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A Remark on Nonclassical Diffusion Equations with Memory

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Abstract The nonclassical diffusion equation with hereditary memory

$$u_t - \Delta u_t - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)\,\mathrm{d}s + \varphi(u) = f$$

on a 3D bounded domain is considered, for a very general class of memory kernels κ. Setting the problem both in the classical past history framework and in the more recent minimal state one, the related solution semigroups are shown to possess finite-dimensional regular exponential attractors.

1 Introduction

1.1 The Equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Introducing the strictly positive Dirichlet operator $A = -\Delta$ with domain

$$\mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega),$$

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we consider for $t > 0$ the equation in the unknown variable $u = u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$

$$u_t + Au_t + Au + \int_0^\infty \kappa(s) Au(t-s)\,\mathrm{d}s + \varphi(u) = f. \tag{1.1}$$

The problem is supplemented with the initial conditions

$$u(0) = u_0 \text{ and } u(-s)_{|s>0} = g_0(s), \tag{1.2}$$

where $u_0 : \Omega \to \mathbb{R}$ and $g_0 : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are given functions.
Equation (1.1) constitutes the memory relaxation of the nonclassical diffusion equation

$$u_t - \Delta u_t - \Delta u + \varphi(u) = f,$$

subject to Dirichlet boundary conditions, arising in the classical diffusion theory under the constitutive assumption that the material u behaves as a linearly viscous fluid [1]. For certain classes of materials such as polymer and high-viscosity liquids, the diffusive process is nontrivially influenced by the past history of u, which is represented in (1.1) by the convolution term against a suitable memory kernel characterizing the diffusive specie, see e.g. [17].

1.2 Assumptions on the Constitutive Terms

The external force $f \in L^2(\Omega)$ is independent of time. The nonlinearity $\varphi \in \mathcal{C}^1(\mathbb{R})$ fulfills $\varphi(0) = 0$, along with the critical growth restriction

$$\left|\varphi'(u)\right| \le c\big(1 + |u|^4\big) \tag{1.3}$$

and the dissipation condition

$$\liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda_1 \tag{1.4}$$

where $\lambda_1 > 0$ is the first eigenvalue of A.

The memory kernel κ is a nonnegative summable function of total mass $\int_0^\infty \kappa(s)\,\mathrm{d}s = 1$ having the explicit form

$$\kappa(s) = \int_s^\infty \mu(y)\,\mathrm{d}y,$$

where $\mu \in L^1(\mathbb{R}^+)$ is a nonincreasing (hence nonnegative) piecewise absolutely continuous function allowed to exhibit (infinitely many) jumps. Moreover, we assume that

$$\kappa(s) \le \Theta\mu(s), \tag{1.5}$$

for every $s > 0$ and some $\Theta > 0$, or equivalently (see [4]), the existence of $M \geq 1$ and $\delta > 0$ such that for every $s \geq 0$ and almost every $\tau > 0$.

$$\mu(s+\tau) \leq M e^{-\delta s} \mu(\tau). \tag{1.6}$$

The well-posedness and the long-term behavior of the solutions to (1.1) have been studied in [22,23] (see also [20] and the references therein for the model without memory) for a smaller class of smooth memory kernels with μ satisfying

$$\mu'(s) + \delta\mu(s) \leq 0. \tag{1.7}$$

Such a condition has been weakened in the recent work [4], where, assuming (1.5), the authors prove the existence of the global attractor of optimal regularity for the semigroup associated to (1.1) in the past history framework of Dafermos [9].

Remark 1.1 It should be stressed that (1.6) with $M = 1$ boils down to (1.7). On the contrary, when (1.6) holds true for some $M > 1$, then it is far more general that (1.7) (see e.g. [4]). For instance, any compactly supported decreasing μ with flat zones fulfills (1.6) for some $M > 1$, but it cannot comply with (1.7).

Let us recall that the global attractor is a suitable object to describe the asymptotic behavior of a dynamical system $S(t)$ acting on a (Banach) phase-space $\mathcal{H}$, since it is the unique compact set $\mathcal{A} \subset \mathcal{H}$ which is at the same time fully invariant and attracting for the semigroup (see e.g. [2,16,21]). Nonetheless, the global attractor is not a completely satisfactory object, due to the fact that the rate of attraction can be arbitrarily small; besides, the global attractor may not be stable under small perturbations of the model, see [12,18].

The aim of this paper is to investigate the existence of *exponential attractors* for the semigroup associated to (1.1), larger objects than the global attractor, able to attract trajectories exponentially fast, and which are usually more stable than global attractors. Precisely, by definition (see [11,18]), an exponential attractor for a semigroup $S(t)$ acting on $\mathcal{H}$ is a compact set $\mathcal{E} \subset \mathcal{H}$ satisfying the following properties:

- $\mathcal{E}$ is positively invariant for the semigroup, namely, $S(t)\mathcal{E} \subset \mathcal{E}$ for all $t \geq 0$.
- $\mathcal{E}$ has finite fractal dimension.[1]
- $\mathcal{E}$ is exponentially attracting for $S(t)$, i.e. there exist an exponential rate $\omega > 0$ and a nondecreasing positive function Q such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{E}) \leq Q(\|\mathcal{B}\|_{\mathcal{H}}) e^{-\omega t}$$

[1] Denoting by $N(r)$ the smallest number of r-balls of $\mathcal{H}$ necessary to cover $\mathcal{E}$, the fractal dimension of $\mathcal{E}$ in $\mathcal{H}$ is defined as

$$\limsup_{r\to 0} \frac{\ln N(r)}{\ln \frac{1}{r}}.$$

for every bounded subset $\mathcal{B} \subset \mathcal{H}$, where

$$\mathrm{dist}_{\mathcal{H}}(\mathcal{B}_1, \mathcal{B}_2) = \sup_{b_1 \in \mathcal{B}_1} \inf_{b_2 \in \mathcal{B}_2} \|b_1 - b_2\|_{\mathcal{H}}$$

denotes the standard Hausdorff semidistance between two subsets.

Proving the existence of exponential attractors for equations with memory may be an hard task, mainly due to the presence of the auxiliary past-history component and to the corresponding structural lack of compactness in the past history framework. This yields to technical difficulties and annoying additional work in the application of the classical techniques, as it is particularly evident in [19], where the case of nonclassical diffusion is discussed for smooth kernels satisfying (1.7) (see also [3] for a related problem).

In this note, we take advantage of the recent studies in [10], where the authors provide a "user friendly" theoretical tool, specifically tailored for equations with memory, establishing sufficient conditions for the existence of a regular exponential attractor. As a result, we obtain a straight and elegant proof of the existence of a regular exponential attractor for the semigroups associated with (1.1), both in the classical past history framework and in the more general minimal state one, recently devised in [14].

