The lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$ generated by a bireversible automaton

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Abstract

We construct a bireversible self-dual automaton with 3 states over an alphabet with 3 letters which generates the lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$.

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1   Introduction

Groups generated by automata (or automaton groups) are interesting from several points of view. First of all, automaton groups provide simple examples of groups with many extraordinary properties: finitely generated infinite torsion groups, groups of intermediate growth, just-infinite groups, groups with non-uniformly exponential growth. At the same time, groups generated by automata arise in various areas of mathematics: in fractal geometry via limit spaces of automaton groups, in complex dynamics via iterated monodromy groups, in graph theory via Schreier graphs of automaton groups, in dynamical systems via limit dynamical systems of automaton groups, in game theory via algebraic models of games (see [1, 8, 12] and the reference therein).

There is an ongoing project to understand which groups can be realized by finite automata. We mention only a few results in this direction relative to the current paper. In [9] Grigorchuk and Żuk showed that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ can be generated by a 2-state automaton over a 2-letter alphabet, which further lead to a negative answer to the strong Atiyah conjecture concerning L^2-Betti numbers [6]. Silva and Steinberg [14] realized the lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ by the so-called reset automata, these automata were further generalized in [3]. Some solvable automaton groups were realized by Bartholdi and Šuniḱ in [2].

There are two standard operations that can be performed on automata: taking dual automaton by interchanging the alphabet with the set of states, and taking inverse automaton by switching input and output letters (in general, the inverse of an automaton may be not well-defined, i.e., it may be not a (deterministic) automaton). By applying these two operations to any automaton one can produce up to eight automata. If all these automata are well-defined, the original automaton is called bireversible. The study of bireversible
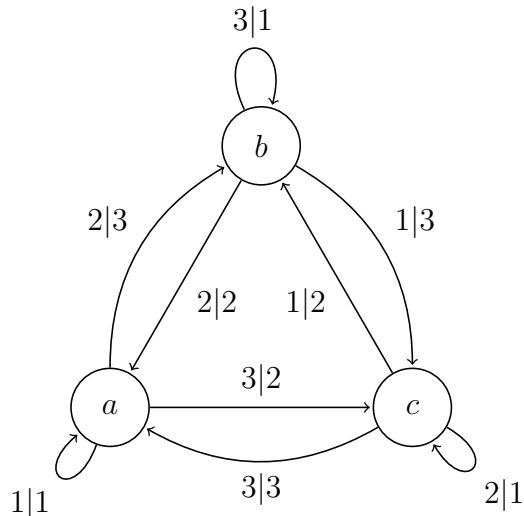
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Figure 1: The automaton A generating the lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$

automata was initiated by Macedońska, Nekrashevych and Sushchansky [10] in connection with the commensurator of a regular (unrooted) tree. New geometric ideas came to the area with the paper [4] of Glasner and Mozes, who associated a square complex to each finite automaton and noticed that an automaton is bireversible if and only if the universal covering of the associated square complex is a topological product of two trees. This approach led to the first automaton realizations of free groups.

It is surprisingly difficult to describe a group generated by a bireversible automaton. For example, there are only two bireversible automata with 3 states over an alphabet with 2 letters generating infinite groups — Aleshin and Bellaterra automata. The Bellaterra automaton generates the free product of three copies of C_2 [12, Theorem 1.10.2]. As concerning Aleshin automaton, it was an open question for a long time whether this automaton generates the free group of rank three, until this was confirmed by Vorobets and Vorobets [16]. Two families of bireversible automata generalizing Aleshin and Bellaterra automata were studied in [17, 15, 13]: the automata in these families generate a free group of finite rank or the free product of copies of C_2. Up to now all investigated bireversible automata generate finitely presented groups, while all the automata generating $(\mathbb{Z}_n)^k \wr \mathbb{Z}$, $k \geq 1$, from [2, 3, 9, 14] are not bireversible as well as all known automata generating infinite torsion groups and groups of intermediate growth.

In this paper we consider the automaton A with 3 states over an alphabet with 3 letters shown in Figure 1. This automaton is bireversible and its dual automaton $\partial(A)$ is equivalent to A: the correspondence $a \mapsto 1$, $b \mapsto 3$, $c \mapsto 2$ converts A to $\partial(A)$. Therefore all eight automata obtained from A by taking dual and inverse automata generate isomorphic groups. Our goal is to prove the following main theorem.

