mathcal{R}^p$ is to be estimated, and $v(t)\in\mathcal{R}^q$ is the noise input signal in $L_2[0,\infty)$. $\omega(t)=[\omega_1(t),\omega_2(t),...,\omega_n(t)]^T$ is the n-dimensional Brownian motion. $\mathcal{B}=diag\{\lfloor_1,...,\lfloor_n\}$ is the positive diagonal matrix. $E,E_1,H_0,H_1,H_2,W_0,W_1,W_2,B_1,B_2,C,D$, and J are known constant matrices with appropriate dimensions. $\vartheta(t)$, $b(t)$, and h respectively represent the time-varying interval and time-varying distributed and leakage delays. These delays are assumed to satisfy

$$0\le\vartheta_1\le\vartheta(t)\le\vartheta_2,\ \dot{\vartheta}(t)\le\phi,\ 0\le b(t)\le d, \quad (2)$$

where $\vartheta_1,\ \vartheta_2,\ \phi,\ d$, and h are positive constants.

Assumption (1): Every $g_i(t)$ for $i=1,2,...,n$ is continuous, bounded and satisfies

$$c_i^-\le\frac{g_i(k_1)-g_i(k_2)}{k_1-k_2}\le c_i^+, \quad (3)$$

with $V^+=diag(c_1^+,c_2^+,...,c_n^+)$ and $V^-=diag(c_1^-,c_2^-,...,c_n^-)$ and $k_1,k_2\in\mathcal{R}$, $k_1\neq k_2$.

We now design the following filter for the NN in (1):

$$d\widehat{u}(t)=\Big[-\mathcal{B}\widehat{u}(t-h)+W_0g(\widehat{u}(t))+W_1g(\widehat{u}(t-\vartheta(t)))+W_2\int_{t-b(t)}^{t}g(\widehat{u}(s))ds\Big]dt+X[x(t)-\widehat{x}(t)]dt$$

$$+\Big[E\widehat{u}(t)+E_1\widehat{u}(t-\vartheta(t))+H_0g(\widehat{u}(t))+H_1g(\widehat{u}(t-\vartheta(t)))+H_2\int_{t-b(t)}^{t}g(\widehat{u}(s))ds\Big]d\omega(t),$$

$$\widehat{x}(t)=C\widehat{u}(t-h)+D\widehat{u}(t-\vartheta(t)),$$

$$\widehat{\beta}(t)=J\widehat{u}(t), \quad (4)$$

where $\widehat{u}(t)\in\mathcal{R}^n$, $\widehat{x}(t)\in\mathcal{R}^m$, and $\widehat{\beta}(t)\in\mathcal{R}^q$ respectively denote the estimates of $u(t)$, $x(t)$, and $\beta(t)$. $X\in\mathcal{R}^m$ is the filter gain matrix to be computed.

Defining $e(t) = u(t) - \widehat{u}(t)$ and $\beta(t) = \overline{\beta}(t) - \widehat{\beta}(t)$, allows us to obtain the following filtering error system from (1) and (4):

$$\begin{aligned} de(t) = \Big[&- (\mathcal{B} + XC)e(t-h) - XDe(t-\vartheta(t)) + W_0 f(e(t)) + W_1 f(e(t-\vartheta(t))) + W_2 \int_{t-b(t)}^{t} f(e(s))ds \\ &+ (B_1 - XB_2)v(t)\Big]dt + \Big[Ee(t) + E_1 e(t-\vartheta(t)) + H_0 f(e(t)) + H_1 f(e(t-\vartheta(t))) \\ &+ H_2 \int_{t-b(t)}^{t} f(e(s))ds\Big]d\omega(t), \\ \beta(t) = Je(t), & \end{aligned} \tag{5}$$

where $e(t) = [e_1(t), e_2(t), ..., e_n(t)]^T \in \mathcal{R}^n$, $f(e(\cdot)) = g(u(\cdot)) - g(\widehat{u}(\cdot))$, and $f(e(\cdot)) = [f(e_1(\cdot)), ..., f(e_n(\cdot))]^T$. Each $f_i(k)$ satisfies

$$c_i^- \leq \frac{f_i(k)}{k} \leq c_i^+, \tag{6}$$

where $k \in \mathcal{R}$, $k \neq 0$.

Definition 2.1. *[34]*

- *Mean-square stability: If there exists a scalar $\varsigma(\epsilon) > 0$ that satisfies $\mathbb{E}\{|u^2(t)|\} < \epsilon$ ($\forall\ t > 0$) when $sup_{-\vartheta \leq s \leq 0}\ \mathbb{E}|\rho^2(t)| < \varsigma$ for any scalar $\epsilon > 0$, the filtering error system (5) with $v(t) = 0$ is called mean-square stable. In addition, if $lim_{t\to\infty}\ \mathbb{E}|u^2(t)| = 0$, then the filtering error system is called asymptotically mean-square stable.*
- *H_∞ filtering: Given a level $\chi > 0$, the filtering error system (6) is called asymptotically mean-square stable with an H_∞ performance χ if the following condition is satisfied:*

$$||\beta(t)||_{\mathbb{E}_2} < \chi||v(t)||_2$$

for all nonzero $v(t) \in L_2[0, \infty)$ under the zero initial condition, where $||\overline{\beta}(t)||_{\mathbb{E}_2} = \mathbb{E}\{\int_0^\infty |\overline{\beta}(t)|^2 dt\}^{1/2}$.

Remark 2.2. *This paper, considers the H_∞ filtering problem for SNNs. Mean-square stability is required for the internal stability of the stochastic filtering error system without stochastic noise. Thus, the mean-square stability criterion is obtained for the concerned SNN.*

3. Main Results

This section establishes a novel sufficient condition for the desired H_∞ filter design based on a linear matrix inequality (LMI) approach.

Theorem 3.1. *For a given matrix X and some scalars $\vartheta_1 > 0$, $\vartheta_2 > 0$, $\phi \geq 0$, $d > 0$, and $h > 0$, the filtering error model (5) is asymptotically stable in the mean-square with H_∞ performance, if there exist matrices $P > 0$, $Q_j = \begin{bmatrix} Q_{11}^j & Q_{12}^j \\ * & Q_{22}^j \end{bmatrix} \geq 0$, $(j = 1,2,3)$, $Q_4 > 0$, $R_\ell > 0$ $(\ell = 1,2,3,4)$, $S_l > 0$, $T_l > 0$ $(l = 1,2,\ldots,8)$, and any appropriate dimensional matrices $\mathcal{P}_s$, $\mathcal{R}_s$, $\mathcal{Q}_s$, $\mathcal{S}_s$, $\mathcal{U}_s$, $\mathcal{T}_s$, $\mathcal{V}_s$, $\mathcal{Y}_s$, $\mathcal{X}_s$, $(s = 1,2)$, such that the LMIs (7) hold:*

$$\widehat{\Pi}_i = \begin{bmatrix} \Pi_{11} & \Pi_{12}^i & \Pi_{13} \\ * & \Pi_{22}^i & 0 \\ * & * & -I \end{bmatrix} < 0, \quad i = 1,2,3,4, \tag{7}$$

where

$$\Pi_{11} = \begin{bmatrix} \Xi_{9\times 9} & \Xi_{1,10} & \Xi_{1,11} \\ * & -\chi^2 I & 0 \\ * & * & -R_1 \end{bmatrix},$$

with

$$\Xi_{11} = Q_{11}^1 + Q_4 + \vartheta_1 R_2 + \vartheta_{21} R_3 + hR_4 + \mathcal{P}_1 + \mathcal{P}_1^T + \vartheta_1 \mathcal{S}_1 + \vartheta_1 \mathcal{S}_1^T + \vartheta_{21}\mathcal{T}_1 + \vartheta_{21}\mathcal{T}_1^T + \frac{\vartheta_1^2}{2}\mathcal{U}_1 + \frac{\vartheta_1^2}{2}\mathcal{U}_1^T$$
$$+ \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1 + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1^T + \frac{\vartheta_1^3}{6}\mathcal{X}_1 + \frac{\vartheta_1^3}{6}\mathcal{X}_1^T + \frac{\vartheta_2^3 - \vartheta_1^3}{6}\mathcal{Y}_1 + \frac{\vartheta_2^3 - \vartheta_1^3}{6}\mathcal{Y}_1^T - 2V^-\Gamma_1 V^+,$$

$$\Xi_{12} = -P(\mathcal{B} + XC),\ \Xi_{13} = -PXD + \mathcal{P}_2^T - \mathcal{Q}_1 + \mathcal{R}_1 + \vartheta_1 \mathcal{S}_2^T + \vartheta_{21}\mathcal{T}_2^T + \frac{\vartheta_1^2}{2}\mathcal{U}_2^T + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_2^T$$
$$+ \frac{\vartheta_1^3}{6}\mathcal{X}_2^T + \frac{\vartheta_2^3 - \vartheta_1^3}{6}\mathcal{Y}_2^T,\ \Xi_{14} = -\mathcal{P}_1 + \mathcal{Q}_1,\ \Xi_{15} = -\mathcal{R}_1,\ \Xi_{16} = PW_0 + Q_{12}^1 + \Gamma_1(V^- + V^+),$$

$$\Xi_{17} = PW_1,\ \Xi_{1,10} = P(B_1 - XB_2),\ \Xi_{1,11} = PW_2,\ \Xi_{22} = -Q_4,\ \Xi_{33} = -(1-\phi)Q_{11}^2 - \mathcal{Q}_2 - \mathcal{Q}_2^T$$
$$+ \mathcal{R}_2 + \mathcal{R}_2^T - 2V^-\Gamma_2 V^+,\ \Xi_{34} = -\mathcal{P}_2 + \mathcal{Q}_2,\ \Xi_{35} = -\mathcal{R}_2,\ \Xi_{37} = -(1-\phi)Q_{12}^2 + \Gamma_2(V^- + V^+),$$

$$\Xi_{44} = -Q_{11}^1 + Q_{11}^2 + Q_{11}^3,\ \Xi_{48} = -Q_{12}^1 + Q_{12}^2 + Q_{12}^3,\ \Xi_{55} = -Q_{11}^3,\ \Xi_{59} = -Q_{12}^3,\ \Xi_{66} = Q_{22}^1 + d^2 R_1 - 2\Gamma_1,$$