1.2.1 Plan of the Paper

In the next section we introduce the proper mathematical setting. In Sect. 3 the original integrodifferential problem (1.1) is translated into a dynamical system within the past history framework, where it is shown to generate a dissipative and Lipschitz continuous semigroup. The main result concerning the existence of a regular exponential attractor is contained in Theorem 3.3, whose proof is carried out in the subsequent Sect. 4 by exploiting some known facts from [4]. The main theoretical tool for the proof is Proposition 3.5, where the abstract hypotheses in [10] are formulated in concrete terms, suitable to handle our model. In the final Sect. 5 we settle the original problem in the so-called minimal state framework, instead of the past history one, and we discuss the validity of corresponding results for the semigroup generated by (1.1) in the new setting.

2 Functional Setting and Notation

We denote by $L^p(\Omega)$ ($p \geq 1$) the usual L^p-space on Ω, with norm $\|\cdot\|_{L^p}$. When $p = 2$, we simplify the notation in $\|\cdot\|$. For $\sigma \in \mathbb{R}$, we define the scale of compactly nested Hilbert spaces

$$\mathrm{H}^\sigma = \mathrm{dom}\big(A^{\sigma/2}\big),$$

with inner products and norms given by

$$\langle u, v\rangle_\sigma = \big\langle A^{\frac{\sigma}{2}} u, A^{\frac{\sigma}{2}} v\big\rangle \text{ and } \|u\|_\sigma = \big\|A^{\frac{\sigma}{2}} u\big\|.$$

We will always omit the index σ whenever $\sigma = 0$. The symbol $\langle \cdot, \cdot \rangle$ besides denoting the usual scalar product in $L^2(\Omega)$, will also stand for the duality product between H^σ and its dual space $\mathrm{H}^{-\sigma}$. We recall the relations

$$\mathrm{H}^{-1} = H^{-1}(\Omega), \quad \mathrm{H} = L^2(\Omega), \quad \mathrm{H}^1 = H_0^1(\Omega), \quad \mathrm{H}^2 = H^2(\Omega) \cap H_0^1(\Omega),$$

and the generalized Poincaré inequalities

$$\sqrt{\lambda_1}\, \|u\|_\sigma \leq \|u\|_{\sigma+1}, \quad \forall u \in \mathrm{H}^{\sigma+1}.$$

For further use, we also remark the existence of $\ell > 0$ such that

$$\left\| (I + A)^{-1} u \right\|_1 \leq \ell \|u\|_{-1}, \quad \forall u \in \mathrm{H}^{-1}. \tag{2.1}$$

2.1 History Spaces

For $\sigma \in \mathbb{R}$ we introduce the *history spaces* (σ is omitted if zero)

$$\hat{\mathcal{M}}^\sigma = L^2_\mu\left(\mathbb{R}^+; \mathrm{H}^{\sigma+1}\right)$$

endowed with the weighted L^2-inner products

$$\langle \eta_1, \eta_2 \rangle_{\hat{\mathcal{M}}^\sigma} = \int_0^\infty \mu(s) \big\langle \eta_1(s), \eta_2(s) \big\rangle_{\sigma+1} \, \mathrm{d}s,$$

and the *extended history spaces*

$$\hat{\mathcal{H}}^\sigma = \mathrm{H}^{\sigma+1} \times \hat{\mathcal{M}}^\sigma.$$

We also consider the infinitesimal generator of the right-translation semigroup on $\hat{\mathcal{M}}$, namely the linear operator

$$T\eta = -\eta' \qquad \text{with domain} \qquad \mathrm{dom}(T) = \left\{ \eta \in \hat{\mathcal{M}} : \eta' \in \hat{\mathcal{M}},\ \eta(0) = 0 \right\},$$

the *prime* standing for weak derivative. The following inequality holds, see [15],

$$\langle T\eta, \eta \rangle_{\hat{\mathcal{M}}} \leq 0, \quad \forall \eta \in \mathrm{dom}(T). \tag{2.2}$$

2.2 Notation

Throughout the paper, given a Banach space $\mathcal{X}$, we denote by $\mathbb{B}_{\mathcal{X}}(r)$ the closed ball in $\mathcal{X}$ of radius $r \geq 0$ centered at zero. We will denote by $c \geq 0$ a generic constant, by $\mathfrak{I}$ the set of nondecreasing functions $Q : [0, +\infty) \to [0, +\infty)$ and by $\mathfrak{D}$ the set of nonincreasing functions $q : [0, +\infty) \to [0, +\infty)$ vanishing at infinity. We will use diffusely the Sobolev embeddings, as well as the Young, the Hölder and the Poincaré inequalities.

3 The Dissipative Semigroup

Following the scheme of Dafermos [9] (see also [5]), the original problem (1.1), (1.2) can be translated into the system in the unknown variables $u = u(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} u_t + A\big[u_t + u + \int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s\big] + \varphi(u) = f, \\ \eta_t(s) = T\eta(s) + u, \end{cases} \tag{3.1}$$

with initial conditions

$$u(0) = u_0 \text{ and } \eta^0(s) = \int_0^s g_0(y)\,\mathrm{d}y.$$

This is obtained by introducing the *integrated past history* η of the variable u, formally defined as

$$\eta^t(s) = \int_0^s u(t-y)\,\mathrm{d}y, \qquad t \geq 0 \text{ and } s > 0,$$

The system has been studied in [4,23] where it is shown to generate a dissipative semigroup on $\hat{\mathcal{H}}$. Precisely, the following result holds.

Theorem 3.1 *System (3.1) generates a strongly continuous semigroup $\hat{S}(t) : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$. Thus, for any $t \geq 0$ and any $z = (u_0, \eta_0) \in \hat{\mathcal{H}}$,*

$$\hat{S}(t)z = \big(u(t), \eta^t\big)$$

is the unique solution at time t to (3.1) with initial datum z. Besides,

$$\|\hat{S}(t)z\|_{\hat{\mathcal{H}}} \leq c_0\big(\|z\|_{\hat{\mathcal{H}}}\mathrm{e}^{-\varepsilon_0 t} + 1\big), \qquad \forall t \geq 0, \tag{3.2}$$

for some positive ε_0 and c_0. In addition,

$$u_t \in L^2_{\mathrm{loc}}\big(0, \infty; \mathrm{H}^1\big),$$

while η fulfills the explicit representation formula

$$\eta^t(s) = \begin{cases} \int_0^s u(t-y)\mathrm{d}y & s \leq t, \\ \eta_0(s-t) + \int_0^t u(t-y)\mathrm{d}y & s > t. \end{cases}$$

with $\eta^0 = \eta_0$.

Actually, as observed in [4], the semigroup $\hat{S}(t)$ maps $\hat{\mathcal{H}}^\sigma$ into $\hat{\mathcal{H}}^\sigma$, for every $\sigma \in [0,1]$. It is worth noticing that, by (3.2), the ball

$$\mathcal{B}_0 = \mathbb{B}_{\hat{\mathcal{H}}}(2c_0)$$

is a bounded *absorbing set* able to capture all trajectories of $\hat{S}(t)$ originating from any given bounded set of initial data in finite time. Namely, for every $r > 0$ there exists an entering time $t_r \geq 0$ such that

$$\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}}(r) \subset \mathcal{B}_0, \qquad \text{for every } t \geq t_r.$$

We also have the following continuous dependence estimate.

