On Circular Disarranged Strings of Sequences

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Abstract. Two sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, sharing $n-1$ elements, are said disarranged if for every non-empty subset $Q \subseteq [n]$, the sets $\{a_i \mid i \in Q\}$ and $\{b_i \mid i \in Q\}$ are different. In this paper we investigate properties of these pairs of sequences. Moreover we extend the definition of disarranged pairs to a circular string of n-sequences and prove that, for every positive integer m, except some initials values for n even, there exists a similar structure of length m.

Keywords: direct product of graphs, adjacent vertex distinguishing chromatic index, cyclic permutation, derangement, disarranged sequences, 1-disarranged sequences, circular disarranged string.

1 Introduction

Let n be a positive integer, $[n] = \{1, 2, \ldots, n\}$ and $R = (a_1, a_2, \ldots, a_n)$ and $S = (b_1, b_2, \ldots, b_n)$ n-sequences of distinct elements, sharing exactly $n - 1$ elements. We associate with R and S the bijection f defined by the relation $f(a_i) = b_i$, $1 \leq i \leq n$, and represented in two line notation by the $2 \times n$ array

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}. \tag{1}$$

Let u and v be the different elements which belong to the first and the second line respectively. The function f is formed by the linear ordering $l(f) = (u, f(u), f^2(u), \ldots, f^{k-1}(u), v)$, where k is the minimum positive integer such that $f^k(u) = v$, and a permutation $\pi(f)$ on the remaining elements. In [2] a similar function, called widened permutation, is investigated in the context of the theory of species of Joyal. A subset $H \subseteq R$ is f-fixed if $f(H) = H$. If we want emphasize the involved sequences, we denote the function f by $f(R, S)$ and $l(f)$ by $l(R, S)$.

Definition 1. *R is said* **disarranged** *with respect to S if for every set $\{i_1, i_2, \ldots, i_r\} \subseteq [n]$ $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \neq \{b_{i_1}, b_{i_2}, \ldots, b_{i_r}\}$.*

From the definition it follows that the relation is symmetric; then we say that R and S are disarranged. We also say that the pair (R, S) and the function $f(R, S)$ are disarranged. The sequences R and S are called **1-disarranged** if there exists an index $i \in [n]$ such that $a_i = b_i$ and the sequences, obtained from R and S after deleting a_i and b_i, are disarranged. In this case we say that the pair (R, S) is 1-disarranged.

Now we extend the definition 1 to a string of n-sequences.

Definition 2. *Let* $n, m \in \mathbb{N}$*; an* m*-string* $(S_1, S_2, \ldots, S_m)$ *of* n*-sequences, is called disarranged if:*

(A1) S_i *is disjoint from* S_{i-1} *and* S_{i+1} *,*
(A2) S_{i-1} *and* S_{i+1} *are disarranged.*

for every $i = 2, \ldots, m-1$ *.*

A disarranged m-string of n-sequences is *circular* when the properties **(A1)** and **(A2)** are satisfied for every $i = 1, 2, \ldots, m$ (taking the indices modulo m).
An m-string of n-sequences is 1-*disarranged* if there exists at least one index i such that S_{i-1} or S_{i+1} form a 1-disarranged pair.
The *concatenation* of two strings of sequences $P = (P_1, P_2, \ldots, P_r)$ and $T = (T_1, T_2, \ldots, T_q)$, is the string $PT = (P_1, P_2, \ldots, P_r, T_1, T_2, \ldots, T_q)$. Assume that P and T are disarranged. If P_r and T_1 are disjoint and in addition (P_{r-1}, T_1) and (P_r, T_2) are disarranged, then we have that PT is disarranged. Moreover, if P_1 and T_q are disjoint and the pairs (P_1, T_{q-1}) and (P_2, T_q) are disarranged, then PT is a disarranged circular string. A similar definition holds in the case of 1-disarranged sequences.