$$\Xi_{77} = -(1-\phi)Q_{22}^2 - 2\Gamma_2,\ \Xi_{88} = -Q_{22}^1 + Q_{22}^2 + Q_{22}^3,\ \Xi_{99} = -Q_{22}^3,\ \vartheta_{21} = \vartheta_2 - \vartheta_1,$$

$$\Pi_{12}^1 = \Big[\ \Theta_1 \quad \Big(-\vartheta_1\widetilde{\mathcal{P}} - \tfrac{\vartheta_1^2}{2}\widetilde{\mathcal{S}} - \tfrac{\vartheta_1^3}{6}\widetilde{\mathcal{U}} - \tfrac{\vartheta_1^4}{24}\widetilde{\mathcal{X}} \Big) \quad \widehat{E}^T P \quad \vartheta_1 \widehat{E}^T T_1 \quad \tfrac{\vartheta_1^2}{2}\widehat{E}^T T_3 \quad \tfrac{\vartheta_1^3}{6}\widehat{E}^T T_5$$

$$\frac{\vartheta_1^4}{24}\widehat{E}^T T_7 \quad \vartheta_1\widehat{\mathcal{B}}^T S_1 \quad \frac{\vartheta_1^2}{2}\widehat{\mathcal{B}}^T S_3 \quad \frac{\vartheta_1^3}{6}\widehat{\mathcal{B}}^T S_5 \quad \frac{\vartheta_1^4}{24}\widehat{\mathcal{B}}^T S_7 \quad \widetilde{\mathcal{P}} \quad \vartheta_1\widetilde{\mathcal{S}} \quad \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}} \quad \frac{\vartheta_1^3}{6}\widetilde{\mathcal{X}} \Big],$$

$$\Pi_{12}^2 = \Big[\ \Theta_2 \quad \Big(-\vartheta_1\widetilde{\mathcal{P}} - \frac{\vartheta_1^2}{2}\widetilde{\mathcal{S}} - \frac{\vartheta_1^3}{6}\widetilde{\mathcal{U}} - \frac{\vartheta_1^4}{24}\widetilde{\mathcal{X}}\Big) \quad \widehat{E}^T P \quad \vartheta_1\widehat{E}^T T_1 \quad \frac{\vartheta_1^2}{2}\widehat{E}^T T_3 \quad \frac{\vartheta_1^3}{6}\widehat{E}^T T_5$$
$$\frac{\vartheta_1^4}{24}\widehat{E}^T T_7 \quad \vartheta_1\widehat{\mathcal{B}}^T S_1 \quad \frac{\vartheta_1^2}{2}\widehat{\mathcal{B}}^T S_3 \quad \frac{\vartheta_1^3}{6}\widehat{\mathcal{B}}^T S_5 \quad \frac{\vartheta_1^4}{24}\widehat{\mathcal{B}}^T S_7 \quad \widetilde{\mathcal{P}} \quad \vartheta_1\widetilde{\mathcal{S}} \quad \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}} \quad \frac{\vartheta_1^3}{6}\widetilde{\mathcal{X}} \Big],$$

$$\Pi_{12}^3 = \Big[\ \Theta_3 \quad \Big(-\vartheta_{21}\widetilde{\mathcal{Q}} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}} - \frac{\vartheta_2^4-\vartheta_1^4}{24}\widetilde{\mathcal{Y}}\Big) \quad \widehat{E}^T P \quad \vartheta_{21}\widehat{E}^T T_2 \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^T T_4 \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^T T_6$$
$$\frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{E}^T T_8 \quad \vartheta_{21}\widehat{\mathcal{B}}^T S_2 \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{\mathcal{B}}^T S_4 \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{\mathcal{B}}^T S_6 \quad \frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{\mathcal{B}}^T S_8 \quad \widetilde{\mathcal{Q}} \quad \vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}} \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}} \Big],$$

$$\Pi_{12}^4 = \Big[\ \Theta_4 \quad \Big(-\vartheta_{21}\widetilde{\mathcal{R}} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}} - \frac{\vartheta_2^4-\vartheta_1^4}{24}\widetilde{\mathcal{Y}}\Big) \quad \widehat{E}^T P \quad \vartheta_{21}\widehat{E}^T T_2 \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^T T_4 \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^T T_6$$
$$\frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{E}^T T_8 \quad \vartheta_{21}\widehat{\mathcal{B}}^T S_2 \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{\mathcal{B}}^T S_4 \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{\mathcal{B}}^T S_6 \quad \frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{\mathcal{B}}^T S_8 \quad \widetilde{\mathcal{R}} \quad \vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}} \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}} \Big],$$

$$\Pi_{22}^1 = \Big[\ -\vartheta_1 R_2 \quad \Big(-\vartheta_1 S_1 - \frac{\vartheta_1^2}{2}S_3 - \frac{\vartheta_1^3}{6}S_5 - \frac{\vartheta_1^4}{24}S_7\Big) \quad -P \quad -\vartheta_1 T_1 \quad -\frac{\vartheta_1^2}{2}T_3 \quad -\frac{\vartheta_1^3}{6}T_5$$
$$-\frac{\vartheta_1^4}{24}T_7 \quad -\vartheta_1 S_1 \quad -\frac{\vartheta_1^2}{2}S_3 \quad -\frac{\vartheta_1^3}{6}S_5 \quad -\frac{\vartheta_1^4}{24}S_7 \quad -T_1 \quad -\vartheta_1 T_3 \quad -\frac{\vartheta_1^2}{2}T_5 \quad -\frac{\vartheta_1^3}{6}T_7 \Big],$$

$$\Pi_{22}^2 = \Big[\ -hR_4 \quad \Big(-\vartheta_1 S_1 - \frac{\vartheta_1^2}{2}S_3 - \frac{\vartheta_1^3}{6}S_5 - \frac{\vartheta_1^4}{24}S_7\Big) \quad -P \quad -\vartheta_1 T_1 \quad -\frac{\vartheta_1^2}{2}T_3 \quad -\frac{\vartheta_1^3}{6}T_5$$
$$-\frac{\vartheta_1^4}{24}T_7 \quad -\vartheta_1 S_1 \quad -\frac{\vartheta_1^2}{2}S_3 \quad -\frac{\vartheta_1^3}{6}S_5 \quad -\frac{\vartheta_1^4}{24}S_7 \quad -T_1 \quad -\vartheta_1 T_3 \quad -\frac{\vartheta_1^2}{2}T_5 \quad -\frac{\vartheta_1^3}{6}T_7 \Big],$$

$$\Pi_{22}^3 = \Big[\ -\vartheta_{21} R_3 \quad \Big(-\vartheta_{21} S_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}S_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6}S_6 - \frac{\vartheta_1^4}{24}S_8\Big) \quad -P \quad -\vartheta_{21} T_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}T_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_6$$
$$-\frac{\vartheta_2^4-\vartheta_1^4}{24}T_8 \quad -\vartheta_{21} S_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}S_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}S_6 \quad -\frac{\vartheta_2^4-\vartheta_1^4}{24}S_8 \quad -T_2 \quad -\vartheta_{21} T_4 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}T_6 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_8 \Big],$$

$$\Pi_{22}^4 = \Big[\ -\vartheta_{21} R_3 \quad \Big(-\vartheta_{21} S_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}S_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6}S_6 - \frac{\vartheta_1^4}{24}S_8\Big) \quad -P \quad -\vartheta_{21} T_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}T_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_6$$
$$-\frac{\vartheta_2^4-\vartheta_1^4}{24}T_8 \quad -\vartheta_{21} S_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}S_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}S_6 \quad -\frac{\vartheta_2^4-\vartheta_1^4}{24}S_8 \quad -T_2 \quad -\vartheta_{21} T_4 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}T_6 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_8 \Big],$$

$$\Theta_1^T = \Big[\ \Big(-\vartheta_1\widetilde{\mathcal{S}_1} - \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}_1} - \frac{\vartheta_1^3}{6}\widetilde{\mathcal{X}_1}\Big)^T \quad 0 \quad \Big(-\vartheta_1\widetilde{\mathcal{S}_2} - \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}_2} - \frac{\vartheta_1^3}{6}\widetilde{\mathcal{X}_2}\Big)^T \quad 0_{1\times 8} \Big],\ \Theta_2^T = \Big[\ 0 \quad h\mathcal{B}P(\mathcal{B}+XC)$$
$$h\mathcal{B}PXD \quad 0 \quad 0 \quad -h\mathcal{B}PW_0 \quad -h\mathcal{B}PW_1 \quad 0 \quad 0 \quad -h\mathcal{B}P(B_1 - XB_2) \quad -h\mathcal{B}PW_2 \Big],$$

$$\Theta_3^T = \Big[\ \Big(-\vartheta_{21}\widetilde{\mathcal{T}_1} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}_1} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}_1}\Big)^T \quad 0 \quad \Big(-\vartheta_1\widetilde{\mathcal{T}_2} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}_2} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}_2}\Big)^T \quad 0_{1\times 8} \Big],$$

$$\Theta_4^T = \Big[\ \Big(-\vartheta_{21}\widetilde{\mathcal{T}_1} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}_1} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}_1}\Big)^T \quad 0 \quad \Big(-\vartheta_1\widetilde{\mathcal{T}_2} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}_2} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}_2}\Big)^T \quad 0_{1\times 8} \Big],$$

$$\widetilde{\mathcal{P}}^T = \Big[\ \mathcal{P}_1^T \quad 0 \quad \mathcal{P}_2^T \quad 0_{1\times 8} \Big],\ \widetilde{\mathcal{Q}}^T = \Big[\ \mathcal{Q}_1^T \quad 0 \quad \mathcal{Q}_2^T \quad 0_{1\times 8} \Big],\ \widetilde{\mathcal{R}}^T = \Big[\ \mathcal{R}_1^T \quad 0 \quad \mathcal{R}_2^T \quad 0_{1\times 8} \Big],$$

$$\widetilde{\mathcal{S}}^T = \Big[\ \mathcal{S}_1^T \quad 0 \quad \mathcal{S}_2^T \quad 0_{1\times 8} \Big],\ \widetilde{\mathcal{T}}^T = \Big[\ \mathcal{T}_1^T \quad 0 \quad \mathcal{T}_2^T \quad 0_{1\times 8} \Big],\ \widetilde{\mathcal{U}}^T = \Big[\ \mathcal{U}_1^T \quad 0 \quad \mathcal{U}_2^T \quad 0_{1\times 8} \Big],$$

$$\widetilde{\mathcal{V}}^T = \Big[\ \mathcal{V}_1^T \quad 0 \quad \mathcal{V}_2^T \quad 0_{1\times 8} \Big],\ \widetilde{\mathcal{X}}^T = \Big[\ \mathcal{X}_1^T \quad 0 \quad \mathcal{X}_2^T \quad 0_{1\times 8} \Big],\ \widetilde{\mathcal{Y}}^T = \Big[\ \mathcal{Y}_1^T \quad 0 \quad \mathcal{Y}_2^T \quad 0_{1\times 8} \Big],$$

$$\widehat{E} = \Big[\ E \quad 0 \quad E_1 \quad 0 \quad 0 \quad H_0 \quad H_1 \quad 0 \quad 0 \quad 0 \quad H_2 \Big],$$

$$\widehat{\mathcal{B}} = \begin{bmatrix} 0 & -(\mathcal{B}+XC) & -XD & 0 & 0 & W_0 & W_1 & 0 & 0 & (B_1-XB_2) & W_2 \end{bmatrix}.$$