Theorem 1. *The group G_A generated by the automaton A is isomorphic to the lamplighter*

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group $\mathbb{Z}_3 \wr \mathbb{Z}$.

In particular, we get an example of a bireversible automaton generating infinitely presented group. While finishing this article, we were informed by D. Savchuk and S. Sidki that they proved that a certain bireversible automaton with 4 states over an alphabet with 2 letters generates the infinitely presented group $((\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbb{Z}) \rtimes \mathbb{Z}_2$.

2 Preliminaries

Let X be a finite alphabet and X^* the free monoid freely generated by X. The elements of X^* are finite words $v = x_1 x_2 \ldots x_n$, $x_i \in X$, $n \in \mathbb{N}$, together with the empty word denoted $\emptyset$. The operation is concatenation of words. The length of a word $v = x_1 x_2 \ldots x_n$ is $|v| = n$. We will also consider the space $X^{\mathbb{N}} = \{x_1 x_2 \ldots : \ x_i \in X\}$ of all right infinite sequences over X with the product topology of discrete sets X.

An automaton A over the alphabet X is a finite directed labeled graph, whose vertices are called the states of the automaton, and where each edge is labeled by a pair $x|y$ for some letters $x, y \in X$ in such a way that for each vertex $s \in A$ and every letter $x \in X$ there exists exactly one arrow outgoing from s and labeled by $x|y$ for some $y \in X$. Such automata are precisely finite complete deterministic Mealy automata with the same input and output alphabets.

The dual automaton $\partial(A)$ is obtained by interchanging the alphabet with the set of states: the states of $\partial(A)$ are the elements of X and the arrows are given by the rule

$$x \xrightarrow{s|t} y \ \text{ in } \partial(A) \quad \text{if} \quad s \xrightarrow{x|y} t \ \text{ in } A.$$

The dual automaton $\partial(A)$ is always well-defined. The inverse automaton $i(A)$ is obtained by switching labels of arrows: the states of $i(A)$ are formal symbols s^{-1} for $s \in A$ and the arrows are given by

$$s^{-1} \xrightarrow{y|x} t^{-1} \ \text{ in } i(A) \quad \text{if} \quad s \xrightarrow{x|y} t \ \text{ in } A.$$

The $i(A)$ is not always an automaton: there may be several arrows with the same left label outgoing from the same vertex. If $i(A)$ is an automaton then A is called invertible. An automaton A is called bireversible if all eight automata are well-defined:

$$A, \ \partial(A), \ i(A), \ i(\partial(A)), \ \partial(i(A)), \ \partial(i(\partial(A))), \ i(\partial(i(A))), \ i(\partial(i(\partial(A)))) = \partial(i(\partial(i(A)))).$$

It is easy to see that A is bireversible if A, $\partial(A)$ and $\partial(i(A))$ are invertible.

Let us describe how to generate groups by automata. Every state s of an automaton A defines the transformation $s : X^* \to X^*$ as follows. Given a word $v = x_1 x_2 \ldots x_n \in X^*$, there exists a unique directed path in the automaton A starting at the state s and labeled by $x_1|y_1, x_2|y_2, \ldots, x_n|y_n$ for some $y_i \in X$. Then the word $y_1 y_2 \ldots y_n$ is called the image of $x_1 x_2 \ldots x_n$ under s, and the vertex at the end of this path is called the section of s at v denoted $s|_v$. The action of an automaton A on the space X^* naturally extends to the action

on the space $X^{\mathbb{N}}$. All the transformations given by the states of A are invertible if and only if A is invertible; in this case these transformations generate a group under composition of functions called the automaton group G_A generated by the automaton A (we will be using left actions).

A convenient way to work with transformations given by the states of an automaton are wreath recursions. Every invertible automaton with states $\{s_1, \ldots, s_m\}$ over the alphabet $X = \{1, 2, \ldots, d\}$ can be uniquely given by the following system called wreath recursion:

$$
\begin{aligned}
s_1 &= (s_{11}, s_{12}, \ldots, s_{1d})\pi_1, \\
s_2 &= (s_{21}, s_{22}, \ldots, s_{2d})\pi_2, \\
&\vdots \\
s_m &= (s_{m1}, s_{m2}, \ldots, s_{md})\pi_m,
\end{aligned}
\tag{1}
$$