The notion of circular disarranged or 1-disarranged string of n-sequences has application in relation to an edge coloring problem of graphs [5], where sequences of sets of colors which satisfy $(A1)$ and $(A2)$ are used; in [4] and [7] a similar structure is investigated.
Recall that an *edge coloring* of a finite simple undirected graph G is a map α from the edge set $E(G)$ of G to a finite set of colors C. The coloring α is *proper* if $\alpha(e_1) \neq \alpha(e_2)$ whenever edges e_1, e_2 are adjacent.
The color set of a vertex $u \in V(G)$ with respect to α is the set $C_\alpha(u) := \{\alpha(uv) : uv \in E(G)\}$ of colors assigned by α to edges incident to u. The coloring α is *adjacent vertex distinguishing* (avd for short) if $uv \in E(G)$ implies $C_\alpha(u) \neq C_\alpha(v)$. The *adjacent vertex distinguishing chromatic index* of the graph G is the minimum number $\chi'_a(G)$ of colors in a proper avd edge coloring of G.
The avd chromatic index was discussed also for graphs resulting from binary graph operations; good information about such operations are contained in [6]. In particular one can mention the direct product ([4], [7], [1]). The *direct product* of graphs G and H is the graph $G \times H$ with $V(G \times H) := V(G) \times V(H)$ and $E(G \times H) := \{(u, x)(v, y) : uv \in E(G), xy \in E(H)\}$ (where $(u, x)(v, y)$ is

a simplified notation for the undirected edge $\{(u,x),(v,y)\}$). This product is commutative and associative (up to isomorphisms).

In order to evaluate the avd chromatic index of the direct product of a simple graph G by a cycle C_m, it is useful to employ the concept of a structure such as a circular disarranged or 1-disarranged string. The main result of this paper is the proof of the existence of circular disarranged strings of every length m, except three initial cases, where the strings are 1-disarranged.

We are interested in investigating disarranged pairs of sequences and the existence and construction of circular disarranged strings of sequences having equal length. This paper is organized as follows. In section 2 we determine a characterization of a pair of disarranged sequences. In section 3 we associate with a pair of disarranged sequences a permutation, which turns out to be formed by one cycle only; in the same section we determine many cases of circular disarranged strings. In sections 4 and 5 we prove by construction that for every positive integer m, there exist circular disarranged m-strings of n-sequences, $n > 2$, of every length, except the cases of n even and $m = 6$, $m = 14$ and $m = 2n + 7$, where the strings are 1-disarranged. All these conclusive results are contained in the last section.

2 Disarranged Pairs of Sequences

In this section we determine a characterization of disarranged pairs of sequences of equal length.

Theorem 1. *Let R and S be n-sequences, sharing exactly $n-1$ elements. They are disarranged if and only if the linear ordering $l(R,S)$ contains all the elements of R and S.*

Proof. Let $R = (a_1, a_2, \dots, a_n)$ and $S = (b_1, b_2, \dots, b_n)$ be n-sequences, sharing exactly $n - 1$ elements. Assume they are disarranged.
If the function $f(R,S)$ contains a cycle $C = \{i_1, i_2, \dots, i_s\}$, where $s > 0$, then $f(C) = (f(i_1), f(i_2), \dots, f(i_s)) = (i_2, i_3, \dots, i_1) = C$. Then C turns out to be f-fixed, a contradiction to the assumption. Thus $s = 0$ and f consists only in the linear ordering $l(f)$.
Now assume that the sequence $l(R,S)$ contains all the elements of R and S. Then, for every subset J of the set of elements of R, represented by the ordered subsequence of $l(f)$ $J = (x_1, x_2, \dots, x_j)$, where $j \geq 1$, we have that $f(x_j) \notin J$. This implies that $f(J) \neq J$; then J is not f-fixed and consequently R and S are disarranged.

In other words R and S are disarranged if the function $f(R,S)$ is a widened permutation which consists only of the linear ordering $l(f)$.
A similar characterization holds in the case of two 1-disarranged sequences.

Proposition 1. *Let $R = (a_1, a_2, \dots, a_n)$ and $S = (b_1, b_2, \dots, b_n)$ sequences, sharing exactly $n-1$ elements. They are 1-disarranged if and only if there exists* *an $i \in [n]$ such that $a_i = b_i$ and the linear ordering $l(R', S')$, where R' and S' are obtained from R and S by deleting a_i, contains all the elements of R' and S'.*

Example. Let $R = (1, 2, 3, 4, 5, 6)$ and $S = (2, 3, 4, 5, 7, 1)$, having different elements respectively 6 and 7. The bijection $f(R, S)$ represented by the array

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 7 & 1 \end{pmatrix} \tag{2}$$

is formed by the linear ordering $(6, 1, 2, 3, 4, 5, 7)$, which contains all the elements of $R \cup S$. Thus R and S are disarranged.
An example of 1-disarranged sequence is given by the two sequences $R = (1, 2, 3, 4, 5, 6, 7)$ and $S = (7, 4, 3, 1, 6, 8, 5)$. In this case 3 is fixed, $u = 2$, $v = 8$ and the function $f(R, S)$ is partitioned into the cycle (3) and the linear ordering $(2, 4, 1, 7, 5, 6, 8)$.