Proof : For simplicity, we now let

$$de(t) = m(t)dt + n(t)d\omega(t), \tag{8}$$

where

$$\begin{aligned} m(t) &= -(\mathcal{B}+XC)e(t-h) - XDe(t-\vartheta(t)) + W_0 f(e(t)) + W_1 f(e(t-\vartheta(t))) \\ &\quad + W_2 \int_{t-b(t)}^{t} f(e(s))ds + (B_1 - XB_2)v(t), \\ n(t) &= Ee(t) + E_1 e(t-\vartheta(t)) + H_0 f(e(t)) + H_1 f(e(t-\vartheta(t))) + H_2 \int_{t-b(t)}^{t} f(e(s))ds. \end{aligned}$$

Define the following LKF candidate:

$$V(e_t) = \sum_{q=1}^{5} V_q(e_t), \tag{9}$$

where

$$\begin{aligned} V_1(e_t) &= \Big(e(t) - \mathcal{B}\int_{t-h}^{t} e(s)ds\Big)^T P \Big(e(t) - \mathcal{B}\int_{t-h}^{t} e(s)ds\Big), \\ V_2(e_t) &= \int_{t-\vartheta_1}^{t} \eta^T(s)Q_1\eta(s)ds + \int_{t-\vartheta(t)}^{t-\vartheta_1} \eta^T(s)Q_2\eta(s)ds \\ &\quad + \int_{t-\vartheta_2}^{t-\vartheta_1} \eta^T(s)Q_3\eta(s)ds + \int_{t-h}^{t} e^T(s)Q_4 e(s)ds, \\ V_3(e_t) &= d\int_{-d}^{0}\int_{t+\delta}^{t} f^T(e(s))R_1 f(e(s))dsd\delta + \int_{-\vartheta_1}^{0}\int_{t+\delta}^{t} e^T(s)R_2 e(s)dsd\delta \\ &\quad + \int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t} e^T(s)R_3 e(s)dsd\delta + \int_{-h}^{0}\int_{t+\delta}^{t} e^T(s)R_4 e(s)dsd\delta, \\ V_4(e_t) &= \int_{-\vartheta_1}^{0}\int_{t+\delta}^{t} m^T(s)S_1 m(s)dsd\delta + \int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t} m^T(s)S_2 m(s)dsd\delta \\ &\quad + \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t} m^T(s)S_3 m(s)dsd\mu d\delta + \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t} m^T(s)S_4 m(s)dsd\mu d\delta \\ &\quad + \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} m^T(s)S_5 m(s)dsd\alpha d\mu d\delta + \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} m^T(s)S_6 m(s)dsd\alpha d\mu d\delta \\ &\quad + \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{\alpha}^{0}\int_{t+\sigma}^{t} m^T(s)S_7 m(s)dsd\sigma d\alpha d\mu d\delta \end{aligned}$$

$$+\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{\alpha}^{0}\int_{t+\sigma}^{t} m^T(s)S_8m(s)dsd\sigma d\alpha d\mu d\delta,$$

$$\begin{aligned}V_5(e_t) &= \int_{-\vartheta_1}^{0}\int_{t+\delta}^{t} n^T(s)T_1n(s)dsd\delta + \int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t} n^T(s)T_2n(s)dsd\delta \\ &+ \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t} n^T(s)T_3n(s)dsd\mu d\delta + \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t} n^T(s)T_4n(s)dsd\mu d\delta \\ &+ \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} n^T(s)T_5n(s)dsd\alpha d\mu d\delta + \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} n^T(s)T_6n(s)dsd\alpha d\mu d\delta \\ &+ \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{\alpha}^{0}\int_{t+\sigma}^{t} n^T(s)T_7n(s)dsd\sigma d\alpha d\mu d\delta \\ &+ \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{\alpha}^{0}\int_{t+\sigma}^{t} n^T(s)T_8n(s)dsd\sigma d\alpha d\mu d\delta,\end{aligned}$$

where

$$\eta(s) = \begin{bmatrix} e^T(s) & f^T(e(s)) \end{bmatrix}^T.$$

Through Ito's formula [19], we get

$$dV(e_t) = \mathcal{L}V(e_t)dt + 2\Big(e(t) - \mathcal{B}\int_{t-h}^{t} e(s)ds\Big)^T Pn(t)d\omega(t), \tag{10}$$

where

$$\begin{aligned}\mathcal{L}V(e_t) &= \sum_{q=1}^{5}\mathcal{L}V_q(e_t), \\ \mathcal{L}V_1(e_t) &= 2\Big(e(t) - \mathcal{B}\int_{t-h}^{t} e(s)ds\Big)^T Pm(t) + n^T(t)Pn(t), \\ \mathcal{L}V_2(e_t) &\le \eta^T(t)Q_1\eta(t) + \eta^T(t-\vartheta_1)[-Q_1+Q_2+Q_3]\eta(t-\vartheta_1) - (1-\phi)\eta^T(t-\vartheta(t))Q_2\eta(t-\vartheta(t)) \\ &\quad - \eta^T(t-\vartheta_2)Q_3\eta(t-\vartheta_2) + e^T(t)Q_4e(t) - e^T(t-h)Q_4e(t-h), \\ \mathcal{L}V_3(e_t) &\le f^T(e(t))[d^2R_1]f(e(t)) - \Big(\int_{t-b(t)}^{t} f(e(s))ds\Big)^T R_1\Big(\int_{t-b(t)}^{t} f(e(s))ds\Big) \\ &\quad + e^T(t)[\vartheta_1R_2 + \vartheta_{21}R_3 + hR_4]e(t) - \int_{t-\vartheta_1}^{t} e^T(s)R_2e(s)ds \\ &\quad - \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)R_3e(s)ds - \int_{t-h}^{t} e^T(s)R_4e(s)ds, \\ \mathcal{L}V_4(x_t) &= m^T(t)\Big[\vartheta_1S_1 + \vartheta_{21}S_2 + \Big(\frac{\vartheta_1^2}{2}\Big)S_3 + \Big(\frac{\vartheta_2^2-\vartheta_1^2}{2}\Big)S_4 + \Big(\frac{\vartheta_1^3}{6}\Big)S_5 + \Big(\frac{\vartheta_2^3-\vartheta_1^3}{6}\Big)S_6\end{aligned}$$

$$+\Big(\frac{\vartheta_1^4}{24}\Big)S_7+\Big(\frac{\vartheta_2^4-\vartheta_1^4}{24}\Big)S_8\Big]m(t)-\int_{t-\vartheta_1}^{t}m^T(s)S_1m(s)ds$$
$$-\int_{t-\vartheta_2}^{t-\vartheta_1}m^T(s)S_2m(s)ds-\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t}m^T(s)S_3m(s)dsd\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t}m^T(s)S_4m(s)dsd\delta-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t}m^T(s)S_5m(s)dsd\mu d\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t}m^T(s)S_6m(s)dsd\mu d\delta-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}m^T(s)S_7m(s)dsd\alpha d\mu d\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}m^T(s)S_8m(s)dsd\alpha d\mu d\delta,$$

$$\mathcal{L}V_5(x_t)=n^T(t)\Big[\vartheta_1T_1+\vartheta_{21}T_2+\Big(\frac{\vartheta_1^2}{2}\Big)T_3+\Big(\frac{\vartheta_2^2-\vartheta_1^2}{2}\Big)T_4+\Big(\frac{\vartheta_1^3}{6}\Big)T_5+\Big(\frac{\vartheta_2^3-\vartheta_1^3}{6}\Big)T_6$$
$$+\Big(\frac{\vartheta_1^4}{24}\Big)T_7+\Big(\frac{\vartheta_2^4-\vartheta_1^4}{24}\Big)T_8\Big]n(t)-\int_{t-\vartheta_1}^{t}n^T(s)T_1n(s)ds$$
$$-\int_{t-\vartheta_2}^{t-\vartheta_1}n^T(s)T_2n(s)ds-\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t}n^T(s)T_3n(s)dsd\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t}n^T(s)T_4n(s)dsd\delta-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t}n^T(s)T_5n(s)dsd\mu d\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t}n^T(s)T_6n(s)dsd\mu d\delta-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}n^T(s)T_7n(s)dsd\alpha d\mu d\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}n^T(s)T_8n(s)dsd\alpha d\mu d\delta.$$