where $s_{ij} = s_i|_j \in \{s_1, \ldots, s_m\}$ is the section of s_i at j, and $\pi_i \in Sym(X)$ is the permutation induced by the action of s_i on X. The tuples (s_{ij}) describe the arrows in automaton while permutations π_i describe the labels of arrows: we have an arrow from the vertex s_i to vertex s_{ij} labeled by $j|\pi_i(j)$. The system (1) defines the action of each state s_i on words over X by the recursive rule:

$$
s_i(xv) = \pi_i(x)s_{ix}(v) \ \text{ for } x \in X, v \in X^* \cup X^{\mathbb{N}} \text{ and } i = 1, \ldots, m.
$$

Similarly, one can use wreath recursions to work with elements of automaton groups. Define the section of a product (or word) $s_1 s_2 \ldots s_n$, $s_i \in A^{\pm 1}$ at $v \in X^*$ by the rule

$$
(s_1 s_2 \ldots s_n)|_v = s_1' s_2' \ldots s_n', \text{ where } s_i' = s_i|_{(s_1 \ldots s_{i-1})(v)}.
$$

Then every element $g \in G_A$ over the alphabet $X = \{1, 2, \ldots, d\}$ can be decomposed as

$$
g = (g|_1, g|_2, \ldots, g|_d)\pi_g,
\tag{2}
$$

where $\pi_g \in Sym(X)$ is the permutation induced by the action of g on X, and $g|_i$ are the sections of g at elements of X. The inverse and multiplication of elements written in this form can be performed by the rules

$$
\begin{aligned}
g^{-1} &= (g|_{\pi_g^{-1}(1)}, g|_{\pi_g^{-1}(2)}, \ldots, g|_{\pi_g^{-1}(d)})\pi_g^{-1}, \\
g \cdot h &= (g|_1 h|_{\pi_g(1)}, g|_2 h|_{\pi_g(2)}, \ldots, g|_d h|_{\pi_g(d)})\pi_g \pi_h.
\end{aligned}
$$

There is a direct connection between sections of words over states of A and the action of the dual automaton $\partial(A)$. Elements of the group generated by $\partial(A)$ act on words over the states of A. If v is a word over alphabet (an element of $G_{\partial(A)}$) and w is a word over states, then the image $v(w)$ is equal to the section $w|_v$.

The terminology of wreath recursions comes from the wreath decomposition of automorphism groups of regular rooted trees. The set X^* can be identified with the vertex set of a rooted tree with empty word as the root and with edges (v, vx) for $x \in X$, $v \in X^*$.

For this reason words over X are usually called vertices. The set X^n of words of lengths n is called the n-th level of the tree X^*. The set $X^{\mathbb{N}}$ can be identified with the boundary of X^*. The transformations defined by invertible automata over X act by automorphisms on the tree X^* and by homeomorphisms on the space $X^{\mathbb{N}}$. The automorphism group $\mathrm{Aut}(X^*)$ can be decomposed as the permutational wreath product $\mathrm{Aut}(X^*) \cong \mathrm{Aut}(X^*) \wr \mathrm{Sym}(X)$, that explains why we have the decomposition (2). The permutation π_g from (2) is called the root permutation of g. Note that any element g can be uniquely given by the collection $(\pi_{g|_v})_{v \in X^*}$ of root permutations of all sections of g.

3   Proof of Theorem 1

Let A be the automaton shown in Figure 1 with the set of states $S = \{a, b, c\}$ over the alphabet $X = \{1, 2, 3\}$, and let G_A be the group generated by A. The automaton A and its dual $\partial(A)$ can be given by the following wreath recursions:

$$
\begin{aligned}
a &= (a, b, c)(2, 3), & 1 &= (1, 3, 2)(b, c), \\
b &= (c, a, b)(1, 3), & 2 &= (3, 2, 1)(a, b), \\
c &= (b, c, a)(1, 2), & 3 &= (2, 1, 3)(a, c).
\end{aligned}
$$

So that

$$
\begin{aligned}
a^{-1} &= (a^{-1}, c^{-1}, b^{-1})(2, 3), & 1^{-1} &= (1^{-1}, 2^{-1}, 3^{-1})(b, c), \\
b^{-1} &= (b^{-1}, a^{-1}, c^{-1})(1, 3), & 2^{-1} &= (2^{-1}, 3^{-1}, 1^{-1})(a, b), \\
c^{-1} &= (c^{-1}, b^{-1}, a^{-1})(1, 2), & 3^{-1} &= (3^{-1}, 1^{-1}, 2^{-1})(a, c).
\end{aligned}
$$