3 Disarranged Permutations

In this section we introduce and investigate the notion of disarranged permutations to show the connection with the notion of disarranged pairs of n-sequences.

Definition 3. *A permutation α of $[n]$ is said disarranged if every proper subset D of $[n]$ is not α-fixed.*

Thus a disarranged permutation turns out to be a generalization of a derangement, which consists in a permutation without fixed points.
Recall that a permutation is *cyclic* if it consist of one cycle only [3].

Theorem 2. *A permutation α of $[n]$ is disarranged if and only if α is cyclic.*

Proof. Let α be a disarranged permutation of $[n]$. Assume that α has more than one cycle; in particular let $C = (j_1, j_2, \ldots, j_s)$ one of its cycles, where $s < n$. Then $\alpha(C) = (\alpha(j_1), \alpha(j_2), \ldots, \alpha(j_s)) = (j_2, j_3, \ldots, j_s, j_1) = C$, a contradiction to the assumption.
Now, assume that α has only one cycle, say H. Let $S = \{i_1, i_2, \ldots, i_r\}$, where $r < n$, a proper subset of $[n]$. Then $\alpha(S) = \{\alpha(i_1), \alpha(i_2), \ldots, \alpha(i_r)\}$ is the set of the elements of H which are consecutive to the elements of S in H. Because $r < n$, it follows that $\alpha(S) \neq S$ and α is disarranged.

Proposition 2. *Let $R = (a_1, a_2, \ldots, a_n)$ and $S = (b_1, b_2, \ldots, b_n)$ sequences sharing the elements of a $(n-1)$-set A and having u and v as different elements respectively. The function $f(R, S)$ is disarranged if and only if the function obtained by replacing v by u in S is a cyclic permutation of $A \cup \{u\}$.*

Proof. Let f be disarranged and let $l(f) = (u, a_{k_1}, a_{k_2}, \ldots, a_{k_{n-1}}, v)$. If we replace v by u, we obtain a permutation of $A \cup \{u\}$, which is one cycle.
On the contrary, let π be a permutation of $A \cup \{u\}$, formed by only one cycle, represented in two-line notation

$$\begin{pmatrix} a_{i_1} & a_{i_2} & \ldots & a_{i_n} \\ a_{i_2} & a_{i_3} & \ldots & a_{i_1} \end{pmatrix}.$$

We replace the element u by v in the second line and we obtain a disarranged function.

Lemma 1. *Let α be a permutation of $A = \{a_1, a_2, \ldots, a_n\}$ such that $\alpha(a_i) = a_{i+k}$ (a_{i-k}), where $i \in [n]$, k is a positive integer coprime with n and the integers $i+k$ $(i-k)$ are modulo n. Then α is cyclic.*

Proof. The permutation α contains the cycle $C = (a_1, a_{1+k}, \ldots, a_{1+(r-1)k})$ $(C' = (a_1, a_{1-k}, \ldots, a_{1-(r-1)k}))$, where r is the minimum positive integer such that $1 + rk \equiv 1$ $(1 - rk \equiv 1)$ mod n. Because k is prime with n, then $r = n$ and C (C') contains all the elements of A.

Let R and S be n-sequences sharing $n-1$ elements; u and v are the different elements respectively. For every $k \in [n-1]$ we say that S is a k**-right** (k-left) shift of R if S is obtained from R by cyclically shifting of k positions to the right (left) all the elements of R and replacing u by v.
Moreover S is a k**-shift** of R when it is either a k-right or a k-left shift of R.
Given the permutation $\sigma = (1, 2, \ldots, n)$ of $[n]$, a k-left shift of $R = (a_1, a_2, \ldots, a_n)$ is $\sigma^k(R) = (a_{1+k}, a_{2+k}, \ldots, a_{n+k})$, while a k-right shift of R is $\sigma^{-k}(R) = (a_{1-k}, a_{2-k}, \ldots, a_{n-k})$, where the indices are modulo n.