From the Newton–Leibniz formula, the following equalities hold for any appropriate dimensional matrices $\mathcal{P}$, $\mathcal{R}$, $\mathcal{Q}$, $\mathcal{T}$, $\mathcal{S}$, $\mathcal{V}$, $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{X}$:

$$0=2\lambda^T(t)\mathcal{P}\Big[e(t)-e(t-\vartheta_1)-\int_{t-\vartheta_1}^{t}m(s)ds-\int_{t-\vartheta_1}^{t}n(s)d\omega(s)\Big],\tag{11}$$
$$0=2\lambda^T(t)\mathcal{Q}\Big[e(t-\vartheta_1)-e(t-\vartheta(t))-\int_{t-\vartheta(t)}^{t-\vartheta_1}m(s)ds-\int_{t-\vartheta(t)}^{t-\vartheta_1}n(s)d\omega(s)\Big],\tag{12}$$
$$0=2\lambda^T(t)\mathcal{R}\Big[e(t-\vartheta(t))-e(t-\vartheta_2)-\int_{t-\vartheta_2}^{t-\vartheta(t)}m(s)ds-\int_{t-\vartheta_2}^{t-\vartheta(t)}n(s)d\omega(s)\Big],\tag{13}$$
$$0=2\lambda^T(t)\mathcal{S}\Big[\vartheta_1e(t)-\int_{t-\vartheta_1}^{t}e(s)ds-\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t}m(s)dsd\delta-\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t}n(s)d\omega(s)d\delta\Big],\tag{14}$$
$$0=2\lambda^T(t)\mathcal{T}\Big[\vartheta_{21}e(t)-\int_{t-\vartheta_2}^{t-\vartheta_1}e(s)ds-\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t}m(s)dsd\delta-\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t}n(s)d\omega(s)d\delta\Big],\tag{15}$$
$$0=2\lambda^T(t)\mathcal{U}\Big[\Big(\frac{\vartheta_1^2}{2}\Big)e(t)-\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t}e(s)dsd\delta-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t}m(s)dsd\mu d\delta$$

$$-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\delta}^{t} n(s)d\omega(s)d\mu d\delta\Big], \tag{16}$$

$$0 = 2\lambda^T(t)\mathcal{V}\Big[\Big(\frac{\vartheta_2^2-\vartheta_1^2}{2}\Big)e(t) - \int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t} e(s)dsd\delta - \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t} m(s)dsd\mu d\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\delta}^{t} n(s)d\omega(s)d\mu d\delta\Big], \tag{17}$$

$$0 = 2\lambda^T(t)\mathcal{X}\Big[\Big(\frac{\vartheta_1^3}{6}\Big)e(t) - \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t} e(s)dsd\mu d\delta - \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} m(s)dsd\alpha d\mu d\delta$$
$$-\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} n(s)d\omega(s)d\alpha d\mu d\delta\Big], \tag{18}$$

$$0 = 2\lambda^T(t)\mathcal{Y}\Big[\Big(\frac{\vartheta_2^3-\vartheta_1^3}{6}\Big)e(t) - \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t} e(s)dsd\mu d\delta - \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} m(s)dsd\alpha d\mu d\delta$$
$$-\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} n(s)d\omega(s)d\alpha d\mu d\delta\Big], \tag{19}$$

where

$$\lambda(t) = \begin{bmatrix} e^T(t) & e^T(t-\vartheta(t)) \end{bmatrix}^T,\ \mathcal{P} = \begin{bmatrix} \mathcal{P}_1^T & \mathcal{P}_2^T \end{bmatrix}^T,\ \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1^T & \mathcal{Q}_2^T \end{bmatrix}^T,$$
$$\mathcal{R} = \begin{bmatrix} \mathcal{R}_1^T & \mathcal{R}_2^T \end{bmatrix}^T,\ \mathcal{S} = \begin{bmatrix} \mathcal{S}_1^T & \mathcal{S}_2^T \end{bmatrix}^T,\ \mathcal{T} = \begin{bmatrix} \mathcal{T}_1^T & \mathcal{T}_2^T \end{bmatrix}^T,\ \mathcal{U} = \begin{bmatrix} \mathcal{U}_1^T & \mathcal{U}_2^T \end{bmatrix}^T,$$
$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1^T & \mathcal{V}_2^T \end{bmatrix}^T,\ \mathcal{X} = \begin{bmatrix} \mathcal{X}_1^T & \mathcal{X}_2^T \end{bmatrix}^T,\ \mathcal{Y} = \begin{bmatrix} \mathcal{Y}_1^T & \mathcal{Y}_2^T \end{bmatrix}^T.$$

It is clear that

$$-2\lambda^T(t)\mathcal{P}\int_{t-\vartheta_1}^{t} n(s)d\omega(s)$$
$$\le \lambda^T(t)\mathcal{P}T_1^{-1}\mathcal{P}^T\lambda(t) + \Big[\int_{t-\vartheta_1}^{t} n(s)d\omega(s)\Big]^T T_1\Big[\int_{t-\vartheta_1}^{t} n(s)d\omega(s)\Big], \tag{20}$$

$$-2\lambda^T(t)\mathcal{Q}\int_{t-\vartheta(t)}^{t-\vartheta_1} n(s)d\omega(s)$$
$$\le \lambda^T(t)\mathcal{Q}T_2^{-1}\mathcal{Q}^T\lambda(t) + \Big[\int_{t-\vartheta(t)}^{t-\vartheta_1} n(s)d\omega(s)\Big]^T T_2\Big[\int_{t-\vartheta(t)}^{t-\vartheta_1} n(s)d\omega(s)\Big], \tag{21}$$

$$-2\lambda^T(t)\mathcal{R}\int_{t-\vartheta_2}^{t-\vartheta(t)} n(s)d\omega(s)$$
$$\le \lambda^T(t)\mathcal{R}T_2^{-1}\mathcal{R}^T\lambda(t) + \Big[\int_{t-\vartheta_2}^{t-\vartheta(t)} n(s)d\omega(s)\Big]^T T_2\Big[\int_{t-\vartheta_2}^{t-\vartheta(t)} n(s)d\omega(s)\Big], \tag{22}$$

$$-2\lambda^T(t)\mathcal{S}\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t} n(s)d\omega(s)d\delta$$

$$\leq \vartheta_1 \lambda^T(t)\mathcal{S}T_3^{-1}\mathcal{S}^T\lambda(t) + \int_{-\vartheta_1}^{0} \Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]^T T_3\Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]d\delta, \tag{23}$$

$$-2\lambda^T(t)\mathcal{T}\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t} n(s)d\omega(s)d\delta$$

$$\leq \vartheta_{21}\lambda^T(t)\mathcal{T}T_4^{-1}\mathcal{T}^T\lambda(t) + \int_{-\vartheta_1}^{0} \Big[\int_{-\vartheta_2}^{-\vartheta_1} n(s)d\omega(s)\Big]^T T_4\Big[\int_{-\vartheta_2}^{-\vartheta_1} n(s)d\omega(s)\Big]d\delta, \tag{24}$$

$$-2\lambda^T(t)\mathcal{U}\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t} n(s)d\omega(s)d\mu d\delta$$

$$\leq \Big(\frac{\vartheta_1^2}{2}\Big)\lambda^T(t)\mathcal{U}T_5^{-1}\mathcal{U}^T\lambda(t) + \int_{-\vartheta_1}^{0}\int_{\delta}^{0} \Big[\int_{t+\mu}^{t} n(s)d\omega(s)\Big]^T T_5\Big[\int_{t+\mu}^{t} n(s)d\omega(s)\Big]d\mu d\delta, \tag{25}$$

$$-2\lambda^T(t)\mathcal{V}\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t} n(s)d\omega(s)d\mu d\delta$$

$$\leq \Big(\frac{\vartheta_2^2-\vartheta_1^2}{2}\Big)\lambda^T(t)\mathcal{V}T_6^{-1}\mathcal{V}^T\lambda(t) + \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0} \Big[\int_{t+\mu}^{t} n(s)d\omega(s)\Big]^T T_6\Big[\int_{t+\mu}^{t} n(s)d\omega(s)\Big]d\mu d\delta, \tag{26}$$

$$-2\lambda^T(t)\mathcal{X}\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} n(s)d\omega(s)d\alpha d\mu d\delta$$

$$\leq \Big(\frac{\vartheta_1^3}{6}\Big)\lambda^T(t)\mathcal{X}T_7^{-1}\mathcal{X}^T\lambda(t) + \int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0} \Big[\int_{t+\alpha}^{t} n(s)d\omega(s)\Big]^T T_7\Big[\int_{t+\alpha}^{t} n(s)d\omega(s)\Big]d\alpha d\mu d\delta, \tag{27}$$

$$-2\lambda^T(t)\mathcal{Y}\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t} n(s)d\omega(s)d\alpha d\mu d\delta$$

$$\leq \Big(\frac{\vartheta_2^3-\vartheta_1^3}{6}\Big)\lambda^T(t)\mathcal{Y}T_8^{-1}\mathcal{Y}^T\lambda(t) + \int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0} \Big[\int_{t+\alpha}^{t} n(s)d\omega(s)\Big]^T T_8\Big[\int_{t+\alpha}^{t} n(s)d\omega(s)\Big]d\alpha d\mu d\delta. \tag{28}$$

Through the Ito isometry in [19], we obtain

$$\mathbb{E}\Big[\int_{t-\vartheta_1}^{t} n(s)d\omega(s)\Big]^T T_1\Big[\int_{t-\vartheta_1}^{t} n(s)d\omega(s)\Big] \leq \mathbb{E}\int_{t-\vartheta_1}^{t} n^T(s)T_1 n(s)ds, \tag{29}$$

$$\mathbb{E}\Big[\int_{t-\vartheta(t)}^{t-\vartheta_1} n(s)d\omega(s)\Big]^T T_2\Big[\int_{t-\vartheta(t)}^{t-\vartheta_1} n(s)d\omega(s)\Big] \leq \mathbb{E}\int_{t-\vartheta(t)}^{t-\vartheta_1} n^T(s)T_2 n(s)ds, \tag{30}$$

$$\mathbb{E}\Big[\int_{t-\vartheta_2}^{t-\vartheta(t)} n(s)d\omega(s)\Big]^T T_2\Big[\int_{t-\vartheta_2}^{t-\vartheta(t)} n(s)d\omega(s)\Big] \leq \mathbb{E}\int_{t-\vartheta_2}^{t-\vartheta(t)} n^T(s)T_2 n(s)ds, \tag{31}$$

$$\mathbb{E}\int_{-\vartheta_1}^{0} \Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]^T T_3\Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]d\delta \leq \mathbb{E}\int_{-\vartheta_1}^{0}\int_{t+\delta}^{t} n^T(s)T_3 n(s)dsd\delta, \tag{32}$$

$$\mathbb{E}\int_{-\vartheta_2}^{-\vartheta_1} \Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]^T T_4\Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]d\delta \leq \mathbb{E}\int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\delta}^{t} n^T(s)T_4 n(s)dsd\delta, \tag{33}$$

$$\mathbb{E}\int_{-\vartheta_1}^{0}\int_{\delta}^{0} \Big[\int_{t+\mu}^{t} n(s)d\omega(s)\Big]^T T_5\Big[\int_{t+\delta}^{t} n(s)d\omega(s)\Big]d\mu d\delta \leq \mathbb{E}\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{t+\mu}^{t} n^T(s)T_5 n(s)dsd\mu d\delta, \tag{34}$$

$$\mathbb{E}\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\Big[\int_{t+\mu}^{t}n(s)d\omega(s)\Big]^T T_6\Big[\int_{t+\delta}^{t}n(s)d\omega(s)\Big]d\mu d\delta$$
$$\leq \mathbb{E}\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{t+\mu}^{t}n^T(s)Z_6n(s)dsd\mu d\delta, \tag{35}$$
$$\mathbb{E}\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\Big[\int_{t+\alpha}^{t}n(s)d\omega(s)\Big]^T T_7\Big[\int_{t+\alpha}^{t}n(s)d\omega(s)\Big]d\alpha d\mu d\delta$$
$$\leq \mathbb{E}\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}n^T(s)T_7n(s)dsd\alpha d\mu d\delta, \tag{36}$$
$$\mathbb{E}\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\Big[\int_{t+\alpha}^{t}n(s)d\omega(s)\Big]^T T_8\Big[\int_{t+\alpha}^{t}n(s)d\omega(s)\Big]d\alpha d\mu d\delta$$
$$\leq \mathbb{E}\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}n^T(s)T_8n(s)dsd\alpha d\mu d\delta. \tag{37}$$