Proposition 3.2 *For every $r > 0$ and every $z_1, z_2 \in \mathbb{B}_{\hat{\mathcal{H}}}(r)$,*

$$\|\hat{S}(t)z_1 - \hat{S}(t)z_2\|_{\hat{\mathcal{H}}} \leq c\mathrm{e}^{Q(r)t}\|z_1 - z_2\|_{\hat{\mathcal{H}}}.$$

Proof The difference

$$\left(\bar{u}(t), \bar{\eta}^t\right) = \hat{S}(t)z_1 - \hat{S}(t)z_2$$

fulfills the problem

$$\begin{cases} \bar{u}_t + A\Big[\bar{u}_t + \bar{u} + \displaystyle\int_0^\infty \mu(s)\bar{\eta}(s)\,\mathrm{d}s\Big] + \varphi(u_1) - \varphi(u_2) = 0, \\ \bar{\eta}_t = T\bar{\eta} + \bar{u}, \\ (\bar{u}(0), \bar{\eta}^0) = z_1 - z_2, \end{cases}$$

where $u_\imath(t)$ is the first component of $\hat{S}(t)z_\imath$. Setting

$$Z(t) = \frac{1}{2}\Big[\|\bar{u}(t)\|_1^2 + \|\bar{u}(t)\|^2 + \|\bar{\eta}^t\|_{\hat{\mathcal{M}}}^2\Big]$$

and multiplying within a regularization scheme the first equation by $\bar{u}$ in H and the second one by $\bar{\eta}$ in $\hat{\mathcal{M}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}Z + \langle\varphi(u_1) - \varphi(u_2), \bar{u}\rangle = -\|\bar{u}\|_1^2 + \langle T\bar{\eta}, \bar{\eta}\rangle_{\hat{\mathcal{M}}} \leq 0,$$

having used (2.2) in the latter inequality. Besides, (1.3) and (3.2) implies that

$$-\langle\varphi(u_1) - \varphi(u_2), \bar{u}\rangle \leq c\big(1 + \|u_1\|_1^4 + \|u_2\|_1^4\big)\|\bar{u}\|_1^2 \leq Q(r)\|\bar{u}\|_1^2.$$

Summarizing, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}Z \leq Q(r)Z.$$

Since the functional Z is equivalent to the norm of $(\bar{u}, \bar{\eta})$ in $\hat{\mathcal{H}}$, the sought inequality follows by applying the Gronwall lemma. □

3.1 The Main Result

We are now in the position to state the main result of this paper.

Theorem 3.3 *The semigroup $\hat{S}(t)$ on $\hat{\mathcal{H}}$ generated by (3.1) possesses an exponential attractor $\hat{\mathcal{E}}$ bounded in $\hat{\mathcal{H}}^1$.*

By standard arguments (cf. [2,16,18,21]), the existence of an exponential attractor implies the existence of the *global attractor*. Since by definition the global attractor is fully invariant, it is contained in every closed attracting set, yielding the next corollary.

Corollary 3.4 *The semigroup $\hat{S}(t)$ possesses the global attractor $\hat{\mathcal{A}}$. Moreover,*

$$\hat{\mathcal{A}} \subset \hat{\mathcal{E}}.$$

In particular, $\hat{\mathcal{A}}$ has finite fractal dimension in $\hat{\mathcal{H}}$.

The proof of Theorem 3.3 will be obtained by direct application of the next proposition, which is a concrete reformulation of the abstract result in [10].

Proposition 3.5 *Assume that the following conditions hold:*

I There is $r_1 > 0$ such that the ball $\mathbb{B}_{\hat{\mathcal{H}}^1}(r_1)$ is exponentially attracting for $\hat{S}(t)$.

II There exists $R_1 > 0$ such that, given any $\rho > 0$,

$$\|\hat{S}(t)z\|_{\hat{\mathcal{H}}^1} \leq q_\rho(t) + R_1$$

for some $q_\rho \in \mathfrak{D}$ and every $z \in \mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$.

III There exists $Q \in \mathfrak{I}$ such that, for any $\rho > 0$ and any $\theta \geq 0$,

$$\|\partial_t u\|_{L^2(\theta,2\theta;\mathrm{H}^1)} \leq Q(\rho + \theta)$$

for all u such that $\hat{S}(t)z = (u(t), \eta^t)$ with $z \in \mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$.

IV For every $\rho > 0$ and for all $z_1, z_2 \in \mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$, we have

$$S(t)z_1 - S(t)z_2 = L(t, z_1, z_2) + K(t, z_1, z_2),$$

where the maps L and K satisfy

$$\begin{aligned} \|L(t, z_1, z_2)\|_{\hat{\mathcal{H}}} &\leq q_\rho(t)\|z_1 - z_2\|_{\hat{\mathcal{H}}}, \\ \|K(t, z_1, z_2)\|_{\hat{\mathcal{H}}^1} &\leq Q_\rho(t)\|z_1 - z_2\|_{\hat{\mathcal{H}}}, \end{aligned}$$

for some $q_\rho \in \mathfrak{D}$ and $Q_\rho \in \mathfrak{I}$. Moreover, assume that the second component of $K(t, z_1, z_2)$, denoted by $\bar{\eta}^t$, fulfills the Cauchy problem

$$\begin{cases} \bar{\eta}_t = T\bar{\eta} + y, \\ \bar{\eta}^0 = 0, \end{cases}$$

for some function y satisfying the estimate

$$\|y(t)\|_1 \le Q_\rho(t)\|z_1 - z_2\|_{\hat{\mathcal{H}}}.$$

Then $\hat{S}(t)$ has an exponential attractor $\hat{\mathcal{E}}$ contained in $\mathbb{B}_{\hat{\mathcal{H}}^1}(r_1)$.

Proof Having in mind the application of Theorem 5.1 in [10], we need to rewrite (3.1) in a proper form. This is obtained by introducing the unknown variables

$$\boldsymbol{x} = \boldsymbol{x}(t) = (I + A)u(t) \text{ and } \boldsymbol{\eta} = \boldsymbol{\eta}^t(s) = A\eta^t(s)$$

and the operators

$$\boldsymbol{A} = A(I + A)^{-1} \text{ and } \boldsymbol{B}\boldsymbol{x} = Au + \varphi(u) - f. \tag{3.3}$$

In this way (3.1) becomes the ODE

$$\begin{cases} \boldsymbol{x}_t + \displaystyle\int_0^\infty \mu(s)\boldsymbol{\eta}(s)\,\mathrm{d}s + \boldsymbol{B}\boldsymbol{x} = 0, \\ \boldsymbol{\eta}_t = T\boldsymbol{\eta} + \boldsymbol{A}\boldsymbol{x}, \end{cases} \tag{3.4}$$

in the phase-space

$$\mathcal{H}^0 = \mathrm{H}^{-1} \times L^2_\mu\big(\mathbb{R}; \mathrm{H}^{-1}\big).$$