Proposition 3. *Let R and S be n-sequences sharing $n-1$ elements and let S be a k-shift of R, where k is a positive integer coprime with n. Then (R, S) is disarranged.*

Proof. It follows from Lemma 1 and Proposition 2.

An example of application of Proposition 3 is given from the circular disarranged string of n-sequences of length $2n+1$, $n > 2$, of the following Proposition.

Proposition 4. *Let $n > 2$ be a positive integer. Then the circular string*

$$C_{2n+1} = (Q_1, Q_2, \ldots, Q_{2n+1}),$$

where $Q_1 = \{1, 2, \ldots, n\}$, $Q_2 = \{n+1, n+2, \ldots, 2n\}$ and every other sequence Q_i, $3 \leq i \leq 2n+1$, is obtained from Q_{i-1} by taking n cyclically consecutive elements of the sequence $\{1, 2, \ldots, 2n+1\}$, is disarranged.

Proof. By construction, every Q_i is 1-right shift of Q_{i-2}, $3 \leq i \leq 2n+1$ and by Proposition 3 the result is done.

Lemma 2. *Let $R = P_1P_2 \dots P_t$ be a concatenation of d-sequences, $d, t > 1$. Then the sequences $S = P_2P_3 \dots P_t\sigma^{\pm k}(P_1)$ and $S' = \sigma^{\pm k}(P_t)P_1P_2 \dots P_{t-1}$, where k is a positive integer coprime with d, form together with R a cyclic permutation.*

Proof. Let $P_i = (p_{i,1}, p_{i,2}, \dots, p_{i,d})$, $1 \leq i \leq t$, and denote $V_j = (p_{1,j}, p_{2,j}, \dots, p_{t,j})$, where $1 \leq j \leq d$.

Note that, since k is coprime with d, the set $\{p_{1,1}, p_{1,1+k}, \dots, p_{1,1+(d-1)k}\}$ coincides with P_1. Thus the permutation having R and S as rows in the line notation

$$\begin{pmatrix} p_{1,1} & \cdots & p_{1,d} \cdots p_{t,1} \cdots p_{t,d} \\ p_{2,1} & \cdots & p_{2,d} \cdots p_{1,1+k} \cdots p_{1,d+k} \end{pmatrix}$$

is formed by the cycle obtained by the concatenation $V_1V_{1+k}V_{1+2k} \dots V_{1+(d-1)k}$, which contain all the elements of R. A similar situation holds for the pair R and S'.

Proposition 5. *Let $R = P_1P_2 \dots P_t$ be a concatenation of d-sequences, $d, t > 1$, $S = P_2P_3 \dots P_t\sigma^{\pm k}(P_1)$ and $S' = \sigma^{\pm k}(P_t)P_1P_2 \dots P_{t-1}$, where k is a positive integer coprime with d. If an element x of R is replaced by y in S or in S', $y \neq x$, then the new pairs of sequences are disarranged.*

Proof. It follows from Lemma 2 and Proposition 2.

Lemma 3. *Let n be a positive even integer, $R = (1, 2, \dots, n)$ and $S = (3, 4, \dots, n, x, 1)$, where $x \notin R$. Then (R, S) is disarranged.*

Proof. It is easy to prove that $l(R, S) = (2, 4, \dots, n, 1, 3, \dots, n-1, x)$ contains all the elements of R and S.

Lemma 4. *Let n be a positive even integer, $R = (1, 2, \dots, n)$ and $S = (3, 4, \dots, n, 2, y)$, where $y \notin R$. Then (R, S) is disarranged.*

Proof. As in the previous lemma $l(R, S) = (1, 3, \dots, n-1, 2, 4, \dots, n, y)$ contains all the elements of R and S.

4 Circular Disarranged Strings of Even Length

In this section we determine a construction of circular disarranged m-strings of n-sequences, where n is a positive integer and $m \in \{8, 10, 12\}$, whose elements belong to the set $[2n + 1]$.
Throughout this section and the following ones we apply properties of Proposition 3 and Proposition 5 without an explicit recall.
Moreover if we want to remark the different elements u and v of a disarranged ordered pair of sequences, we write $[u \hookrightarrow v]$.
Recall that Q_i, $1 \leq i \leq 2n + 1$, denotes the i-th sequence of the circular string C_{2n+1} of Proposition 4.