From (6), the following properties hold for positive diagonal matrices $\Gamma_1 > 0$ and $\Gamma_2 > 0$ (see [20]):

$$0 \leq -2f^T(e(t))\Gamma_1 f(e(t)) + 2e^T(t)\Gamma_1(V^- + V^+)f(e(t)) - 2e^T(t)V^-\Gamma_1V^+e(t), \tag{38}$$
$$\begin{aligned}0 \leq\ & -2f^T(e(t-\vartheta(t)))\Gamma_2 f(e(t-\vartheta(t))) + 2e^T(t-\vartheta(t))\Gamma_2(V^- + V^+)f(e(t-\vartheta(t)))\\ & -2e^T(t-\vartheta(t))V^-\Gamma_2V^+e(t-\vartheta(t)).\end{aligned} \tag{39}$$

Define

$$\begin{aligned}\xi^T(t) = \Big[\ & e^T(t) \quad e^T(t-h) \quad e^T(t-\vartheta(t)) \quad e^T(t-\vartheta_1) \quad e^T(t-\vartheta_2) \quad f^T(e(t)) \quad f^T(e(t-\vartheta(t)))\\ & f^T(e(t-\vartheta_1)) \quad f^T(e(t-\vartheta_2)) \quad v^T(t) \quad \Big(\textstyle\int_{t-b(t)}^{t} f(e(s))ds\Big)^T \quad e^T(s) \quad m^T(s)\ \Big].\end{aligned}$$

Combining equations (10)-(39) and using the Schur complement, allows us to deduce that

$$\begin{aligned}\mathbb{E}\mathcal{L}V(x_t) \leq\ & \mathbb{E}\Big(\frac{24}{\vartheta_1^5}\Big)\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}\int_{t-\vartheta_1}^{t}\xi^T(s)\overline{\Pi}_1\xi(s)dsd\sigma d\alpha d\mu d\delta\\ & + \mathbb{E}\Big(\frac{24}{h\vartheta_1^4}\Big)\int_{-\vartheta_1}^{0}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}\int_{t-h}^{t}\xi^T(s)\overline{\Pi}_2\xi(s)dsd\sigma d\alpha d\mu d\delta\\ & + \mathbb{E}\Big(\frac{24}{\vartheta_{21}(\vartheta_2^4-\vartheta_1^4)}\Big)\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}\int_{t-\vartheta(t)}^{t-\vartheta_1}\xi^T(s)\overline{\Pi}_3\xi(s)dsd\sigma d\alpha d\mu d\delta\\ & + \mathbb{E}\Big(\frac{24}{\vartheta_{21}(\vartheta_2^4-\vartheta_1^4)}\Big)\int_{-\vartheta_2}^{-\vartheta_1}\int_{\delta}^{0}\int_{\mu}^{0}\int_{t+\alpha}^{t}\int_{t-\vartheta_2}^{t-\vartheta(t)}\xi^T(s)\overline{\Pi}_4\xi(s)dsd\sigma d\alpha d\mu d\delta,\end{aligned} \tag{40}$$

where

$$\overline{\Pi}_i = \begin{bmatrix} \overline{\Pi}_{11} & \Pi_{12}^i \\ * & \Pi_{22}^i \end{bmatrix}, \quad i = 1, 2, 3, 4,$$

$$\overline{\Pi}_{11} = \begin{bmatrix} \Xi_{9\times 9} & \Xi_{1,10} & \Xi_{1,11} \\ * & \mathbf{0} & 0 \\ * & * & -R_1 \end{bmatrix},$$

and Π_{12}^i, Π_{22}^i, $(i = 1, 2, 3, 4)$ are the same as those defined in LMI (7). Note that,

$$\beta^T(t)\beta(t) = \xi^T(t)\Pi_{13}\Pi_{13}^T\xi(t), \tag{41}$$

where $\Pi_{13}^T = \begin{bmatrix} J & \underbrace{0\cdots 0}_{11 \text{ times}} \end{bmatrix}$. We now set

$$\mathbb{J}(t) = \mathbb{E}\Big\{\int_0^t [\beta^T(t)\beta(t) - \chi^2 v^T(t)v(t)]dt\Big\},\ t > 0.$$

Through Ito's formula, we then obtain

$$\begin{aligned} \mathbb{J}(t) &= \mathbb{E}\Big\{\int_0^t [\beta^T(t)\beta(t) - \chi^2 v^T(t)v(t) + \mathcal{L}V(e_t)]dt\Big\} - \mathbb{E}\Big\{\mathcal{L}V(e(t), t)\Big\}, \\ &\leq \mathbb{E}\Big\{\int_0^t [\beta^T(t)\beta(t) - \chi^2 v^T(t)v(t) + \mathcal{L}V(e_t)]dt\Big\}, \end{aligned}$$

which leads to

$$\mathbb{J}(t) \leq \mathbb{E}\Big\{\int_0^t \xi^T(t)\widehat{\Pi}_i\xi(t)dt\Big\},$$

where

$$\widehat{\Pi}_i = \begin{bmatrix} \Pi_{11} & \Pi_{12}^i \\ * & \Pi_{22}^i \end{bmatrix} + \Pi_{13}\Pi_{13}^T, \quad i = 1, 2, 3, 4.$$

From (7) and the Schur complement, it then follows from $\widehat{\Pi}_i < 0$, that

$$\mathbb{E}[\mathcal{L}V(x_t)dt] = \mathbb{E}[dV(x_t)dt] < 0, \forall\ e(t) \neq 0.$$

Hence, $||\beta||_2 \leq \chi||v||_2$ holds.

4. H_∞ FILTERING

Theorem 4.1. *Given scalars $\vartheta_1 > 0,\ \vartheta_2 > 0,\ \phi \geq 0,\ d > 0,$ and $h > 0$, the filtering error system (5) is asymptotically stable in the mean-square with an H_∞ performance χ if there exist matrices $P > 0,\ Q_j = \begin{bmatrix} Q_{11}^j & Q_{12}^j \\ * & Q_{22}^j \end{bmatrix} \geq 0,\ (j = 1,2,3),\ Q_4 > 0,\ R_\ell > 0\ (\ell = 1,2,3,4),\ S_l > 0,\ T_l > 0\ (l = 1,2,\dots,8)$, and any appropriate dimensional matrices $\mathcal{P}_s,\ \mathcal{R}_s,\ \mathcal{Q}_s,\ \mathcal{S}_s,\ \mathcal{T}_s,\ \mathcal{V}_s,\ \mathcal{U}_s,\ \mathcal{Y}_s,\ \mathcal{X}_s,\ (s = 1,2)$, such that the LMIs (42) hold:*

$$\widehat{\Sigma}_i = \begin{bmatrix} \Sigma_{11} & \Sigma_{12}^i & \Sigma_{13} \\ * & \Sigma_{22}^i & 0 \\ * & * & -I \end{bmatrix} < 0, \quad i = 1,2,3,4, \tag{42}$$

where

$$\Sigma_{11} = \begin{bmatrix} \Phi_{9\times 9} & \Phi_{1,10} & \Phi_{1,11} \\ * & -\chi^2 I & 0 \\ * & * & -R_1 \end{bmatrix},$$

with

$$\Phi_{11} = Q_{11}^1 + Q_4 + \vartheta_1 R_2 + \vartheta_{21} R_3 + h R_4 + \mathcal{P}_1 + \mathcal{P}_1^T + \vartheta_1 \mathcal{S}_1 + \vartheta_1 \mathcal{S}_1^T + \vartheta_{21}\mathcal{T}_1 + \vartheta_{21}\mathcal{T}_1^T + \frac{\vartheta_1^2}{2}\mathcal{U}_1 + \frac{\vartheta_1^2}{2}\mathcal{U}_1^T$$
$$+ \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1 + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1^T + \frac{\vartheta_1^3}{6}\mathcal{X}_1 + \frac{\vartheta_1^3}{6}\mathcal{X}_1^T + \frac{\vartheta_2^3 - \vartheta_1^3}{6}\mathcal{Y}_1 + \frac{\vartheta_2^3 - \vartheta_1^3}{6}\mathcal{Y}_1^T - 2V^-\Gamma_1 V^+,$$

$$\Phi_{12} = -P\mathcal{B} - GC,\ \Phi_{13} = -GD + \mathcal{P}_2^T - \mathcal{Q}_1 + \mathcal{R}_1 + \vartheta_1 \mathcal{S}_2^T + \vartheta_{21}\mathcal{T}_2^T + \frac{\vartheta_1^2}{2}\mathcal{U}_2^T + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_2^T + \frac{\vartheta_1^3}{6}\mathcal{X}_2^T$$
$$+ \frac{\vartheta_2^3 - \vartheta_1^3}{6}\mathcal{Y}_2^T,\ \Phi_{14} = -\mathcal{P}_1 + \mathcal{Q}_1,\ \Phi_{15} = -\mathcal{R}_1,\ \Phi_{16} = PW_0 + Q_{12}^1 + \Gamma_1(V^- + V^+),\ \Phi_{17} = PW_1,$$

$$\Phi_{1,10} = PB_1 - GB_2,\ \Phi_{1,11} = PW_2,\ \Phi_{22} = -Q_4,\ \Phi_{33} = -(1-\phi)Q_{11}^2 - \mathcal{Q}_2 - \mathcal{Q}_2^T + \mathcal{R}_2 + \mathcal{R}_2^T - 2V^-\Gamma_2 V^+,$$