By the well-posedness result Theorem 3.1 we infer that (3.4) generates a semigroup

$$S(t) : \mathcal{H}^0 \to \mathcal{H}^0.$$

Besides, owing to Proposition 3.2, $S(t)$ satisfies the following Lipschitz continuity property

$$\big\|S(t)\boldsymbol{z}_1 - S(t)\boldsymbol{z}_2\big\|_{\mathcal{H}^0} \le c\mathrm{e}^{Q(r)t}\|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\mathcal{H}^0}, \tag{3.5}$$

whenever $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{B}_{\mathcal{H}^0}(r)$.
Now note that, if $\mathcal{E} \subset \mathcal{H}^0$ is an exponential attractor for $S(t)$ acting on $\mathcal{H}^0$, then, the set

$$\hat{\mathcal{E}} = \big\{(u, \eta) \,:\, u = (I + A)^{-1}\boldsymbol{x},\ \eta = A^{-1}\boldsymbol{\eta} \text{ with } (\boldsymbol{x}, \boldsymbol{\eta}) \in \mathcal{E}\big\} \subset \hat{\mathcal{H}}$$

is an exponential attractor for $\hat{S}(t)$ acting on $\hat{\mathcal{H}}$. In light of (2.1), we get that

$$\frac{1}{c}\|(\boldsymbol{x}, \boldsymbol{\eta})\|_{\mathcal{H}^0} \le \big\|\big((I + A)^{-1}\boldsymbol{x}, A^{-1}\boldsymbol{\eta}\big)\big\|_{\hat{\mathcal{H}}} \le c\|(\boldsymbol{x}, \boldsymbol{\eta})\|_{\mathcal{H}^0},$$

for some positive c, implying that $\mathcal{E}$ and $\hat{\mathcal{E}}$ share the same fractal dimension. Besides, if $\mathcal{E}$ is a bounded subset of

$$\mathcal{H}^1 = \mathrm{H} \times L^2_\mu(\mathbb{R}; \mathrm{H}) \subset \mathcal{H}^0,$$

then $\hat{\mathcal{E}}$ is bounded in $\hat{\mathcal{H}}^1$.
Thanks to (3.5), in order to prove the existence of an exponential attractor $\mathcal{E}$ for $S(t)$, which is bounded in $\mathcal{H}^1$, we can now rely on Theorem 5.1 in [10], once the sufficient conditions (i)–(v) therein are satisfied for the concrete choice of spaces

$$X^0 = Y^0 = \mathrm{H}^{-1} \text{ and } X^1 = Y^1 = \mathrm{H}$$

and operators $\boldsymbol{A}$ and $\boldsymbol{B}$ as in (3.3). Note that, if $z \in \mathrm{H}$, by virtue of (2.1) we get

$$\|\boldsymbol{A}z\|_{-1} = \big\|(I + A)^{-1} z\big\|_1 \leq \ell \|z\|_{-1} \leq c\|z\|,$$

so that condition (iii) is automatically satisfied (even in greater generality). The proof of Proposition 3.5 is now completed by observing that the assumptions (i), (ii) and (iv), (v) there, written for the semigroup $S(t)$, when translated in terms of $\hat{S}(t)$ simply becomes I, II and III, IV here. □

4 Proof of Theorem 3.3

The verification of the sufficient conditions in Proposition 3.5 will be obtained through several lemmas.

To start with, we recall some known results from [4] that will play a crucial role in the course of our investigation. The first result provides a basic tool for proving energy estimates for the semigroup $\hat{S}(t)$, see [4, Lemma A.1].

Lemma 4.1 *Let $\sigma \in [0, 1]$ be fixed. For a sufficiently regular function $\gamma = \gamma(t)$ on $[0, \infty)$, let us consider the Cauchy problem in the unknowns $(u(t), \eta^t)$*

$$\begin{cases} u_t + A\Big[u_t + u + \displaystyle\int_0^\infty \mu(s)\eta(s)\,\mathrm{d}s\Big] = \gamma, \\ \eta_t = T\eta + u, \\ \big(u(0), \eta^0\big) = z \in \hat{\mathcal{H}}^\sigma, \end{cases}$$

with the associated σ-energy

$$E_\sigma(t) = \frac{1}{2}\Big[\big\|u(t)\big\|^2_{\sigma+1} + \big\|u(t)\big\|^2_\sigma + \big\|\eta^t\big\|^2_{\hat{\mathcal{M}}^\sigma}\Big].$$

Assume that

$$\big\langle \gamma(t), A^\sigma u(t)\big\rangle \leq \big(a + h(t)\big)\|u(t)\|^2_{\sigma+1} + b,$$

for some $0 \le a < 1$, *a nonnegative locally summable function* h *on* $\mathbb{R}^+$, *and* $b \ge 0$. *Then, there exists* $\varepsilon > 0$ *small such that*

$$E_\sigma(t) \le 2I(t)\mathrm{e}^{-\varepsilon t} E_\sigma(0) + \frac{1}{\varepsilon} I(t) b,$$

where

$$I(t) = \exp\left[\int_0^t 2h(\tau)\,\mathrm{d}\tau\right].$$

The second result borrowed by [4] (see the proof of Lemma 5.3 therein) establishes the existence of $r_\star > 0$ such that

$$\mathcal{B}_\star = \mathbb{B}_{\hat{\mathcal{H}}^{1/3}}(r_\star)$$

attracts exponentially the bounded absorbing set $\mathcal{B}_0$, namely, for every $t \ge 0$,

$$\mathrm{dist}_{\hat{\mathcal{H}}}\big(\hat{S}(t)\mathcal{B}_0, \mathcal{B}_\star\big) \le c_\star \mathrm{e}^{-\omega_\star t},$$

for some $c_\star \ge 0$ and $\omega_\star > 0$. Then, it follows by a standard argument that every bounded set of $\hat{\mathcal{H}}$ is exponentially attracted by $\mathcal{B}_\star$. To this aim, let $r > 0$ be given. By exploiting the semigroup property $\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}}(r) = \hat{S}(t - t_r)\hat{S}(t_r)\mathbb{B}_{\hat{\mathcal{H}}}(r)$, where $t_r > 0$ is the entering time of $\mathbb{B}_{\hat{\mathcal{H}}}(r)$ in $\mathcal{B}_0$, we deduce

$$\mathrm{dist}_{\hat{\mathcal{H}}}\big(\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}}(r), \mathcal{B}_\star\big) \le \mathrm{dist}_{\hat{\mathcal{H}}}\big(\hat{S}(t - t_r)\mathcal{B}_0, \mathcal{B}_\star\big) \le c_\star \mathrm{e}^{-\omega_\star (t - t_r)}, \qquad \forall t \ge t_r.$$

By this, we easily conclude that

$$\mathrm{dist}_{\hat{\mathcal{H}}}\big(\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}}(r), \mathcal{B}_\star\big) \le Q(r)\mathrm{e}^{-\omega_\star t}, \qquad \forall t \ge 0.$$

As a consequence, for every $z \in \mathbb{B}_{\hat{\mathcal{H}}}(r)$, we can find a suitable decomposition of the semigroup $\hat{S}(t)z = (u(t), \eta^t)$, with the first component of the form

$$u(t) = v_\star(t) + w_\star(t) \tag{4.1}$$

where, for every $t \ge 0$,

$$w_\star(t) \in \mathcal{B}_\star \quad \text{and} \quad \|v_\star(t)\|_1 \le Q(r)\mathrm{e}^{-\omega_\star t}.$$

Indeed, it is enough to choose $w_\star(t)$ as the first component of the projection of $\hat{S}(t)z$ on the closed convex set $\mathcal{B}_\star$.
Owing to the decomposition (4.1), we can show that $\hat{S}(t)\mathcal{B}_\star$ remains bounded in $\hat{\mathcal{H}}^{1/3}$ for all times.