Proposition 6. *Let n be a positive integer. Then the circular 8-string of n - sequences*

$$E_8^n : T_i = Q_i, 1 \leq i \leq 5, T_6 = (n+1, \ldots, 2n-1, n-1), T_7 = (n, 1, \ldots, n-2, 2n),$$

$$T_8 = (2n+1, n+1, \ldots, 2n-1)$$

is disarranged.

Proof. By Proposition 4 the sequences T_3, T_4, T_5 are 1-right shift of T_1, T_2, T_3 respectively.
Moreover it is easy to see that T_6 is 1-left shift of $T_4 = (n, n+1, \ldots, 2n-1)$ ($[n \hookrightarrow n-1]$), T_7 is 1-left shift of $T_5 = (2n, 2n+1, 1, 2, \ldots, n-2)$ ($[2n+1 \hookrightarrow n]$) and 1-right shift of T_1 ($[n-1 \hookrightarrow 2n]$)and, finally, T_8 is 1-right shift of T_6 ($[n \hookrightarrow 2n+1]$) and T_2 ($[2n \hookrightarrow 2n+1]$).

Proposition 7. *Let n be a positive odd integer. Then the circular 10-string of n-sequences*

$$E_{10}^n : Z_i = Q_i, 1 \leq i \leq 4, Z_5 = (1, 2, \ldots, n-1, 2n),$$

$$Z_6 = (2n+1, n+2, \ldots, 2n-1, n), Z_7 = (2, \ldots, n-1, 2n, n+1),$$

$$Z_8 = (n+2, \ldots, 2n-1, 1, 2n+1), Z_9 = (3, \ldots, n+1, 2), Z_{10} = (2n, 2n+1, n+2, \ldots, 2n-1),$$

is disarranged,

Proof. By Proposition 4 the sequences Z_3, Z_4 are 1-right shift of Z_1, Z_2 respectively. Moreover the sequences Z_i, for $5 \leq i \leq 8$ are 1-left shift of Z_{i-2}, Z_9 is 1-left shift of Z_7 and 2-left shift of Z_1. Finally Z_{10} is 2-right shift of Z_8 and 1-right shift of Z_2. Note that because n is odd, then 2 is coprime with n.

Proposition 8. *Let $n = 2h$ be a positive even integer, $h > 2$. Then the circular 10-string of n-sequences*

$$E_{10}'^n : Z_i' = Q_i, 1 \leq i \leq 4, Z_5' = (1, 2, \ldots, n-1, 2n),$$

$$Z_6' = (2n+1, n+2, \ldots, 2n-1, n), Z_7' = (h+1, \ldots, n-1, n+1; h, 1, 2, \ldots, h-1),$$

$$Z_8' = (n+h+1, \ldots, 2n-1, n, n+2, \ldots, n+h, 2n),$$

$$Z_9' = (n, 1, 2, \ldots, h-1, h+2, n-1, n+1, h+1),$$

$$Z_{10}' = (n+2, \ldots, n+h, 2n, 2n+1, n+h+1, \ldots, 2n-1),$$

is disarranged.

Proof. Recall that the sequences Z'_3, Z'_4 are 1-right shift of Z'_1, Z'_2 respectively. The sequence Z'_5 is 1-left shift of Z'_3. Moreover, denoted $Z'_5 = P_1P_2$, where $P_1 = (1, 2, \ldots, h)$, $P_2 = (h+1, \ldots, 2h-1, 2n)$, we obtain $Z'_7 = P_2\sigma^{-1}(P_1)$, with $[2n \hookrightarrow n+1]$. Now, denoted $Z'_7 = R_1R_2$, where $R_1 = (h+1, \ldots, n-1, n+1)$ and $R_2 = (h, 1, 2, \ldots, h-1)$ we have $Z'_9 = R_2\sigma(R_1) = (n, 1, \ldots, h-1; h+2, \ldots, n-1, n+1, h+1)$ with $[h \hookrightarrow n]$. Then $(Z'_9, Z'_1) = (n+1, n-1, \ldots, h+1, n, 1, 2, \ldots, h)$ contains all the elements of Z'_1 and Z'_9.
Now we establish that $Z'_6 = (2n+1, n+2, \ldots, 2n-1, n)$ is 1-left shift of $Z_4 = (n, \ldots, 2n-1)$ with $[n+1 \hookrightarrow 2n+1]$.
Denoted $Z'_6 = A_1A_2$, where $A_1 = (2n+1, n+2, \ldots, n+h)$ and $A_2 = (n+h+1, \ldots 2n-1, n)$, then $Z'_8 = A_2\sigma(A_1) = (n+h+1, \ldots, 2n-1, n; n+2, \ldots, n+h, 2n) = T_1T_2$ with $[2n+1 \hookrightarrow 2n]$. It follows $Z'_{10} = T_2\sigma^{-1}(T_1) = (n+2, \ldots, n+h, 2n; 2n+1, n+h+1, \ldots, 2n-1)$ with $[n \hookrightarrow 2n+1]$. Finally $(Z'_{10}, Z'_2) = (2n+1, n+h+1, \ldots, 2n, n+h, \ldots, n+1)$, where the sequence contains all the elements of Z'_{10} and Z'_2.
By all previous motivations the string E'^{n}_{10} is disarranged.