$$\Phi_{34} = -\mathcal{P}_2 + \mathcal{Q}_2,\ \Phi_{35} = -\mathcal{R}_2,\ \Phi_{37} = -(1-\phi)Q_{12}^2 + \Gamma_2(V^- + V^+),\ \Phi_{44} = -Q_{11}^1 + Q_{11}^2 + Q_{11}^3,$$

$$\Phi_{48} = -Q_{12}^1 + Q_{12}^2 + Q_{12}^3,\ \Phi_{55} = -Q_{11}^3,\ \Phi_{59} = -Q_{12}^3,\ \Phi_{66} = Q_{22}^1 + d^2 R_1 - 2\Gamma_1,$$

$$\Phi_{77} = -(1-\phi)Q_{22}^2 - 2\Gamma_2,\ \Phi_{88} = -Q_{22}^1 + Q_{22}^2 + Q_{22}^3,\ \Phi_{99} = -Q_{22}^3,\ \vartheta_{21} = \vartheta_2 - \vartheta_1,$$

$$\Sigma_{12}^1 = \Big[\ \Theta_1 \quad \Big(-\vartheta_1\widetilde{\mathcal{P}} - \tfrac{\vartheta_1^2}{2}\widetilde{\mathcal{S}} - \tfrac{\vartheta_1^3}{6}\widetilde{\mathcal{U}} - \tfrac{\vartheta_1^4}{24}\widetilde{\mathcal{X}}\Big) \quad \widehat{E}^T P \quad \vartheta_1\widehat{E}^T T_1 \quad \tfrac{\vartheta_1^2}{2}\widehat{E}^T T_3 \quad \tfrac{\vartheta_1^3}{6}\widehat{E}^T T_5$$
$$\tfrac{\vartheta_1^4}{24}\widehat{E}^T T_7 \quad \vartheta_1\widehat{\mathcal{B}}^T P \quad \tfrac{\vartheta_1^2}{2}\widehat{\mathcal{B}}^T P \quad \tfrac{\vartheta_1^3}{6}\widehat{\mathcal{B}}^T P \quad \tfrac{\vartheta_1^4}{24}\widehat{\mathcal{B}}^T P \quad \widetilde{\mathcal{P}} \quad \vartheta_1\widetilde{\mathcal{S}} \quad \tfrac{\vartheta_1^2}{2}\widetilde{\mathcal{U}} \quad \tfrac{\vartheta_1^3}{6}\widetilde{\mathcal{X}}\ \Big],$$

$$\Sigma_{12}^2 = \Big[\ \Psi_2 \quad \Big(-\vartheta_1\widetilde{\mathcal{P}} - \tfrac{\vartheta_1^2}{2}\widetilde{\mathcal{S}} - \tfrac{\vartheta_1^3}{6}\widetilde{\mathcal{U}} - \tfrac{\vartheta_1^4}{24}\widetilde{\mathcal{X}}\Big) \quad \widehat{E}^T P \quad \vartheta_1\widehat{E}^T T_1 \quad \tfrac{\vartheta_1^2}{2}\widehat{E}^T T_3 \quad \tfrac{\vartheta_1^3}{6}\widehat{E}^T T_5$$

$$\frac{\vartheta_1^4}{24}\widehat{E}^TT_7 \quad \vartheta_1\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_1^2}{2}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_1^3}{6}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_1^4}{24}\widehat{\mathcal{B}}^TP \quad \widetilde{\mathcal{P}} \quad \vartheta_1\widetilde{\mathcal{S}} \quad \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}} \quad \frac{\vartheta_1^3}{6}\widetilde{\mathcal{X}} \Big],$$

$$\Sigma_{12}^3 = \Big[\Theta_3 \quad \Big(-\vartheta_{21}\widetilde{\mathcal{Q}} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}} - \frac{\vartheta_2^4-\vartheta_1^4}{24}\widetilde{\mathcal{Y}}\Big) \quad \widehat{E}^TP \quad \vartheta_{21}\widehat{E}^TT_2 \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^TT_4 \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^TT_6$$
$$\frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{E}^TT_8 \quad \vartheta_{21}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{\mathcal{B}}^TP \quad \widetilde{\mathcal{Q}} \quad \vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}} \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}} \Big],$$

$$\Sigma_{12}^4 = \Big[\Theta_4 \quad \Big(-\vartheta_{21}\widetilde{\mathcal{R}} - \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}} - \frac{\vartheta_2^4-\vartheta_1^4}{24}\widetilde{\mathcal{Y}}\Big) \quad \widehat{E}^TP \quad \vartheta_{21}\widehat{E}^TT_2 \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^TT_4 \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^TT_6$$
$$\frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{E}^TT_8 \quad \vartheta_{21}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{\mathcal{B}}^TP \quad \frac{\vartheta_2^4-\vartheta_1^4}{24}\widehat{\mathcal{B}}^TP \quad \widetilde{\mathcal{R}} \quad \vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}} \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{Y}} \Big],$$

$$\Sigma_{22}^1 = \Big[-\vartheta_1R_2 \quad \Big(-\vartheta_1S_1 - \frac{\vartheta_1^2}{2}S_3 - \frac{\vartheta_1^3}{6}S_5 - \frac{\vartheta_1^4}{24}S_7\Big) \quad -P \quad -\vartheta_1T_1 \quad -\frac{\vartheta_1^2}{2}T_3 \quad -\frac{\vartheta_1^3}{6}T_5 \quad -\frac{\vartheta_1^4}{24}T_7$$
$$\vartheta_1(S_1-2P) \quad \frac{\vartheta_1^2}{2}(S_3-2P) \quad \frac{\vartheta_1^3}{6}(S_5-2P) \quad \frac{\vartheta_1^4}{24}(S_7-2P) \quad -T_1 \quad -\vartheta_1T_3 \quad -\frac{\vartheta_1^2}{2}T_5 \quad -\frac{\vartheta_1^3}{6}T_7 \Big],$$

$$\Sigma_{22}^2 = \Big[-hR_4 \quad \Big(-\vartheta_1S_1 - \frac{\vartheta_1^2}{2}S_3 - \frac{\vartheta_1^3}{6}S_5 - \frac{\vartheta_1^4}{24}S_7\Big) \quad -P \quad -\vartheta_1T_1 \quad -\frac{\vartheta_1^2}{2}T_3 \quad -\frac{\vartheta_1^3}{6}T_5 \quad -\frac{\vartheta_1^4}{24}T_7$$
$$\vartheta_1(S_1-2P) \quad \frac{\vartheta_1^2}{2}(S_3-2P) \quad \frac{\vartheta_1^3}{6}(S_5-2P) \quad \frac{\vartheta_1^4}{24}(S_7-2P) \quad -T_1 \quad -\vartheta_1T_3 \quad -\frac{\vartheta_1^2}{2}T_5 \quad -\frac{\vartheta_1^3}{6}T_7 \Big],$$

$$\Sigma_{22}^3 = \Big[-\vartheta_{21}R_3 \quad \Big(-\vartheta_{21}S_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}S_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6}S_6 - \frac{\vartheta_1^4}{24}S_8\Big) \quad -P \quad -\vartheta_{21}T_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}T_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_6$$
$$-\frac{\vartheta_2^4-\vartheta_1^4}{24}T_8 \quad \vartheta_{21}(S_2-2P) \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}(S_4-2P) \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}(S_6-2P) \quad \frac{\vartheta_2^4-\vartheta_1^4}{24}(S_8-2P) \quad -T_2 \quad -\vartheta_{21}T_4$$
$$-\frac{\vartheta_2^2-\vartheta_1^2}{2}T_6 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_8 \Big],$$

$$\Sigma_{22}^4 = \Big[-\vartheta_{21}R_3 \quad \Big(-\vartheta_{21}S_2 - \frac{\vartheta_2^2-\vartheta_1^2}{2}S_4 - \frac{\vartheta_2^3-\vartheta_1^3}{6}S_6 - \frac{\vartheta_1^4}{24}S_8\Big) \quad -P \quad -\vartheta_{21}T_2 \quad -\frac{\vartheta_2^2-\vartheta_1^2}{2}T_4 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_6$$
$$-\frac{\vartheta_2^4-\vartheta_1^4}{24}T_8 \quad \vartheta_{21}(S_2-2P) \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}(S_4-2P) \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}(S_6-2P) \quad \frac{\vartheta_2^4-\vartheta_1^4}{24}(S_8-2P) \quad -T_2 \quad -\vartheta_{21}T_4$$
$$-\frac{\vartheta_2^2-\vartheta_1^2}{2}T_6 \quad -\frac{\vartheta_2^3-\vartheta_1^3}{6}T_8 \Big], \; \Psi_2^T = \Big[0 \quad h\mathcal{B}P\mathcal{B} + h\mathcal{B}GC \quad h\mathcal{B}GD \quad 0 \quad 0 \quad -h\mathcal{B}PW_0 \quad -h\mathcal{B}PW_1$$
$$0 \quad 0 \quad -h\mathcal{B}PB_1 + h\mathcal{B}GB_2 \quad -h\mathcal{B}PW_2 \Big], \; \widehat{E} = \Big[E \quad 0 \quad E_1 \quad 0 \quad 0 \quad H_0 \quad H_1 \quad 0 \quad 0 \quad 0 \quad H_2 \Big],$$
$$\widehat{\mathcal{B}} = \Big[0 \quad -(\mathcal{B} + P^{-1}GC) \quad -P^{-1}GD \quad 0 \quad 0 \quad W_0 \quad W_1 \quad 0 \quad 0 \quad (B_1 - P^{-1}GB_2) \quad W_2 \Big].$$

In addition, the gain matrix X is computed from $X = P^{-1}G$.

Proof : Pre- and post- multiply $\widehat{\Pi}_1$ and $\widehat{\Pi}_2$ in (7), respectively, by $diag\{\overbrace{I,...,I}^{18 \text{ times}}, PS_1^{-1}, PS_3^{-1}, PS_5^{-1}, PS_7^{-1}, I, I, I, I, I\}$ and $diag\{\overbrace{I,...,I}^{18 \text{ times}}, S_1^{-1}P, S_3^{-1}P, S_5^{-1}P, S_7^{-1}P, I, I, I, I, I\}$. Next, pre- and post-multiply $\widehat{\Pi}_3$ and $\widehat{\Pi}_4$ in (7), respectively, by $diag\{\overbrace{I,...,I}^{18 \text{ times}}, PS_2^{-1}, PS_4^{-1}, PS_6^{-1}, PS_8^{-1}, I, I, I, I, I\}$ and $diag\{\overbrace{I,...,I}^{18 \text{ times}}, S_2^{-1}P, S_4^{-1}P, S_6^{-1}P, S_8^{-1}P, I, I, I, I, I\}$. Then, we obtain $\widehat{\Sigma}_i$, $(i = 1, 2, 3, 4)$ in (42) using the inequality $(S_l - 2P) \geq PS_l^{-1}P$ $(l = 1, 2, ..., 8)$ (see [35]). The proof of Theorem 4.1 completes from Theorem 3.1.