Lemma 4.2 *For every $t \geq 0$,*

$$\hat{S}(t)\mathcal{B}_\star \subset \mathbb{B}_{\hat{\mathcal{H}}^{1/3}}\big(Q(r_\star)\big).$$

Proof Let $z \in \mathcal{B}_\star$ and $\hat{S}(t)z = (u(t), \eta^t)$. In light of (1.3) we have

$$\big\langle -\varphi(u), A^{1/3}u\big\rangle \leq c\int_\Omega \big(1 + |u|^5\big)|A^{1/3}u|\mathrm{d}x,$$

where, exploiting (4.1),

$$\int_\Omega |u|^5|A^{1/3}u|\mathrm{d}x \leq 2\int_\Omega |u|^3|v_\star|^2\big|A^{1/3}u\big|\mathrm{d}x + 2\int_\Omega |u|^3|w_\star|^2\big|A^{1/3}u\big|\mathrm{d}x.$$

Then, thanks to the Sobolev embeddings

$$\mathrm{H}^{2/3} \subset L^{18/5}(\Omega), \qquad \mathrm{H}^1 \subset L^6(\Omega), \qquad \mathrm{H}^{4/3} \subset L^{18}(\Omega), \tag{4.2}$$

and recalling (3.2), we can compute as follows

$$\begin{aligned}\int_\Omega |u||u|^2|v_\star|^2\big|A^{1/3}u\big|\mathrm{d}x &\leq \|u\|_{L^{18}}\|u\|_{L^6}^2\|v_\star\|_{L^6}^2\big\|A^{1/3}u\big\|_{L^{18/5}}\\ &\leq c\|u\|_{4/3}\|u\|_1^2\|v_\star\|_1^2\|A^{1/3}u\|_{2/3}\\ &\leq Q(r_\star)\mathrm{e}^{-\omega_\star t}\|u\|_{4/3}^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_\Omega |u|^3|w_\star|^2\big|A^{1/3}u\big|\mathrm{d}x &\leq c\|u\|_{L^6}^3\|w_\star\|_{L^{18}}^2\big\|A^{1/3}u\big\|_{L^{18/5}}\\ &\leq c\|u\|_1^3\|w_\star\|_{4/3}^2\|u\|_{4/3}\\ &\leq Q(r_\star)\|u\|_{4/3}.\end{aligned}$$

Hence, we obtain

$$\big\langle -\varphi(u) + f, u\big\rangle_{1/3} \leq \big(1/2 + Q(r_\star)\mathrm{e}^{-\omega_\star t}\big)\|u\|_{4/3}^2 + Q(r_\star),$$

and a direct application of Lemma 4.1 with $\sigma = 1/3$ and $\gamma = -\varphi(u) + f$, noticing that $\int_0^\infty \mathrm{e}^{-\omega_\star t}\,\mathrm{d}t < \infty$ provides the bound

$$\|\hat{S}(t)z\|_{\hat{\mathcal{H}}^{1/3}} \leq Q(r_\star),$$

as claimed. □

Lemma 4.3 *There exists* $r_1 > 0$ *such that the ball* $\mathcal{B}_1 = \mathbb{B}_{\hat{\mathcal{H}}^1}(r_1)$ *is exponentially attracting for* $\hat{S}(t)$*, namely, for some* $\omega_1 > 0$ *and every* $r > 0$*,*

$$\mathrm{dist}_{\hat{\mathcal{H}}}\big(\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}}(r), \mathcal{B}_1\big) \le Q(r)\mathrm{e}^{-\omega_1 t}, \qquad \textit{for every } t \ge 0.$$

Proof In view of the transitivity property of exponential attraction, see [13], it is enough to prove the existence of a set $\mathcal{B}_1 = \mathbb{B}_{\hat{\mathcal{H}}^1}(r_1)$ satisfying, for some positive ω_0,

$$\mathrm{dist}_{\hat{\mathcal{H}}}\big(\hat{S}(t)\mathcal{B}_\star, \mathcal{B}_1\big) \le Q(r_\star)\mathrm{e}^{-\omega_0 t}, \qquad \text{for every } t \ge 0. \tag{4.3}$$

Indeed, the property applies due to the Lipschitz continuity of $\hat{S}(t)$ proved in Proposition 3.2, and since the absorbing set $\mathcal{B}_0$ is exponentially attracted by $\mathcal{B}_\star$, we infer that $\mathcal{B}_0$ is exponentially attracted by $\mathcal{B}_1$. It is now a standard matter to conclude that indeed every bounded set of $\hat{\mathcal{H}}$ is exponentially attracted by $\mathcal{B}_1$.

In order to prove (4.3), for every $z \in \mathcal{B}_\star$ we split $\hat{S}(t)z$ into the sum

$$\hat{S}(t)z = L(t)z + K(t)z,$$

where

$$L(t)z = \big(v(t), \xi^t\big) \quad \text{and} \quad K(t)z = \big(w(t), \zeta^t\big)$$

are solutions (respectively) to

$$\begin{cases} v_t + A\Big[v_t + v + \displaystyle\int_0^\infty \mu(s)\xi(s)\,\mathrm{d}s\Big] = 0, \\ \xi_t = T\xi + v, \\ L(0)z = z, \end{cases}$$

and

$$\begin{cases} w_t + A\Big[w_t + w + \displaystyle\int_0^\infty \mu(s)\zeta(s)\,\mathrm{d}s\Big] + \varphi(u) = f, \\ \zeta_t = T\zeta + w \\ K(0)z = 0. \end{cases}$$

A straightforward application of Lemma 4.1 with $\sigma = 0$ and $\gamma = 0$, provides $\omega_0 > 0$ such that

$$\|L(t)z\|_{\hat{\mathcal{H}}} \le Q(r_\star)\mathrm{e}^{-\omega_0 t}, \qquad \forall t \ge 0. \tag{4.4}$$