For $n \geq 3$, consider the $(n-1)$-sequences $A = \{1, 2, \ldots, n-1\}$ and $B = \{n+1, n+2, \ldots, 2n-1\}$ of the set $[2n+1]$; for every $x \in \{n, 2n, 2n+1\}$ denote Ax the ordered set obtained by inserting x after last element of A. A similar definition holds for xA, Bx and xB.
The following sequence is an example of disarranged 6-string of n-sequences:

$$P_1 = (An, B2n, (2n+1)A, nB, A2n, B(2n+1)). \tag{3}$$

Indeed the third and the fourth sequences are 1-right shift of the first and the second respectively, while the fifth and the sixth are 1-left shift of the third and the fourth respectively.
A similar situation holds for the following 6-string of n-sequences.

$$P_2 = (nA, 2nB, A(2n+1), Bn, 2nA, (2n+1)B). \tag{4}$$

From previous disarranged strings it is immediate to obtain the following circular disarranged string.

Proposition 9. *The concatenation P_1P_2 is a disarranged circular 12-string of n-sequences.*

Example 1. For $n = 3$, we have the following disarranged 6-strings of 3-sequences

$$P_1 = (123, 456, 712, 345, 126, 457)$$

$$P_2 = (312, 645, 127, 453, 612, 745)$$

and the following circular disarranged 12-string of 3-sequences

$$P_1P_2 = (123, 456, 712, 345, 126, 457, 312, 645, 127, 453, 612, 745).$$

4.1 The circular strings of length 6 and 14

Now we consider the particular case of circular strings of length 6 or 14 of n-sequences.

Proposition 10. *Let n be a positive integer. Then, for n odd, the circular 6-string*

$$E_6^n : V_i = Q_i, 1 \le i \le 4, V_5 = (2, \dots, n{-}1, 2n, 1), V_6 = (n{+}2, \dots, 2n{-}1, 2n{+}1, n{+}1),$$

is disarranged, while, for n even, the circular 6-string

$$E_6'^n : V_i, 1 \le i \le 4, V_5' = (1, n{-}1, 2n, 2, \dots, n{-}2), V_6' = (n{+}1, 2n{-}1, 2n{+}1, n{+}2, \dots, 2n{-}2),$$

is 1-disarranged. In particular the sequences V_1 and V_5' have the element 1 fixed, while V_2 and V_6' have $n+1$ fixed.

Proof. Let n be odd. By Proposition 4 (V_3, V_1) and (V_4, V_2) are disarranged; moreover V_5 and V_6 are 1-left shift of V_1 and V_2 respectively, while they are 2-left shift of V_3 and V_4 respectively. Being 2 coprime with n, by Proposition 3 (V_3, V_5) and (V_4, V_6) are disarranged. By Lemmas 3 and 4 E_6^n is disarranged.
Let n be even. By the same previous motivation (V_3, V_1) and (V_4, V_2) are disarranged. Moreover $V_5' \setminus \{1\}$ and $V_6' \setminus \{n+1\}$ are 2-right shift of $V_1 \setminus \{1\}$ and $V_2 \setminus \{n+1\}$ respectively. Finally we have that the sequence

$$l(V_3, V_5') = (2n+1, 1, n-1, \dots, 2, 2n)$$

contains all the elements of V_3 and V_5' and the sequence

$$l(V_4, V_6') = (n, n+1, 2n-1, \dots, n+2, 2n+1)$$

contains all the elements of V_4 and V_6'. By Theorem 1 the pairs (V_3, V_5') and (V_4, V_6') are disarranged.