Remark 4.2. The problem of H_∞ filtering for SNNs is presented in terms of LMIs using a new quintuple integral LKF, that guarantees the mean-square asymptotic stability of the resulting filtering error system. In Theorems 3.1 and 4.1, the derivative of the LKF can finally be expressed as the sum of four parts (40). This method is generally different from [7, 39].

Remark 4.3. In [13], the authors examined the H_∞ state estimation problem for static NNs. The H_∞ state estimation of SNNs with mixed delays was established in [34]. However, the H_∞ filtering of SNNs with leakage-delayed and mixed time-varying delayed signals was not presented in the existing work. Hence, the results of our work have great scope.

Remark 4.4. The problem of H_∞ state estimation for SNNs using a new LKF with triple integral terms was investigated for the first time in [34]. No other H_∞ state estimation result is available for SNNs in the literature to the authors' best knowledge. Quadruple integral approaches have mostly been used for deterministic delayed NN models in recent years. There are many methods of solving quadruple integral LKFs in deterministic NN cases. However, it is difficult to use quadruple integral approaches for SNNs. Nevertheless, the quadruple integral approach for SNNs was introduced and successfully solved in [38]. Next, we proposed a new quintuple integral LKF for solving the H_∞ filtering problem for SNNs. Additionally, multiple time-varying delays have been added in this paper: time-varying interval delays, leakage delays, and time-varying distributed delays. It will be possible to investigate the existence of neutral-type time-varying delays in the model under consideration in this paper in future works.

Remark 4.5. The distributed H_∞ state estimation problem was examined for stochastic delayed $2D$ systems in [21]. The state estimation problem for asynchronous multi-rate multi-smart sensors was studied in [27]. Global μ-stability analysis for quaternion-valued NNs with unbounded time-varying delays was presented in [23]. The L_∞ performance problem for single and interconnected NNs with time-varying delays is considered in [3]. From the perspective of the authors, state estimation and filtering problems have not been completely studied for SNNs in literature, which partly motivates us to consider such problems in this paper.

Remark 4.6. Generally, H_∞ control, H_∞ state estimation, and H_∞ filtering problems are not simply applied to SNNs. Some research publications have tackled such problems. However, the authors used very simple LKFs to solve the stability problems in those articles. A new LKF with quintuple integrals is proposed for the stability analysis of SNNs in this paper, considering that some computational complexity can occur in our method. However, the H_∞ filtering problem was completely studied for SNNs with mixed time delays, which is the main contribution and motivation of our work.

5. Numerical Examples

We have provided both numerical and practical examples to show the superiority and usefulness of our approach in this article.

Example 5.1. Now, consider the filtering error system in (5) with the following matrix parameters:

$$\mathcal{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ W_0 = \begin{bmatrix} -0.5 & 0.2 \\ -0.2 & 0.5 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.3 & 0.2 \end{bmatrix},\ W_2 = \begin{bmatrix} 0.2 & -0.5 \\ -0.4 & 1.2 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 & 0.2 \\ -0.05 & -0.1 \end{bmatrix},\ E_1 = \begin{bmatrix} -0.05 & 0.1 \\ 0.01 & -0.1 \end{bmatrix},\ H_0 = \begin{bmatrix} 0.05 & 0.05 \\ 0.01 & -0.01 \end{bmatrix},\ H_1 = \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & 0.01 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.02 & -0.01 \\ 0.01 & 0.02 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.4 \\ 0.4 \end{bmatrix},\ C = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix},\ D = \begin{bmatrix} 0.05 & 0.1 \end{bmatrix},$$

$$B_2 = 0.8,\ J = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix},\ V^- = 0_2,\ V^+ = I_2.$$

We obtain the desired gain matrix by solving Theorem 4.1 with $\vartheta_1 = 0.4,\ \vartheta_2 = 1,\ d = 0.3,\ h = 0.2$ and $\phi = 0.4$:

$$X = \begin{bmatrix} -0.5051 \\ 0.4964 \end{bmatrix}.$$

Table 1. Comparison of minimum χ for various values of ϑ_2, ϕ, when $\vartheta_1 = 1,\ d = 0.3,\ h = 0.2$.

ϑ_2 \ ϕ	0.1	0.2	0.3	0.4	0.5
1.5	0.2421	0.2658	0.2901	0.3124	0.3367
2.0	0.4124	0.4347	0.4604	0.4873	0.5162

TABLE 2. Comparison of the maximum allowable delay upper bound ϑ_2 for various values of ϑ_1, ϕ, when $d = 0.3$, $h = 0.2$.

ϑ_1 \ ϕ	0.1	0.2	0.3	0.4	0.5
1.5	3.2143	3.1623	3.0538	2.9143	2.7865
2.0	3.8132	3.5876	3.4687	3.3587	3.2786

The state trajectories of $u_1(t)$ and $u_2(t)$ and their respective estimations $\widehat{u}_1(t)$ and $\widehat{u}_2(t)$ are shown in Figure 1. Figure 2 represents the filtering error, $e(t) = u(t) - \widehat{u}(t)$. Finally, Figure 3 shows the output error, $\beta(t) = \overline{\beta}(t) - \widehat{\beta}(t)$. It is easy to observe the usefulness of Theorem 4.1 for the H_∞ filter design of delayed SNN from the simulation results.

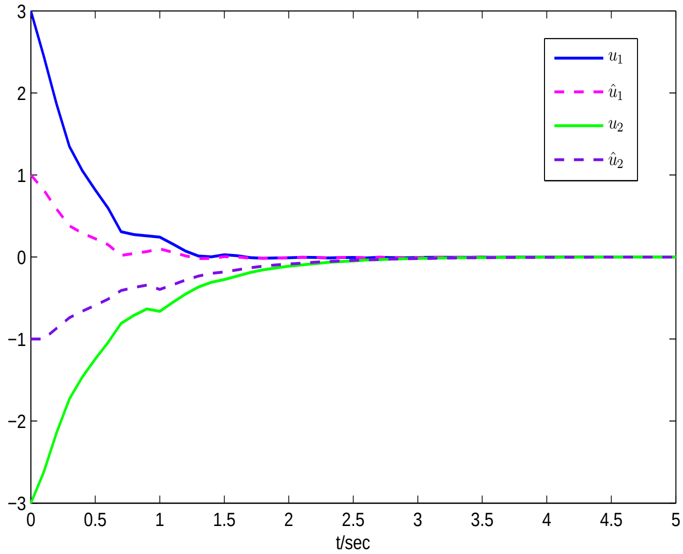

FIGURE 1. State response $u(t)$ and its estimation $\widehat{u}(t)$.

Example 5.2. Consider the following matrix parameters:

$$\mathcal{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \ W_0 = \begin{bmatrix} 0.4 & -0.2 & 0.3 \\ -0.2 & 0.2 & 0.4 \\ 0.2 & 0.3 & -0.3 \end{bmatrix}, \ W_1 = \begin{bmatrix} 0.2 & -0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \\ 0 & -0.2 & 0 \end{bmatrix}, \ W_2 = \begin{bmatrix} 0.3 & -0.2 & 0.4 \\ -0.2 & 0.4 & 0.2 \\ 0.3 & 0.1 & -0.1 \end{bmatrix},$$

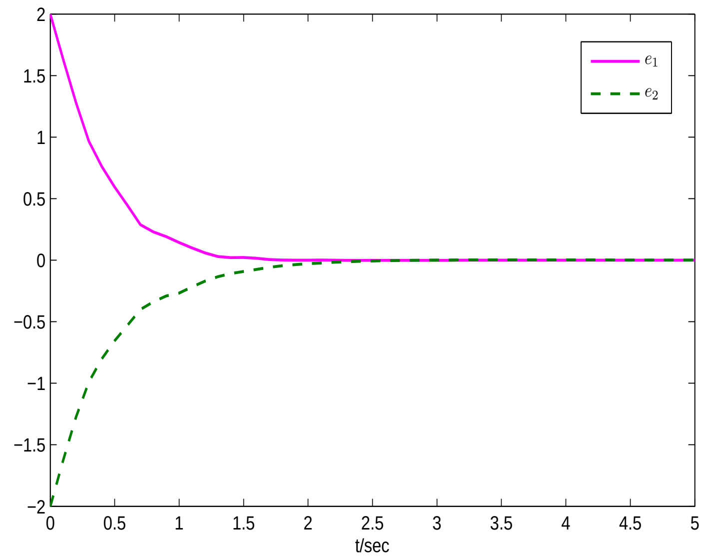

FIGURE 2. Filtering error $e(t)$.

