Calculating the joint distribution of n batch delivery spans in a stochastic permutation flow shop

Alessio Angius[1], András Horváth[2] and Marcello Urgo[1]

[1] Politecnico di Milano, Italy
alessio.angius@polimi.it,marcello.urgo@polimi.it
[2] Università di Torino, Italy
horvath@di.unito.it

Keywords: Flow shop, Batch delivery, Markov processes, PH-type distribution.

1 Introduction

In a manufacturing environment, the execution of a generic process usually consists of a set of process steps linked by proper precedence constraints. It is rather common that these process steps are arranged in a fixed sequence to be executed on a set of resources, (e.g., a machine tool, a measurement station, an inspection, etc). In such cases, the manufacturing system is said to be organized as a flow shop. i.e., a processing system in which the task sequence of each job is fully specified (chain precedence) and all jobs visit the work stations in the same order (Emmons & Vairaktarakis 2013). We consider a *permutation* flow shop problem, i.e., a flow shop where the jobs are executed in the same order on each of the machines. This class of problems arises when the possibility for one job to overtake another one is difficult or even impossible, due to the dimension of the product undergoing the process or to the characteristics of the production system (e.g., fixed transportation path). After being processed in the flow shop, the products must undergo an additional process step executed on a batch, e.g., a transport, a heat treatment, etc. We take in consideration a set of processes having stochastic durations which has been divided in batches. Given a batch i, we focus on the distribution of the time between the completion of the first and last part of the same batch; let us refer to this measure as *"delivery span of batch i"*. In this setting, the paper provides the mathematical description of the joint distribution of n batch delivery spans by starting from a pre-scheduled set of processes.

The definition of this stochastic flow shop problem stems from the production of large parts (e.g., in the production of aircrafts). These components (e.g., wings) are assembled in production lines according to a given sequence in all the steps, due to the extremely large dimension not allowing swapping among the parts. At the end of the assembling, a batch of parts is loaded on special vehicles (special airplanes or ships) and transferred to the aircraft assembly plant. The time these vehicle must spend waiting for the batch to be completed has a cost (they are waiting without transporting anything) hence, minimizing this time (or an associated measure of risk in the stochastic version of the problem) aims at improving their profitability. This leads to a scheduling problem that aims to optimize the distribution of the batch delivery spans according to some cost function. It is clear that, in this scenario, the computation of the whole joint distribution plays a crucial role since any cost function can be directly computed from it.

In literature, the stochastic flow shop problem is often considered for the 2-machine case with random processing times, due to the fact that significant results can be provided. In such cases, the Talwar's Rule assures that the expected makespan is minimized by sequencing the jobs in non-increasing order of $(\mu_{i1}^{-1} - \mu_{i2}^{-1})$ where μ_{ij}^{-1} is the mean of the processing time of job i on machine j (Cunningham & Dutta 1973). In addition, various heuristics have been developed to deal with general distribution of the processing times and the minimization of the expected makespan (Baker & Trietsch 2010). Moreover, if the processing time distributions are exponential, (Gourgand, Grangeon & Norre 2003) show that the expected makespan calculation can be carried out analytically using a Markovian approach even if some limit occurs for larger problems. To optimize the expected makespan with general probability distributions for processing times, the available approaches deal with heuristic procedures to or good sequences and/or simulation procedures (Emmons & Vairaktarakis 2013) (Baker & Altheimer 2012). No other results are available for objective functions different from the expected makespan or total completion time.

2 Evaluation of a fixed scheduling

We consider a pre-scheduled set of processes $\mathcal{P}$ where every process $p \in \mathcal{P}$ must finish N steps to be completed. As further assumptions, we have that: i) the order of the steps is the same for any process p but the distribution can be different; ii) there is at the most one active process for every step. Since the order with which the processes enter into the system is pre-determined, such scenario is equivalent to consider a multi-class queuing network composed of N first-in-first-out single server stations in which $K = |\mathcal{P}|$ jobs are served. At the beginning job j_1 starts service at the first station and jobs $j_2, ..., j_K$ are waiting in the queue of the first station. A state of the system can be represented by a sequence that contains the jobs $j_1, ..., j_K$ and N "separator" symbols that indicate the position of the jobs in the serial line of servers. For example, with $K = 4$ and $N = 3$, in the state described by the sequence $j_4 j_3 | j_2 || j_1$ jobs j_4 and j_3 are at the first station, j_2 is at the second station, there is no job at the third station and job j_1 has already gone through every station. Since the order of the jobs in the state descriptor is fixed, the number of states is equal to the number of ways that the separator symbols can be positioned, i.e., the number of state is $S = \binom{K+N}{N}$. Let us assume first that service times are exponential and thus the behavior of the system can be captured by a continuous time Markov chain with S states. The service rate of station i will be denoted by λ_i. In Figure 1 we show a portion of this Markov chain, namely, we depict the predecessor states and the successor states of state $j_4 j_3 | j_2 || j_1$ highlighting on the edges the transition rates. Note that, since each job receives service at each station exactly once, the Markov chains is acyclic. Non-exponential service