Now let us consider the case of $m = 14$. Since $14 = 8 + 6$, by concatenating a circular disarranged string of length 8 from Proposition 6 and a 1-disarranged string of length 6 from Proposition 10, we obtain that there exists a 1-disarranged string of length 14.

Particular case. Take the case $n = 4$. From the circular strings E_8^4 of Proposition 6 and the circular 1-disarranged string $E_6'^4$ of Proposition 10 we obtain the circular strings $E_6'^4 : (1234, 5678, 9123, 4567, 1382, 5796)$ and
$E_8'^4 : (1234, 5678, 9123, 4567, 8912, 5673, 4128, 9567)$.
Note that $E_6'^4$ is 1-disarranged. Indeed last two strings have the elements 1 and 5 fixed with respect the first and the second sequence respectively. By concatenation we obtain the 1-disarranged circular string $E_{14}'^4 : (1234, 5678, 9123, 4567, 8912, 5673, 4128, 9567, 1234, 5678, 9123, 4567, 1382, 5796)$.

5 Circular Disarranged Strings of Odd Length

In this section we give a construction of circular disarranged m-strings of n-sequences, where $n > 2$ and $m \in \{2n+3, 2n+5\}$. Recall that by Proposition 4 there exists a circular disarranged $(2n+1)$-string of n-sequences, denoted Q_i, $1 \le i \le 2n+1$.

Proposition 11. *For $n > 4$ be a positive integer. Then, for n odd, the circular string of length $(2n+3)$*

$$C_{2n+3} = (Q_1, \dots, Q_{2n-1}, W_{2n}, W_{2n+1}, W_{2n+2}, W_{(2n+3)}),$$

where $W_{2n} = (n+2, 3, \dots, n, 1)$, $W_{2n+1} = (2, n+3, \dots, 2n+1)$, $W_{2n+2} = (3, 4, \dots, n, 1, n+1)$ and $W_{2n+3} = (n+3, \dots, 2n+1, n+2)$, and, for $n > 4$, even, the circular string

$$C'_{2n+3} = (Q_1, \dots, Q_{2n-3}, W'_{2n-2}, W'_{2n-1}, W'_{2n}, W'_{2n+1}, W'_{2n+2}, W'_{2n+3}),$$

where $W'_{2n-2} = (3, 4, \dots, n+2)$ $W'_{2n-1} = (n+5, \dots, 2n+1, 1, n+3, n+4)$ $W'_{2n} = (4, 5, \dots, n+1, 2, 3)$ $W'_{2n+1} = (n+4, \dots, 2n+1, 1, n+2)$ $W'_{2n+2} = (3, \dots, n+1, 1)$, $W'_{2n+3} = (n+2, n+3, \dots, 2n+1, n+3)$, are disarranged.

Proof. Let n be odd. We see that W_{2n} is 1-right shift of $Q_{2n-2} = (3, \dots, n+2)$, W_{2n+2} is 1-left shift of W_{2n} and it is 2-left shift of Q_1.
Moreover W_{2n+1} is 1-right shift of $Q_{2n-1} = (n+3, \dots, 2n+1, 1)$, W_{2n+3} is 1-left shift of W_{2n+1} and it is 2-left shift of Q_2.
Let n be even. Recall that $Q_{2n-4} = (4, 5, \dots, n+2)$ and $Q_{2n-3} = (n+4, \dots, 2n+1, 1, 2)$; and We have that W'_{2n-2} is 1-right shift of W'_{2n-4} with $n+3 \hookrightarrow 3$, W'_{2n} is 1-left shift of W'_{2n-2} with $n+2 \hookrightarrow 2$, W'_{2n+2} is 1-right shift of W'_{2n} $2 \hookrightarrow 1$. Moreover W'_{2n-1} is 1-left shift of W'_{2n-3} with $2 \hookrightarrow n+3$, W'_{2n+1} is 1-right shift of W'_{2n-1} with $2 \hookrightarrow n+3$, W'_{2n+3} is 1-right shift of W'_{2n+1} with $1 \hookrightarrow n+3$.
Moreover W'_{2n+2} and Q_1 are disarranged because there is the linear ordering $(n+1, n-1, \dots, 1, n, n-2, \dots, 2)$ which contain all the elements of W'_{2n+2} and Q_1.
Finally W'_{2n+3} and Q_2 are disarranged because there is the linear ordering $(2n+1, 2n-1, \dots, n+3, 2n, 2n-2, \dots, n+2, n+1)$ which contain all the elements of W'_{2n+3} and Q_2.