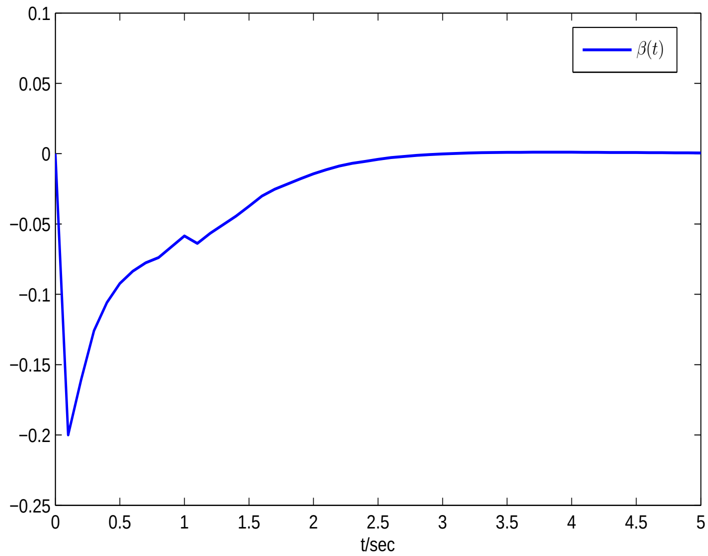

FIGURE 3. Output error $\beta(t)$.

$$E = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \; H_0 = \begin{bmatrix} -0.05 & 0 & -0.01 \\ -0.02 & -0.03 & 0 \\ 0.05 & 0.04 & -0.04 \end{bmatrix}, \; H_1 = \begin{bmatrix} 0.01 & -0.05 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0.05 & 0.05 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} -0.05 & 0.1 & 0.1 \\ 0.1 & 0.05 & 0.1 \\ -0.1 & 0.1 & -0.1 \end{bmatrix}, \; B_1 = \begin{bmatrix} -0.3 \\ 0.1 \\ -0.1 \end{bmatrix}, \; C = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.1 & 0.4 & 0.2 \end{bmatrix}, \; D = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.4 \\ 0.3 \end{bmatrix}, \; J = \begin{bmatrix} 0.3 & 0.4 & 0.5 \end{bmatrix}, \; E_1 = V^- = 0_3, \; V^+ = I_3.$$

Choosing $V_1(e_t) = e^T(t)Pe(t)$ instead of $V_1(e_t) = \Big(e(t) - \mathcal{B}\int_{t-h}^{t} e(s)ds\Big)^T P\Big(e(t) - \mathcal{B}\int_{t-h}^{t} e(s)ds\Big)$ in LKF (9) implies $\widehat{\Sigma}_2 = 0$. Then, we obtain the minimum H_∞ performance index χ for different values of delay ϑ_2 and ϕ by solving LMI $\widehat{\Sigma}_i$ $(i = 1, 3, 4)$ in Theorem 4.1 with $\vartheta_1 = 0.5, \; d = 0.3, \; h = 0.$ The comparison results are listed in Table 3. Thus, our results are less conservative than the method proposed in [34].

TABLE 3. Comparison of minimum χ for various values of $\vartheta_2, \; \phi$.

$(\vartheta_2, \; \phi)$	(0.8,0.4)	(0.9,0.3)	(0.9,0.5)	(1.0,0.5)	(1.0,0.7)
[34]	0.8478	0.8796	0.8974	0.9278	0.9537
Our result	0.3154	0.3485	0.3705	0.4163	0.4673

Example 5.3. Artificial NNs have the characteristics of functionally related biological neurons in a nervous system. Moreover, NNs can be used in some practical systems. In this example, we choose the quadruple-tank process (QTP), as shown in Fig. 4. The main objective was to stabilize the liquid level in tanks 1 and 2 using pumps 1 and 2, respectively. Therefore, v_1 and v_2 are the respective input voltages to pumps 1 and 2 and the outputs are γ_1 and γ_2 (voltages from level measurement devices). The QTP can be represented using the NN. The differential equation for the mass balances in the QTP can be expressed as follows (see [12]):

$$\dot{\widetilde{x}}(t) = \widetilde{A}_0\widetilde{x}(t) + \widetilde{A}_1\widetilde{x}(t - \vartheta_1) + \widetilde{B}_0\widetilde{u}(t - \vartheta_2) + \widetilde{B}_1\widetilde{u}(t - \vartheta_3), \tag{43}$$

where

$$\widetilde{A}_0 = \begin{bmatrix} -0.0021 & 0 & 0 & 0 \\ 0 & -0.0021 & 0 & 0 \\ 0 & 0 & -0.0424 & 0 \\ 0 & 0 & 0 & -0.0424 \end{bmatrix}, \; \widetilde{A}_1 = \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{B}_0 = \begin{bmatrix} 0.1113\gamma_1 & 0 & 0 & 0 \\ 0 & 0.1042\gamma_2 & 0 & 0 \end{bmatrix}, \ \widetilde{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0.1113\gamma_1 \\ 0 & 0 & 0.1042\gamma_2 & 0 \end{bmatrix},$$

$$\widetilde{K} = \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix}, \ \gamma_1 = 0.333, \ \gamma_2 = 0.307, \ \widetilde{u}(t) = \widetilde{K}\widetilde{x}(t).$$

Mass balances in delayed equations can be represented by differential equations. Transport delays are

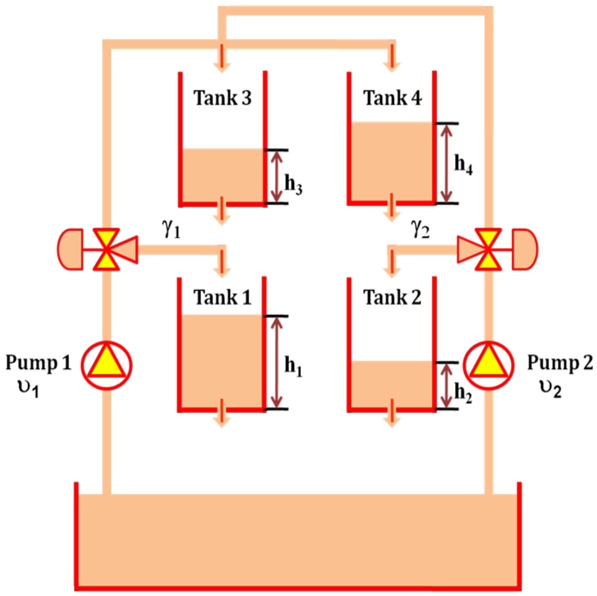

FIGURE 4. Schematic representation of QTP (see [12])

usually included with the delay phenomenon in the tanks' water inlets. Additionally, the transport delays between tanks and valves vary with respect to time. It is assumed that $\vartheta_1 = 0$, $\vartheta_2 = 0$, and $\vartheta_3 = \vartheta(t)$. The control input variable $\widetilde{u}(t)$ denotes the amount of water provided by the pumps. Hence, it is easy to see that $\widetilde{u}(t)$ exhibits a threshold value because of the limited capacity area and pump hoses. Thus, we can consider the nonlinear function $\widetilde{u}(t)$ as follows:

$$\widetilde{u}(t) = \widetilde{K}\widetilde{f}(\widetilde{x}(t)),$$

$$\widetilde{f}(\widetilde{x}(t)) = [\widetilde{f}_1(\widetilde{x}_1(t)), ...\widetilde{f}_4(\widetilde{x}_4(t))]^T,$$

$$\widetilde{f}_i(\widetilde{x}_i(t)) = 0.01(|\widetilde{x}_i(t) + 1| - |\widetilde{x}_i(t) - 1|), \ i = 1, ..., 4.$$

TABLE 4. Upper bounds of ϑ_2 for various values of γ_1 and γ_2.

γ_2 \ γ_1	0.3	0.4	0.5
0.3	1.2456	1.3258	1.3691
0.4	1.3361	1.3812	1.4305
0.5	1.3615	1.4083	1.4538

The QTP (43) can be transferred to SNN (1), without stochastic disturbance, as follows:

$$\dot{u}(t) = -\mathcal{B}u(t) + W_0 f(u(t)) + W_1 f(u(t-\vartheta(t))) + J, \tag{44}$$
$$x(t) = Cu(t),$$
$$u(t) = \rho(t), t \in [-\vartheta_2, 0],$$

where $\mathcal{B} = -\widetilde{A}_0 - \widetilde{A}_1$, $W_0 = \widetilde{B}_0\widetilde{K}$, $W_1 = \widetilde{B}_1\widetilde{K}$, and $f(\cdot) = \widetilde{f}(\cdot)$. Additionally, by taking the values $J = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$, $V^- = 0_4$, $V^+ = 0.01I_4$, $\phi = 0.5$, $\vartheta_1 = 1.0$, we obtained the delay bound $\vartheta_2 = 1.5142$, which ensures the feasibility of the LMI (with $\widetilde{\Pi}_2 = 0$) in Theorem 3.1. In addition, the upper bounds of ϑ_2 for various values of γ_1 and γ_2 are given in Table 4 when $\vartheta_2 = 1$.

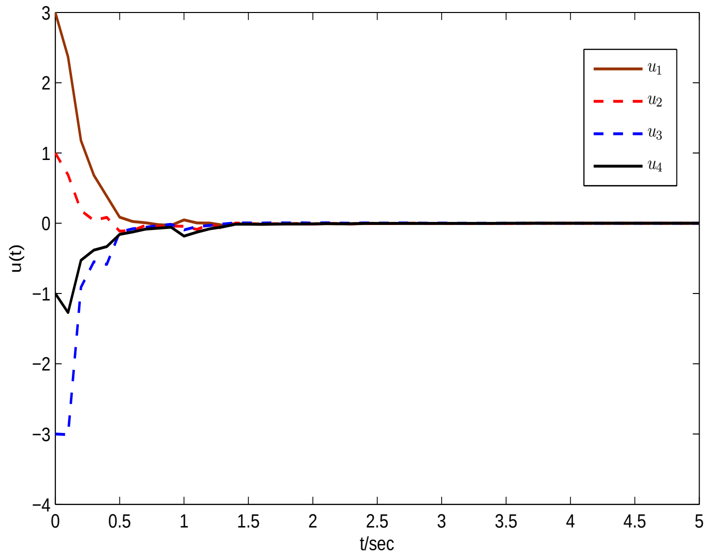

FIGURE 5. The state trajectories of the system in the Example (5.3).

Remark 5.1. Example 5.3 is motivated by a modified version of the quadruple-tank benchmark, which considers the transport delays in the process variables. This QTP system was selected to illustrate that the presented technique can be applied to real-life problems. The QTP is easy to use, very interesting, and gives effective dynamic properties. Thus, this standard problem has recently received a great deal of attention in both research and control education. This system is composed of four interconnected water tanks and two pumps (Figure 4). A comparison table and figure have been added to show the feasibility and convergence of our results (see Table 4 and Figure 5).

Remark 5.2. There are many published papers in the literature related to H_∞ filtering and state estimation problems. However, to the best of the authors' knowledge, H_∞ filtering and state estimation problems for SNNs are not considered in depth in the literature except in Ref. [34]. Moreover, the results in Ref. [34] are a special case in our work. Considering that our work is more general than [34], however, this reference is suitable for comparison purposes with our results.

6. Summary

This paper investigates the problem of H_∞ filtering for SNNs with leakage delay, mixed time-varying interval, and distributed delays. The proposed problem has been examined in terms of LMIs by establishing a new LKF with quintuple integral terms for delayed NNs. Additionally, the desired gain matrix has been estimated with feasible LMIs. Practical and numerical examples have been provided to show the feasibility and superiority of the presented results. An extension of the proposed results to the existence of neutral-type time-varying delays in the model under consideration will be investigated in further work.

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