Now, we prove that

$$\sup_{t\ge 0} \|K(t)z\|_{\hat{\mathcal{H}}^1} \le Q(r_\star). \tag{4.5}$$

To this aim, note first that $\hat{S}(t)\mathcal{B}_\star$ remains bounded in $\hat{\mathcal{H}}^{1/3}$ by Lemma 4.2, yielding in particular

$$\|u(t)\|_{4/3} \le Q(r_\star), \qquad \forall t \ge 0.$$

Hence, owing to the Sobolev embedding $\mathrm{H}^{4/3} \subset \mathrm{H}^{6/5} \subset L^{10}(\Omega)$ we get

$$\big\langle f - \varphi(u), Aw \big\rangle \le \|f\| \|w\|_2 + c\big(1 + \|u\|_{6/5}^5\big)\|w\|_2 \le \frac{1}{2}\|w\|_2^2 + Q(r_\star).$$

Now a further application of Lemma 4.1 with $\sigma = 1$ and $\gamma = -\varphi(u) + f$, together with the condition $K(0)z = 0$, entails the bound (4.5), which together with (4.4) proves (4.3). □

4.2 Proof of Condition II

Lemma 4.4 *There exists $R_1 > 0$ such that, given any $\rho \ge 0$,*

$$\big\|\hat{S}(t)z\big\|_{\hat{\mathcal{H}}^1} \le q_\rho(t) + R_1, \tag{4.6}$$

for some $q_\rho \in \mathfrak{D}$ and every $z \in \mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$.

Proof Assume that $z \in \mathbb{B}_{\hat{\mathcal{H}}}(r) \cap \mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$, and let $\hat{S}(t)z = (u(t), \eta^t)$. In light of (1.3) we have

$$\big\langle -\varphi(u), Au\big\rangle = -\big\langle \varphi'(u)A^{1/2}u, A^{1/2}u\big\rangle \le c\int_\Omega \big(1 + |u|^4\big)\big|A^{1/2}u\big|^2 \mathrm{d}x,$$

where, exploiting (4.1),

$$\int_\Omega |u|^4\big|A^{1/2}u\big|^2\mathrm{d}x \le 2\int_\Omega |u|^2|v_\star|^2\big|A^{1/2}u\big|^2\mathrm{d}x + 2\int_\Omega |u|^2|w_\star|^2\big|A^{1/2}u\big|^2\mathrm{d}x.$$

Then, in light of (3.2),

$$\int_\Omega |u|^2|v_\star|^2\big|A^{1/2}u\big|^2\mathrm{d}x \le c\|u\|_1^2\|v_\star\|_1^2\|u\|_2^2 \le Q(r)\mathrm{e}^{-\omega_\star t}\|u\|_2^2,$$

while, thanks to the Sobolev embeddings (4.2) and interpolation

$$\begin{aligned}\int_\Omega |u|^2|w_\star|^2\big|A^{1/2}u\big|^2\mathrm{d}x &\le \|u\|_{L^6}^2\|w_\star\|_{L^{18}}^2\big\|A^{1/2}u\big\|_{L^{18/5}}^2\\ &\le Q(r)\|w_\star\|_{4/3}^2\|u\|_{5/3}^2\\ &\le \frac{1}{4}\|u\|_2^2 + Q(r).\end{aligned}$$

In this way, we obtain that

$$\big\langle -\varphi(u)+f, Au\big\rangle \le \big(1/2+Q(r)\mathrm{e}^{-\omega_\star t}\big)\|u\|_2^2+Q(r),$$

so we are in the position to apply Lemma 4.1 with $r=1$ and $\gamma=-\varphi(u)+f$. Noticing that $\int_0^\infty \mathrm{e}^{-\omega_\star t}\,\mathrm{d}t<\infty$ and recalling that $\|z\|_2\le\rho$, we get the bound

$$\big\|\hat{S}(t)z\big\|_{\hat{\mathcal{H}}^1}\le q_\rho(t)+Q(r). \tag{4.7}$$

The above estimate ensures that, under the extra assumption $z\in\mathcal{B}_0\cap\mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$, where $\mathcal{B}_0$ is the absorbing set, (4.6) holds true with $R_1=Q(\|\mathcal{B}_0\|_{\hat{\mathcal{H}}})$. In the general case, let $t_\rho\ge 0$ be such that

$$\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)\subset\mathcal{B}_0,\qquad\text{for every } t\ge t_\rho.$$

Then, it is enough to show that

$$\hat{S}(t)\mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)\subset\mathbb{B}_{\hat{\mathcal{H}}^1}\big(Q(\rho)\big),\qquad\text{for every } t\le t_\rho, \tag{4.8}$$

since, in this case, for every $t\ge t_\rho$,

$$\hat{S}(t)z=\hat{S}(t-t_\rho)\tilde{z},\ \text{ with } \tilde{z}=\hat{S}(t_\rho)z\in\mathbb{B}_{\hat{\mathcal{H}}^1}\big(Q(\rho)\big)\cap\mathcal{B}_0,$$

implying

$$\big\|\hat{S}(t)z\big\|_{\hat{\mathcal{H}}^1}\le q_\rho(t-t_\rho)+Q\big(\|\mathcal{B}_0\|_{\hat{\mathcal{H}}}\big)\le q_\rho(t)+Q\big(\|\mathcal{B}_0\|_{\hat{\mathcal{H}}}\big).$$

Now note that (4.8) is a straightforward consequence of (4.7), which ends the proof.□

4.3 Proof of Condition III

Lemma 4.5 *For every $r>0$ and every $\theta\ge 0$,*

$$\int_\theta^{2\theta}\big(\|u_t(y)\|^2+\|u_t(y)\|_1^2\big)\,\mathrm{d}y\le Q(r+\theta),\qquad\forall z\in\mathbb{B}_{\hat{\mathcal{H}}}(r). \tag{4.9}$$

Proof First note that, from (3.2),

$$\big\|\hat{S}(t)z\big\|_{\hat{\mathcal{H}}}\le Q(r),\qquad\text{for every } z\in\mathbb{B}_{\hat{\mathcal{H}}}(r) \text{ and every } t\ge 0. \tag{4.10}$$

In order to control the integral in (4.9), we multiply the first equation of (3.1) by u_t in H and we get

$$2\|u_t\|^2 + 2\|u_t\|_1^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 = -2\int_0^\infty \mu(s)\langle\eta(s), u_t\rangle_1 \,\mathrm{d}s + 2\langle f - \varphi(u), u_t\rangle \tag{4.11}$$

$$\leq 2\kappa(0)\|\eta\|^2_{\hat{\mathcal{M}}} + \frac{1}{2}\|u_t\|_1^2 + 2\langle f - \varphi(u), u_t\rangle.$$

In light of (1.3) and (4.10), we deduce that

$$2\langle f - \varphi(u), u_t\rangle \leq c\big(1 + \|u\|^5_{L^6}\big)\|u_t\|_{L^6} + 2\|f\|\|u_t\|$$

$$\leq Q(r)\|u_t\|_1 + 2\|f\|\|u_t\| \leq Q(r) + \frac{1}{2}\|u_t\|_1^2 + \|u_t\|^2.$$