The wreath recursions for $ab^{-1}, bc^{-1}, ca^{-1}$:

$$
\begin{aligned}
ab^{-1} &= (ab^{-1}, bc^{-1}, ca^{-1})(1, 3, 2), \\
bc^{-1} &= (ca^{-1}, ab^{-1}, bc^{-1})(1, 3, 2), \\
ca^{-1} &= (bc^{-1}, ca^{-1}, ab^{-1})(1, 3, 2),
\end{aligned}
$$

imply the relations $ab^{-1} = bc^{-1} = ca^{-1}$. We denote $\alpha = ab^{-1}$. Then α has order 3 and satisfies the wreath recursion

$$\alpha = (\alpha, \alpha, \alpha)(1, 3, 2).$$

Analogously one gets relations

$$\alpha^{-1} = ac^{-1} = ba^{-1} = cb^{-1}, \quad a^{-1}b = b^{-1}c = c^{-1}a, \quad a^{-1}c = b^{-1}a = c^{-1}b.$$

Now it is clear that the group G_A is generated by a and α. Our goal is to prove that G_A has the following presentation:

$$G_A = \langle a, \alpha \mid \alpha^3, [a^{-n}\alpha a^n, a^{-m}\alpha a^m], n, m \in \mathbb{Z} \rangle \cong \mathbb{Z}_3 \wr \mathbb{Z}. \tag{3}$$

Let us define the subgroup W of $\mathrm{Aut}(X^*)$ consisting of elements g such that the root permutations of all sections of g belong to $Alt_3 = \{\varepsilon, (1,2,3), (1,3,2)\}$ and for each $n \in \mathbb{N}$ the root permutations of $g|_v$ at all vertices v of n-th level are equal. In other words, each element g of W can be given by a sequence $(\pi_1, \pi_2, \ldots)$, $\pi_i \in Alt_3$, where π_i is the root permutation of $g|_v$ for all vertices $v \in X^{i-1}$; such element acts on X^* as follows:

$$g(x_1 x_2 \ldots x_n) = \pi_1(x_1)\pi_2(x_2)\ldots\pi_n(x_n)$$

for any $x_i \in X$ and $n \in \mathbb{N}$. It is easy to see that the group W is abelian of exponent 3. Notice that the elements α, α^{-1}, $a^{-1}b$, $a^{-1}c$ belong to W and that every $g \in W$ can be decomposed as $g = (h, h, h)\pi$ for some $h \in W$ and $\pi \in Alt_3$.

Lemma 1. *For any $g \in W$ and $x, y \in \{a, b, c\}$ the elements $x^{-1}gy$ and xgy^{-1} belong to W.*

Proof. For every $g \in W$, we split the elements $x^{-1}gy$ on three types:

$$T_1(g) = \{a^{-1}ga, b^{-1}gb, c^{-1}gc\},$$
$$T_2(g) = \{a^{-1}gb, b^{-1}gc, c^{-1}ga\},$$
$$T_3(g) = \{a^{-1}gc, b^{-1}ga, c^{-1}gb\}.$$

We will prove that for each $n \in \mathbb{N}$ the set of sections of $x^{-1}gy$ at vertices of n-th level is equal to the set $T_i(g_n)$ for some $i \in \{1, 2, 3\}$ depending on n, where g_n is the section of g at some vertex of n-th level (they are all equal), i.e.,

$$\{(x^{-1}gy)|_v : v \in X^n\} = T_i(g_n).$$

For $n = 1$, let us write $g = (h, h, h)\pi$, $h \in W$ and note that $\{(x^{-1}gy)|_v : v \in X\}$ is equal to $T_i(h)$ for some $i \in \{1, 2, 3\}$. Assume inductively that the claim holds for level n. By direct computations shown in Table 1 we see that for every $i \in \{1, 2, 3\}$

$$\{f|_v : f \in T_i(g) \text{ and } v \in X\} = T_j(h)$$

for some $j \in \{1, 2, 3\}$ depending only on i and π. Therefore

$$\{(x^{-1}gy)|_v : v \in X^{n+1}\} = \{f|_v : f \in T_i(g_n) \text{ and } v \in X\} = T_j(g_{n+1}),$$

and the claim is proved.