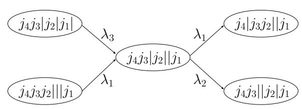


Fig. 1. Portion of the system with exponential service times and $K = 4, N = 3$

times can be modeled as well by using phase type distributions (Horváth & Telek 2002) (Bobbio, Horváth & Telek 2005) which preserve the Markovian property. The acyclicity property is preserved between blocks of states that represent the same state of the system at different distribution aging. The infinitesimal generator of the Markov chain can be built following the method described in (Angius, Horváth & Urgo 2014); from now-on it will be denoted with Q and the entry in position (k, l) will be referred to as $Q(k, l)$.

Consider an ordered set of batches $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ that partitions $\mathcal{P}$ in such a way that $\bigcup_i^n B_i = \mathcal{P}$ and $k > k'$ if $k \in B_i \wedge k' \in B_{i+1}$, $1 \leq i \leq n-1$. Since every B_i represents a set of jobs that have been scheduled one after the other, $\mathcal{B}$ is conveniently described by means of a set of pairs $\{(i_1, i_2), (i_3, i_4), ..., (i_{2n-1}, i_{2n})\}$ where each pair (i_{2k-1}, i_{2k}) determines the indices of those jobs whose completion delimit the delivery span distribution of batch B_k. For example, given $j_1, \ldots, j_5$, the set $\mathcal{B} = \{(1, 3), (4, 5)\}$ means that there are two batches: the first that considers j_1, j_2 and j_3; the second that considers j_4 and j_5.

A random variable X_k is associated to the time elapsed since the completion of job i_{2k-1} and the completion of job i_{2k}. Consequently, the joint probability density function (pdf) of n batch delivery spans is determined by the random variables $X_1, ..., X_n$ and will be denoted as

$$f(x_1, ..., x_n) = \frac{\partial^n \text{Prob}(X_1 \leq x_1, ..., X_n \leq x_n)}{\partial x_1 \cdots \partial x_n}$$

In order to calculate $f(x_1, ..., x_n)$ we partition the state space of the Markov chain as follows. Set S_0 contains those states of the Markov chains in which job i_1 has not been delivered yet. The set S_i with $1 \leq i \leq 2n$ contains those states in which jobs $1, ..., i_i$ have been already delivered (i.e., they have gone through all station) and jobs $i_i + 1, ..., K$ are at one of the stations. The infinitesimal generator can be partitioned into blocks according to the partitioning of the state space: the entries of matrix Q_{ij} are given as

$$Q_{ij}(k,l) = \begin{cases} Q(k,l) & k \in S_i \wedge l \in S_j \\ 0 & \text{otherwise} \end{cases}$$

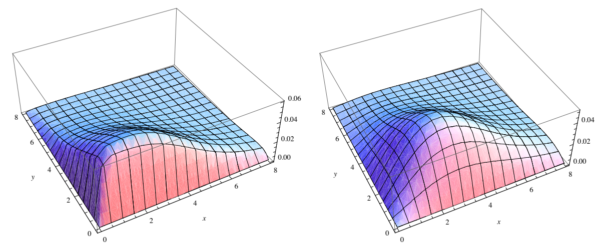
Consequently, Q_{ii} contains the intensities of the transitions that are inside S_i and Q_{ij} contains the intensities of the transitions that move the Markov chain from S_i to S_j. Note that Q_{ij} is a non-zero matrix only if $j = i$ or $j = i + 1$. With the above notation, it can be shown that the joint pdf of $X_1, ..., X_n$ can be written as

$$f(x_1, \ldots, x_n) = \pi_0 (-Q_{00})^{-1} Q_{01} \exp(x_1 Q_{11}) Q_{12} (-Q_{22})^{-1} Q_{23} \cdots \exp(x_n Q_{2n-1,2n-1}) Q_{2n-1,2n} \mathbf{1}$$