Plugging this estimate in (4.11) and using again (4.10), we end up with

$$\|u_t\|^2 + \|u_t\|_1^2 \leq -\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 + Q(r).$$

An integration in the time interval $(\theta, 2\theta)$ and a further application of (4.10), allow to conclude. □

4.4 Proof of Condition IV

Lemma 4.6 *For every $\rho > 0$, there are $Q_\rho \in \mathfrak{I}$ and $q_\rho \in \mathfrak{D}$ with the following property: for all $z_1, z_2 \in \mathbb{B}_{\hat{\mathcal{H}}^1}(\rho)$,*

$$\hat{S}(t)z_1 - \hat{S}(t)z_2 = L(t, z_1, z_2) + K(t, z_1, z_2)$$

where the maps L and K satisfy

$$\|L(t, z_1, z_2)\|_{\hat{\mathcal{H}}} \leq q_\rho(t)\|z_1 - z_2\|_{\hat{\mathcal{H}}} \quad \text{and} \quad \|K(t, z_1, z_2)\|_{\hat{\mathcal{H}}^1} \leq Q_\rho(t)\|z_1 - z_2\|_{\hat{\mathcal{H}}}. \tag{4.12}$$

Moreover, the second component of $K(t, z_1, z_2)$, called ζ, fulfills the Cauchy problem

$$\begin{cases} \zeta_t = T\zeta + y, \\ \zeta^0 = 0 \end{cases} \tag{4.13}$$

for some y satisfying the estimate

$$\|y(t)\|_1 \leq Q_\rho(t)\|z_1 - z_2\|_{\hat{\mathcal{H}}}. \tag{4.14}$$

Proof In order to prove (4.12), set $S(t)z_i = (u_i(t), \eta_i^t)$ and split the difference as

$$\hat{S}(t)z_1 - \hat{S}(t)z_2 = L(t, z_1, z_2) + K(t, z_1, z_2),$$

where $L(t, z_1, z_2) = (v(t), \xi^t)$ and $K(t, z_1, z_2) = (w(t), \zeta^t)$ solve respectively

$$\begin{cases} v_t + A\Big[v_t + v + \int_0^\infty \mu(s)\xi(s)\,\mathrm{d}s\Big] = 0, \\ \xi_t = T\xi + v, \\ (v(0), \xi^0) = z_1 - z_2 \end{cases} \tag{4.15}$$

and

$$\begin{cases} w_t + A\Big[w_t + w + \int_0^\infty \mu(s)\zeta(s)\,\mathrm{d}s\Big] = \varphi(u_2) - \varphi(u_1), \\ \zeta_t = T\zeta + w \\ (w(0), \zeta^0) = 0. \end{cases} \tag{4.16}$$

Applying Lemma 4.1 to problem (4.15), with $\sigma = 0$ and $\gamma = 0$, we immediately obtain the exponential decay

$$\big\|L(t, z_1, z_2)\big\|_{\hat{\mathcal{H}}} \le c\mathrm{e}^{-\omega t}\|z_1 - z_2\|_{\hat{\mathcal{H}}},$$

for some $\omega > 0$. To study the behavior of $K(t, z_1, z_2)$, note that assumption (1.3), Lemma 4.4 and Proposition 3.2 entail the estimate

$$\big\|\varphi(u_2(t)) - \varphi(u_1(t))\big\|^2 \le c\Big(1 + \|u_1(t)\|_{L^\infty}^4 + \|u_2(t)\|_{L^\infty}^4\Big)^2\|u_1(t) - u_2(t)\|^2 \le c\|\bar{z}\|_{\hat{\mathcal{H}}}^2\mathrm{e}^{Q(\rho)t}.$$

Hence, multiplying (4.16) by (w, ζ) in $\mathrm{H}^1 \times \hat{\mathcal{M}}^1$ and taking into account (2.2), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\|K(t, z_1, z_2)\|_{\hat{\mathcal{H}}^1}^2 + \|w(t)\|_1^2\Big) &\le -2\|w(t)\|_2^2 + 2\big\langle\varphi(u_2(t)) - \varphi(u_1(t)), Aw(t)\big\rangle \\ &\le -\|w(t)\|_2^2 + \big\|\varphi(u_2(t)) - \varphi(u_1(t))\big\|^2 \le c\|\bar{z}\|_{\hat{\mathcal{H}}}^2\mathrm{e}^{Q(\rho)t}. \end{aligned}$$

for some positive c. Integrating in time we get

$$\big\|K(t, z_1, z_2)\big\|_{\hat{\mathcal{H}}^1}^2 \le c\|\bar{z}\|_{\hat{\mathcal{H}}}^2\mathrm{e}^{Q(\rho)t},$$

proving (4.12). Finally, the second component of $K(t, z_1, z_2)$ fulfills the Cauchy problem (4.13), and by the last estimate, (4.14) immediately follows. This ends the proof.□

5 The Equation in the Minimal State Framework

To complete the study of (1.1), we analyze the problem in an alternative setting, the *minimal state framework*. This further scheme, developed in [14], was proposed in order to overcome an intrinsic weakness in Dafermos' theory, where two different past histories (considered as initial data) may lead to the same solution. As outlined in the literature on the subject, see e.g. [6–8,10,14], the new setting is obtained by introducing an alternative variable to η, the so-called *minimal state* ξ, accounting for the past history of u, and yielding the desired property that different initial states entail different evolutions. The new auxiliary variable is formally defined as

$$\xi^t(s) = \int_0^\infty \mu(s+y)u(t-y)\mathrm{d}y$$

and the suitable phase spaces where translating the problem are the *extended state spaces*, defined as

$$\mathcal{V}^\sigma = \mathrm{H}^{\sigma+1} \times \mathcal{S}^\sigma, \qquad \|(u,\xi)\|^2_{\mathcal{V}^\sigma} = \|u\|^2_{\sigma+1} + \|\xi\|^2_{\mathcal{S}^\sigma},$$

with

$$\mathcal{S}^\sigma = L^2_\nu\big(\mathbb{R}^+; \mathrm{H}^{\sigma+1}\big), \qquad \|\xi\|^2_{\mathcal{S}^\sigma} = \int_0^\infty \nu(s)\|\xi(s)\|^2_{\sigma+1}\mathrm{d}s,$$

for $\sigma \in [0,1]$ (omitting σ in the notation when 0). Here, the weight function ν is given by

$$\nu(s) = \begin{cases} 1/\mu(s) & \text{if } \mu(s) \neq 0 \\ 0 & \text{if } \mu(s) = 0 \\ \lim_{y\to 0} 1/\mu(y) & \text{if } s = 0. \end{cases}$$

Within this framework, problem (1.1)–(1.2) can be recast as the system of two variables $u = u(t)$ and $\xi = \xi^t(s)$

$$\begin{cases} u_t + Au_t + Au + \displaystyle\int_0^\infty A\xi(s)\,\mathrm{d}s + \varphi(u) = f \\ \xi_t(s) = P\xi(s) + \mu(s)u, \end{cases} \tag{5.1}$$

where P is the infinitesimal generator of the left-translation semigroup $\mathcal{S}$, namely

$$P\xi = \xi' \qquad \text{with domain} \qquad \mathrm{dom}(P) = \big\{\xi \in \mathcal{S} : \xi' \in \mathcal{S}\big\},$$

satisfying

$$\langle P\xi, \xi\rangle_{\mathcal{S}} \leq 0, \quad \forall \xi \in \mathrm{dom}(P). \tag{5.2}$$

The corresponding initial conditions become

$$u(0) = u_0 \text{ and } \xi^0(s) = \int_0^\infty \mu(s+y) g_0(y)\,\mathrm{d}y.$$

A well-posedness result holds also within this scheme.