Notice that for each $i \in \{1, 2, 3\}$ and any $g \in W$ all elements of $T_i(g)$ have the same root permutations which belong to Alt_3. Hence for each $n \in \mathbb{N}$ the root permutations of $(x^{-1}gy)|_v$ at the vertices v of n-th level are all equal. Therefore $x^{-1}gy \in W$ for any $x, y \in \{a, b, c\}$.

Analogously one can show that $xgy^{-1} \in W$ for any $x, y \in \{a, b, c\}$. $\qquad\square$

Remark 1. It follows from the lemma that actually all elements in each $T_i(g)$ are equal, because they have the same set of permutation on each level:

$$a^{-1}ga = b^{-1}gb = c^{-1}gc, \qquad a^{-1}gb = b^{-1}gc = c^{-1}ga, \qquad a^{-1}gc = b^{-1}ga = c^{-1}gb.$$

Therefore T_1, T_2, T_3 can be considered as transformations of W. Since every element in W can be uniquely represented by a sequence of permutations $(\pi_1, \pi_2, \ldots)$, $\pi_i \in Alt_3$, i.e., an element from $Alt_3^{\mathbb{N}}$, the T_1, T_2, T_3 can be viewed as transformations of $Alt_3^{\mathbb{N}}$. Table 1 shows that these transformations satisfy the wreath recursion

$$T_1 = (T_1, T_2, T_3)(\tau_2, \tau_3)$$
$$T_2 = (T_3, T_1, T_2)(\tau_1, \tau_3),$$
$$T_3 = (T_2, T_3, T_1)(\tau_1, \tau_3)$$

where $\tau_1 = \varepsilon, \tau_2 = (1,2,3), \tau_3 = (1,3,2)$. Interestingly, this recursion repeats the wreath recursion for the automaton A if we identify τ_i with i.

We can apply this observation to elements of G_A of the form $u^{-1}v$ for $u, v \in \{a, b, c\}^n$. In this case $u^{-1}v$ corresponds to a word w in the alphabet $\{T_1, T_2, T_3\}$ of length n, because we can write $u^{-1}v$ as the progressive composition of some conjugations of the form $x^{-1}y$. For example:

$$a^{-1}b^{-1}b^{-1}c^{-1}abaa = T_2 T_1 T_3 T_1(e).$$

The trivial element $e \in W$ is represented by the sequence $\tau_1^{\infty} = \tau_1 \tau_1 \ldots$. Therefore the sequence of permutations representing $u^{-1}v$ is equal to the image $w(\tau_1^{\infty})$.

For a word w over $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ we denote by $ord(w)$ the sum of exponents of letters in w. For example, $ord(a^{-1}b^3 cb^{-2}) = -1 + 3 + 1 - 2 = 1$.

Lemma 2. *If an element $g \in G_A$ can be represented by a word over $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ with $ord(w) = 0$, then $g \in W$. In particular, $g^3 = e$ and such elements commute with each other.*

Proof. We prove the lemma by induction on the length of g. The statement holds for elements $x^{-1}y$ and xy^{-1} for all $x, y \in \{a, b, c\}$. If we assume that the statement holds for elements g of length $\leq 2n$, then it holds for elements $x^{-1}gy$ and xgy^{-1} by Lemma 1. So let us prove the statement for words starting and ending with either both letters in X or both letters in X^{-1}. These words are either of type xvy with $ord(v) = -2$ or of type $x^{-1}vy^{-1}$ with $ord(v) = 2$. It is easy to show by induction, that if $|w| = 2(n+1)$, $w = xvy$ with $ord(v) = -2$ (resp. $w = x^{-1}vy^{-1}$ with $ord(v) = 2$) then w is a concatenation of words of the form xuy^{-1} and $x^{-1}uy$ (resp. $x^{-1}uy$ and xuy^{-1}) for some u with $ord(u) = 0$ and length $\leq 2n$. The statement follows. $\qquad \square$

Notice that all elements $a^{-n}\alpha a^n$ from presentation (3) satisfy the condition of Lemma 2.