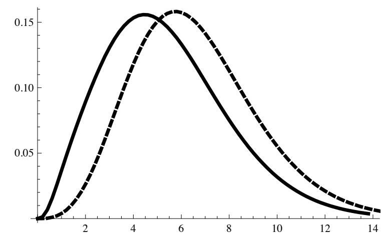
where π_0 denotes the initial probability vector of the Markov chain and $\mathbf{1}$ denotes the column vector with all entries equal to 1. Due to space limitations, we give only a short description of the above formula. Terms $(-Q_{2i,2i})^{-1}$ with $i = 0, ..., n-1$ capture the way the Markov chain goes through the states in S_{2i} without taking into account the elapsed time. Terms $\exp(x_i Q_{2i-1,2i-1})$ with $i = 1, ..., n$ capture the way the Markov chain goes through the states in S_{2i-1} given that the time spent in the states in S_{2i-1} is x_i time units. The multiplication by $\mathbf{1}$ is necessary in order to get a scalar quantity. Given a cost function $c(x_1, ..., _n)$ and joint density $f(x_1, ..., x_n)$ it is straightforward to determine the distribution (or the mean, or the variance) of the cost.

3 Numerical example

In this section we provide a simple numerical example to illustrate the kind of measures that one can obtain by the calculations. We consider a system with four stations and six jobs organized into two batches of three jobs each. Consequently, we calculate the joint pdf of the time elapsed between the delivery of the first and the third job (X_1) and the time elapsed between the delivery of the fourth and the sixth jobs (X_2). We assume exponential service times at each station for each jobs. The service rate at the first three stations is equal to 1 for all jobs. At the last station, jobs j_1 and j_2 have service rate equal to 1, jobs j_3 and j_4 have service rate equal to 2, and jobs j_5 and j_6 have service rate equal to 4. We assume that the cost function is given simply by $X_1 + X_2$. We evaluate two orders: $j_1, j_3, j_5, j_2, j_4, j_6$ (i.e., in both batches, the jobs that are faster on the last server are executed later) and $j_3, j_1, j_5, j_4, j_2, j_6$. The joint pdf of X_1 and X_2 is depicted in Fig. 2(a) for both cases. It can be seen that the order has a strong impact on the joint pdf of X_1 and X_2. In particular, in case of the second order the peak of the pdf is further from the origin. In Fig. 2(b) we depicted the pdf of the cost (i.e., the pdf of $X_1 + X_2$). One can see the cost in case of the second order is probably higher. The mean cost for the first order is 5.33 and it is 6.63 for the second order.



(a) Joint pdf of the two involved times for the two orders: $j_1, j_3, j_5, j_2, j_4, j_6$ on the left and $j_3, j_1, j_5, j_4, j_2, j_6$ on the right. The axes x and y refers to X_1 and X_2 respectively.



(b) Cost pdf: solid line for the first order and dashed line for the second..

4 Conclusive Remarks and Future Works

In this paper we introduced the notion of “batch delivery span” which corresponds to the time that passes from the moment that the first part and the last part of a batch are completed. Then, we provided the mathematical description of the joint distribution of n batch delivery spans by starting from a pre-scheduled set of processes. At the best of our knowledge, this is the first time that this measure has been introduced in the context of stochastic flow shop problem. We deal with a multivariate distribution but our definition is simple enough to permit the computation of not trivial cases. Furthermore, since the problem has been defined on the basis of acyclic block matrices, more sophisticated techniques can be used to speed up the computation; see for example (Ballarini & Horváth 2008).

The work leads to possible future works in the context of scheduling optimization problems. In fact, the search of the scheduling that minimize a cost function based on several batch delivery spans has never been investigated. In this context, the definition of branching schemes and the formulation of lower and upper bounds will be investigated in the near future. As last, strategies to re-use part of the computation of a given scheduling during the computation of another are currently under investigation.

Acknowledgements

This research was supported by the EU projects "ProRegio", Grant 636966.

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