Proposition 5.1 *System* (5.1) *generates a strongly continuous semigroup*[2] $\tilde{S}(t) : \mathcal{V} \to \mathcal{V}$. *Furthermore, the following uniform-in-time estimate holds, for every* $r > 0$ *and every* $\tilde{z} \in \mathbb{B}_{\mathcal{V}}(r)$,

$$\left\| \tilde{S}(t)\tilde{z} \right\|_{\mathcal{V}} \leq Q(r). \tag{5.3}$$

Proof We limit ourselves to the validity of estimate (5.3), since the well-posedness is obtained recasting in the minimal state setting the proof performed in the memory framework, see [4,23]. Testing within a regularization scheme system (5.1) with $(u, \xi) \in \mathrm{H} \times \mathcal{S}$, and appealing to (5.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|_1^2 + \|u\|^2 + \|\xi\|_{\mathcal{S}}^2 \Big) + 2\|u(t)\|_1^2 \leq 2\big\langle f - \varphi(u), u \big\rangle.$$

Now, recalling assumption (1.4), we deduce the existence of $\varepsilon > 0$ (possibly very small), such that

$$\big\langle f - \varphi(u), u \big\rangle \leq \|f\|\|u\| + (1-2\varepsilon)\|u\|_1^2 + c \leq (1-\varepsilon)\|u\|_1^2 + c$$

yielding to the control

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|_1^2 + \|u\|^2 + \|\xi\|_{\mathcal{S}}^2 \Big) + 2\varepsilon\|u(t)\|_1^2 \leq c.$$

The Gronwall lemma allows finally to deduce estimate (5.3). □

As devised in [6,8,10], there is a natural connection between the history and the state frameworks, which allows to obtain an asymptotic analysis of $\tilde{S}(t)$ leaning on the known results for $\hat{S}(t)$. The link is provided by the following lemma (see [6,8] for the proof) which entails in particular that the state approach is more general.

Lemma 5.2 *There exists a bounded linear operator of unitary norm* $\mathbf{\Lambda} : \hat{\mathcal{H}}^\sigma \to \mathcal{V}^\sigma$ *such that, for every* $z \in \hat{\mathcal{H}}^\sigma$, *the following equality holds*

$$\tilde{S}(t)\mathbf{\Lambda} z = \mathbf{\Lambda}\hat{S}(t)z.$$

Besides, once the existence of an exponential attractor $\hat{\mathcal{E}}$ is attained in the memory space for $\hat{S}(t)$, the corresponding result for $\tilde{S}(t)$ in the minimal state framework can be easily deduced relying on the existence of the map $\mathbf{\Lambda}$. This is established in [10] for abstract equations with memory satisfying a rather mild assumption, see Condition 6.1

[2] Actually, $\tilde{S}(t)$ maps $\mathcal{V}^\sigma$ into $\mathcal{V}^\sigma$ for every $\sigma \in [0, 1]$.

therein. Recalling the functional setting in the proof of Theorem 3.5, it is immediate to check that its concrete reformulation for our model is

Condition 6.1. For every initial datum $\tilde{z} \in \mathbb{B}_{\mathcal{V}}(r)$, we define

$$\psi^t(s) = \int_0^{\min\{t,s\}} u(t-y)\,\mathrm{d}y,$$

where u is the first component of the solution $\tilde{S}(t)\tilde{z} = (u(t), \xi^t)$. Then, the function $(u(t), \psi^t)$ belongs to $\hat{\mathcal{H}}$ for every $t \geq 0$ and the following estimate

$$\sup_{t\geq 0} \|(u(t), \psi^t)\|_{\hat{\mathcal{H}}} \leq Q(r) \tag{5.4}$$

holds.

Accordingly, the abstract Theorem 6.3 in [10] becomes

Theorem 5.3 *In addition to the general assumptions, let Condition 6.1 hold. If the semigroup $\hat{S}(t) : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ possesses an exponential attractor $\hat{\mathcal{E}}$, then the set*

$$\tilde{\mathcal{E}} = \mathbf{\Lambda}\hat{\mathcal{E}}$$

is an exponential attractor for the semigroup $\tilde{S}(t) : \mathcal{V} \to \mathcal{V}$.

As a consequence we have

Corollary 5.4 *The semigroup $\tilde{S}(t)$ on $\mathcal{V}$ possesses an exponential attractor $\tilde{\mathcal{E}}$ bounded in $\mathcal{V}^1$, which is the set $\tilde{\mathcal{E}} = \mathbf{\Lambda}\hat{\mathcal{E}}$.*

Proof We only need to check the validity of Condition 6.1 above. In order to prove (5.4), note that (5.3) yields

$$\left\|\psi^t(s)\right\|_1 = \int_0^{\min\{t,s\}} \|u(t-y)\|_1\,\mathrm{d}y \leq Q(r)s,$$

and the estimate $\|\psi^t\|_{\hat{\mathcal{M}}} \leq Q(r)$ immediately follows by (1.6). Owing to Theorem 3.3, we can thus apply Theorem 5.3, implying that the set $\tilde{\mathcal{E}} = \mathbf{\Lambda}\hat{\mathcal{E}}$ is an exponential attractor for the semigroup $\tilde{S}(t) : \mathcal{V} \to \mathcal{V}$. Besides, since $\hat{\mathcal{E}}$ is bounded in $\hat{\mathcal{H}}^1$, then $\tilde{\mathcal{E}}$ is bounded in $\mathcal{V}^1$ by Lemma 5.2.

Remark 5.5 By the same argument leading to Corollary 3.4, the semigroup $\tilde{S}(t)$ on $\mathcal{V}$ possesses the global attractor $\tilde{\mathcal{A}}$ which is contained in $\tilde{\mathcal{E}}$, hence bounded in $\mathcal{V}^1$ and with finite fractal dimension. Actually, the relation

$$\tilde{\mathcal{A}} = \mathbf{\Lambda}\hat{\mathcal{A}}$$

holds. Indeed, the compactness and invariance of $\mathbf{\Lambda}\hat{\mathcal{A}}$ immediately follows by Lemma 5.2. The fact that $\mathbf{\Lambda}\hat{\mathcal{A}}$ is also attracting can be verified arguing as in the proof of [6, Theorem 7.2].

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