Corollary 1.1. *The relations $[a^{-n}\alpha a^n, a^{-m}\alpha a^m] = e$ hold in the group G_A.*

	$g = (h,h,h)\pi$	
$\pi = \varepsilon$	Type I :	$\begin{aligned} a^{-1}ga &= (a^{-1}ha, c^{-1}hc, b^{-1}hb) \\ b^{-1}gb &= (b^{-1}hb, a^{-1}ha, c^{-1}hc) \\ c^{-1}gc &= (c^{-1}hc, b^{-1}hb, a^{-1}ha) \end{aligned}$
	Type II :	$\begin{aligned} a^{-1}gb &= (a^{-1}hc, c^{-1}hb, b^{-1}ha)(1,3,2) \\ b^{-1}gc &= (b^{-1}ha, a^{-1}hc, c^{-1}hb)(1,3,2) \\ c^{-1}ga &= (c^{-1}hb, b^{-1}ha, a^{-1}hc)(1,3,2) \end{aligned}$
	Type III :	$\begin{aligned} a^{-1}gc &= (a^{-1}hb, c^{-1}ha, b^{-1}hc)(1,2,3) \\ b^{-1}ga &= (b^{-1}hc, a^{-1}hb, c^{-1}ha)(1,2,3) \\ c^{-1}gb &= (c^{-1}ha, b^{-1}hc, a^{-1}hb)(1,2,3) \end{aligned}$
$\pi = (1,2,3)$	Type I :	$\begin{aligned} a^{-1}ga &= (a^{-1}hb, c^{-1}ha, b^{-1}hc)(1,3,2) \\ b^{-1}gb &= (b^{-1}hc, a^{-1}hb, c^{-1}ha)(1,3,2) \\ c^{-1}gc &= (c^{-1}ha, b^{-1}hc, a^{-1}hb)(1,3,2) \end{aligned}$
	Type II :	$\begin{aligned} a^{-1}gb &= (a^{-1}ha, c^{-1}hc, b^{-1}hb)(1,2,3) \\ b^{-1}gc &= (b^{-1}hb, a^{-1}ha, c^{-1}hc)(1,2,3) \\ c^{-1}ga &= (c^{-1}hc, b^{-1}hb, a^{-1}ha)(1,2,3) \end{aligned}$
	Type III :	$\begin{aligned} a^{-1}gc &= (a^{-1}hc, c^{-1}hb, b^{-1}ha) \\ b^{-1}ga &= (b^{-1}ha, a^{-1}hc, c^{-1}hb) \\ c^{-1}gb &= (c^{-1}hb, b^{-1}ha, a^{-1}hc) \end{aligned}$
$\pi = (1,3,2)$	Type I :	$\begin{aligned} a^{-1}ga &= (a^{-1}hc, c^{-1}hb, b^{-1}ha)(1,2,3) \\ b^{-1}gb &= (b^{-1}ha, a^{-1}hc, c^{-1}hb)(1,2,3) \\ c^{-1}gc &= (c^{-1}hb, b^{-1}ha, a^{-1}hc)(1,2,3) \end{aligned}$
	Type II :	$\begin{aligned} a^{-1}gb &= (a^{-1}hb, c^{-1}ha, b^{-1}hc) \\ b^{-1}gc &= (b^{-1}hc, a^{-1}hb, c^{-1}ha) \\ c^{-1}ga &= (c^{-1}ha, b^{-1}hc, a^{-1}hb) \end{aligned}$
	Type III :	$\begin{aligned} a^{-1}gc &= (a^{-1}ha, c^{-1}hc, b^{-1}hb)(1,3,2) \\ b^{-1}ga &= (b^{-1}hb, a^{-1}ha, c^{-1}hc)(1,3,2) \\ c^{-1}gb &= (c^{-1}hc, b^{-1}hb, a^{-1}ha)(1,3,2) \end{aligned}$

Table 1: Decomposition of $x^{-1}gy$ for $x, y \in \{a, b, c\}$ and $g \in W$

Lemma 3. *The group G_A acts transitively on X^n for every $n \in \mathbb{N}$.*

Proof. We will prove that the stabilizer $St_{G_A}(1^n)$ of the vertex 1^n acts transitively on $1^n X$ for each $n \in \mathbb{N}$. The statement immediately follows from this claim. Indeed, let $u, v \in X^n$

and we want to find an element $g \in G_A$ such that $g(u) = v$. We proceed by induction on the length $|u| = |v| = n$. If $n = 1$ the claim follows from the transitivity of G_A on X. Let $|u| = |v| = n + 1 > 1$ with $u = u'x$ and $v = v'y$, where $u', v' \in X^n$ and $x, y \in X$. By induction there exist $h, k \in G_A$ such that $h(u') = k(v') = 1^n$. Let $x' = h|_{u'}(x)$ and $y' = k|_{v'}(y)$. Since $St_{G_A}(1^n)$ is transitive on $1^n X$, there is $s \in St_{G_A}(1^n)$ such that $s(1^n x') = 1^n y'$. Then if we put $g = hsk^{-1}$, we have $g(u) = v$.