Proposition 12. *For $n > 4$, the circular string*

$$C_{2n+5} = (U_1, \dots, U_{2n}, U_{2n+1}, U_{2n+2}, U_{2n+3}, U_{2n+4}, U_{2n+5}),$$

where $U_i = Q_i, 1 \le i \le 2n$,

$$U_{2n+1} = (1, n+3, \dots, 2n, n+2), U_{2n+2} = (2n+1, 2, \dots, n)$$

$$U_{2n+3} = (n+2, 1, n+3, \dots, 2n-1, n+1), U_{2n+4} = (2, \dots, n, 2n),$$

$$U_{2n+5} = (2n+1, n+3, \dots, 2n-1, n+1, n+2)),$$

is disarranged.

Proof. Recall that $U_{2n-1} = (n+3, \dots, 2n+1, 1)$ and $U_{2n} = (2, \dots, n+1)$. It is easy to prove that U_i, for $3 \leq i \leq 2n+3$ is a 1-right shift of U_{i-2}, U_{2n+5} is 1-left shift of U_{2n+3} and compared with U_2 determines the sequence

$$l(U_2, U_{2n+5}) = (2n, n+2, n+3, \dots, 2n-1, n+1, 2n+1)$$

which contains all the elements of U_2 and U_{2n+5}. By Theorem 1 (U_2, U_{2n+5}) is disarranged. Moreover U_{2n+4} is 1-left shift of U_{2n+2} and of U_1.

6 Conclusive results

In this last section we summarize previous results; in particular we prove the existence of circular disarranged m-strings, for every positive integer $m > 2$, except few initial values.

Lemma 5. *Let $m > 6$ and $m \neq 14$ be a positive even integer. Then m may be represented as integral linear combination of the integers $8, 10, 12$.*

Proof. It is sufficient to prove that every integer $10a \leq m \leq 10(a+1)$, where a is a positive integer, satisfy the request. It is immediate that the integers $16, 18$, $10a, 10a+8, 10(a+1)$ satisfy the condition. Assume that $a \geq 2$. We write the integers $10a+2, 10a+4, 10a+6$, as $10(a-1)+12, 10(a-2)+24$ and $10(a-1)+16$ respectively, thus proving the result.

Proposition 13. *Let m be a positive even integer and n a positive integer. For every $m \geq 8$ and $m \neq 14$, if n is even, and $m \geq 4$, if n is odd, there exists a circular disarranged m-string of n sequences. For $m = 6$ and $m = 14$ there exists a circular 1-disarranged m-string of n sequences.*

Proof. By Propositions 6, 7, 8 and 9, there are circular disarranged sequences for $m = 8, 10, 12$. By Lemma 5 for every even integer $m \geq 8$ and $m \neq 14$, by concatenating suitable strings whose elements belong to $[2n+1]$, we obtain a circular disarranged m-string. In the remaining initial cases we have a 1-disarranged m-string.

Lemma 6. *Let $m \geq 2n+1$ be a positive odd integer, where $n \geq 3$. Then m may be represented as sum of $2n+1$ or $2n+5$ and an even integer $k \geq 8$.*

Proof. Every integer $h = 2n + h'$, where $h' \geq 9$ is odd may be represented as $h = 2n + 1 + t$, where $t \geq 8$ is a positive even integer. If $t = 14$, we write $h = 2n + 5 + 10$.

Proposition 14. *Let $m \geq 2n+1$, where $n \geq 3$, be a positive odd integer. Then there exists a circular disarranged m-string of n sequences, except $m = 2n+7$ where there exists a 1-disarranged circular string of n sequences.*

Proof. By Lemma 6 every integer m, except $m = 2n+7$, is a sum of $2n+1$ or $2n+5$ and an even integer ≥ 8. Then by concatenating suitable strings we obtain the result. In the case of $m = 2n+7$ by concatenating a $(2n+1)$-string and a circular 6-string, we obtain a circular 1-disarranged m-string.

In conclusion we obtain the following result.

Theorem 3. *Let m, n be positive integers. For n odd and every $m > 2$ or for n even and $m > 6$ even ($m \neq 14$) or for $m \geq 2n+1$ odd ($m \neq 2n+7$), there exists a circular disarranged m-string. For the remaining cases, there exists a circular 1-disarranged m-string.*

A further development concerns the problem of establishing whether some 1-disarranged strings are disarranged.

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