Let us prove the claim. Notice that a^3, b^3, c^3 preserve the vertex set $X11 = \{x_1 = 111, y_1 = 211, z_1 = 311\}$, and the action restricted to $\{x_1, y_1, z_1\}$ can be described by the wreath recursion

$$a^3 = (a^3, b^3, c^3)(y_1, z_1),$$
$$b^3 = (c^3, a^3, b^3)(x_1, z_1),$$
$$c^3 = (b^3, c^3, a^3)(x_1, y_1).$$

which repeat the wreath recursion for the automaton A. It immediately follows by induction that if we denote

$$x_k = 1^{3^k}, \ y_k = 21^{3^{k-1}+2}, \ z_k = 31^{3^{k-1}+2} \quad \text{and} \quad a_k = a^{3^k}, b_k = b^{3^k}, c_k = c^{3^k},$$

then a_k, b_k, c_k preserve the vertex set $\{x_k, y_k, z_k\}$ with the recursion

$$a_k = (a_k, b_k, c_k)(y_k, z_k),$$
$$b_k = (c_k, a_k, b_k)(x_k, z_k),$$
$$c_k = (b_k, c_k, a_k)(x_k, y_k).$$

It follows that
$$b_k^2(x_k) = x_k \quad \text{and} \quad b_k^2|_{x_k} = c_k b_k = c^{3^k} b^{3^k}.$$

The element $c^{3^k} b^{3^k}$ acts as permutation $(1, 2, 3)$ on X. Therefore b_k^2 stabilizes x_k and acts transitively on the set $x_k X$. This means that the stabilizer of 1^n acts transitively on $1^n X$ for each $n = 3^k$, $k \in \mathbb{N}$. By taking sections at vertices 1^i we get that $St_{G_A}(1^{3^k - i})$ is transitive on $1^{3^k - i} X$ which implies our claim for each $n \in \mathbb{N}$. $\square$

Corollary 1.2. *Any word of length n over $\{a, b, c\}$ is a section of any other word of length n over $\{a, b, c\}$.*

Proof. Since the automaton A is equivalent to its dual automaton $\partial(A)$, the group generated by $\partial(A)$ acts transitively on $\{a, b, c\}^n$ for each $n \in \mathbb{N}$. Therefore the semigroup generated by X acts transitively on $\{a, b, c\}^n$. This means that for any $w, w' \in \{a, b, c\}^n$ there exists $v \in X^*$ such that $w|_v = w'$. $\square$

Corollary 1.3. *The semigroup generated by a, b, c is free.*

Proof. Every non-empty word w over $\{a, b, c\}$ is a non-trivial element, because w has a non-trivial section $a^n b$, $n = |w| - 1$.

Assume by contradiction that $w =_{G_A} v$ for a word v over $\{a, b, c\}$ different from w. If $|v| > |w|$ then $v_1^{-1}w =_G v_2$, where $v = v_1v_2$ with $|v_1| = |w|$ and non-empty v_2. Since $ord(v_1^{-1}w) = 0$ we get $v_2^3 = e$ by Lemma 2 contradicting our first statement.

We may assume that $|v| = |w|$, and let us take a shortest such relation $v =_{G_A} w$. By Corollary 1.2 there is $s \in X^*$ such that $v|_s = a^n$. Let $u := w|_s$. Since A is bireversible, $a^n \neq u$. Furthermore $a^n =_{G_A} u$ is a relation where the first letter of u is b or c, because of minimality. Taking the section at 11 of this relation, we get $a^n = a^n|_{11} =_{G_A} u|_{11} =_{G_A} u$. The first letters of u and $u|_{11}$ are equal since $b|_{11} = b$ and $c|_{11} = c$. After canceling these letters we get a shorter relation. Therefore u and $u|_{11}$ are equal as words. However, this is impossible already for the first two letters of w as the following computations show:

$$ba|_{11} = bc, \quad bb|_{11} = ba, \quad bc|_{11} = bb,$$
$$ca|_{11} = cb, \quad cb|_{11} = cc, \quad cc|_{11} = ba.$$

$\square$

We are ready to prove Theorem 1.

Proof of Theorem 1. Let N be the subgroup of G_A generated by elements $a^{-n}\alpha a^n$, $n \in \mathbb{Z}$. Then N is a normal abelian subgroup of exponent 3 by Lemma 2. Let us show that $N = \bigoplus_{\mathbb{Z}}\langle a^{-n}\alpha a^n\rangle \cong \bigoplus_{\mathbb{Z}}\mathbb{Z}_3$. Suppose there is a relation

$$(a^{-n_1}\alpha^{\varepsilon_1}a^{n_1})(a^{-n_2}\alpha^{\varepsilon_2}a^{n_2})\ldots(a^{-n_k}\alpha^{\varepsilon_k}a^{n_k}) = e,$$

where $\varepsilon_i \in \{\pm 1\}$ and $n_1 < n_2 < \ldots < n_k$. Substituting $\alpha = ab^{-1}$ and $\alpha^{-1} = ac^{-1}$ in the previous expression, and making free cancelations we get a relation of the form $a^{-n_1+1}w^{-1}a^{n_k} = e$, where w is a word over $\{a, b, c\}$ with at least one occurrence of b or c. We get a contradiction with Corollary 1.3, which proves our claim.

By Corollary 1.3 the element a has infinite order and $N \cap \langle a\rangle = \{e\}$. Since $G_A = N\langle a\rangle$ and a acts on N by conjugation via the shift, we get the statement of the theorem. $\square$

Remark 2. In the automaton realizations of $\mathbb{Z}_m \wr \mathbb{Z}$ from [9, 14] elements of the subgroup $\bigoplus_{\mathbb{Z}}\mathbb{Z}_m$ are finitary transformations. An element g is called finitary whenever there exists $n \in \mathbb{N}$ such that $g(uv) = g(u)v$ for all $u \in X^n$ and $v \in X^*$, or equivalently, the sections of g at all vertices of n-th level are trivial. In our case, non-trivial elements of the subgroup $N = \bigoplus_{\mathbb{Z}}\mathbb{Z}_3$ are not finitary.

The stabilizers $St_G(w)$ of points $w \in X^{\mathbb{N}}$ are known as parabolic subgroups of G. For every bireversible automaton, almost every point of $X^{\mathbb{N}}$ with respect to the uniform measure on $X^{\mathbb{N}}$ has trivial stabilizer (see [15]). For our group G_A there are points with a non-trivial stabilizer, for example $a \in St_{G_A}(11\ldots)$. Moreover, using the results obtained in [5] it is not difficult to prove that the stabilizer $St_{G_A}(w)$ is non-trivial exactly when the sequence w is eventually periodic. We omit the proof because it uses quite different technique. Instead we just prove the analog of Proposition 4.6 from [14].

Proposition 2. *The stabilizer $St_{G_A}(w)$ of every point $w \in X^{\mathbb{N}}$ is a cyclic group.*

Proof. Note that a non-trivial element $g \in W \supset N$ has no fixed points in $X^{\mathbb{N}}$, since g corresponds to a non-trivial sequence of permutations $(\pi_1, \pi_2, \ldots) \neq (\varepsilon, \varepsilon, \ldots)$, $\pi_i \in Alt_3$ and $g(w) = g(x_1 x_2 \ldots) = \pi_1(x_1)\pi_2(x_2)\ldots$.

Let $g = na^k$, where $n \in N$ and $k \in \mathbb{N}$, be an element of $St_{G_A}(w)$ with the smallest k. We prove that g is a generator of $St_{G_A}(w)$. Take any $h \in St_{G_A}(w)$, $h = n'a^m$, where $n' \in N$ and $m \in \mathbb{Z}$. Then m is a multiple of k; so let $m = kl$ with $l \in \mathbb{Z}$. We have $g^l = (na^k)^l = \overline{n}a^{kl}$ for an opportune $\overline{n} \in N$. Then $hg^{-l} = n'\overline{n}^{-1} \in N \cap St_{G_A}(w) = \{e\}$. Hence $h = g^l$. $\qquad\square$

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