ENERGY STABLE AND CONVERGENT FINITE ELEMENT SCHEMES FOR THE MODIFIED PHASE FIELD CRYSTAL EQUATION

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Abstract. We propose a space semi-discrete and a fully discrete finite element scheme for the modified phase field crystal equation (MPFC). The space discretization is based on a splitting method and consists in a Galerkin approximation which allows low order (piecewise linear) finite elements. The time discretization is a second-order scheme which has been introduced by Gomez and Hughes for the Cahn-Hilliard equation. The fully discrete scheme is shown to be unconditionally energy stable and uniquely solvable for small time steps, with a smallness condition independent of the space step. Using energy estimates, we prove that in both cases, the discrete solution converges to the unique energy solution of the MPFC equation as the discretization parameters tend to 0. Using a Łojasiewicz inequality, we also establish that the discrete solution tends to a stationary solution as time goes to infinity. Numerical simulations illustrate the theoretical results.

Keywords: Finite elements, second-order schemes, gradient-like systems, Łojasiewicz inequality.
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1. Introduction

In this paper, we analyze finite element discretizations of the modified phase field crystal (MPFC) equation

$$\beta u_{tt} + u_t = \Delta[\Delta^2 u + 2\Delta u + f(u)], \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

with periodic boundary conditions on the d-parallelepiped $\Omega = \Pi_{k=1}^d (0, L_k)$ ($L_k > 0$ for $k = 1, \dots, d$) in space dimension $d = 1$, 2 or 3. Equation (1.1) is endowed with initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x), \quad x \in \Omega. \tag{1.2}$$

The unknown function u is the phase function and $\beta > 0$ is a relaxation parameter. The nonlinearity f is the derivative of a polynomial potential F (see (2.1)-(2.2) for a precise definition). A relevant example in applications is given by

$$f(s) = s^3 + (1 - \varepsilon)s, \tag{1.3}$$

where $\varepsilon \in \mathbb{R}$ is constant.

When $\beta = 0$, equation (1.1) is known as the phase field crystal (PFC) equation: it has been employed to model and simulate the dynamics of crystalline materials, including crystal growth in a supercooled liquid, dendritic and eutectic solidification, epitaxial growth [6, 7, 11, 12, 31, 33]. In the phase field approach, the number density of atoms is approximated by the phase function u. The parameter ε in (1.3) is proportional to the undercooling i.e. $\varepsilon \sim T_e - T$, T_e being the equilibrium temperature at which the phase transition occurs. The PFC equation is a gradient flow for the free energy

$$E(u) = \int_\Omega \left(\frac{1}{2}|\Delta u|^2 - |\nabla u|^2 + F(u) \right) dx. \tag{1.4}$$

It preserves the total mass and can be viewed as an analog of the Swift-Hohenberg equation [36].

The MPFC equation (1.1) (with $\beta > 0$) was recently proposed in [34] (cf. also [12, 15, 16, 35]) in order to account for elastic interactions. Equations like (1.1) have also been derived in [14] to take large deviations from thermodynamic equilibrium into account.The MPFC equation is no longer a gradient flow for (1.4), but it is possible to associate to a solution (u, u_t) a "pseudo-energy", obtained by adding to E a kinetic energy term (see (2.7)). This leads in a natural way to the notion of *energy solution* introduced by Grasselli and Wu [25] for the MPFC equation, following a terminology used for the modified Cahn-Hilliard equation [22, 23, 24]. Existence of a unique energy solution was proved by Grasselli and Wu in [25] as well as the convergence of single trajectories to single stationary states. The analysis of global dynamics (that is, existence of global and exponential attractors) for the MPFC equation was carried out in [25, 26].

From the numerical analysis point of view, the MPFC equation has been studied in [3, 4, 17, 37, 38], while the literature on the PFC equation is more abundant (see, e.g., [2, 5, 10, 19, 28, 39]). In [3, 4, 38], the authors proposed unconditionally energy stable and unconditionally uniquely solvable finite difference schemes. The time discretization was based on a convex splitting of the pseudo-energy and was either first order or second order. A priori error estimates were proved assuming enough regularity on the solution. A time semi-discrete scheme was used in [37] to establish the existence of a weak solution and of a unique strong solution to the MPFC equation up to any positive final time $T > 0$. In [17], an unconditionally energy stable finite element scheme was derived, but no proof of convergence was given.

The main purpose of this paper is to derive and analyze a second order (in time) fully discrete finite element scheme for the MPFC equation. For the space discretization we use a splitting approach which is well known in the context of phase field models (cf., e.g., [9, 21]). This argument allows to consider low order (piecewise linear) finite elements, although the analysis is carried out in a more general setting, namely a Galerkin approximation. For the time discretization, we use a modified Crank-Nicolson scheme which was introduced by Gomez and Hughes for the Cahn-Hilliard equation [18]; their approach represents an interesting alternative to secant schemes (cf. the discussion in [40]).

We prove that our scheme is unconditionally energy stable, solvable for any time step, and uniquely solvable for small enough time steps, with a smallness assumption independent of the space step. Using the energy estimate, and assuming only some natural conditions on the initial conditions, we show that the solution of the fully discrete scheme converges to the energy solution of problem (1.1)-(1.2) as the time step τ and the space step h tend to 0. Finally, we prove that the discrete solution tends to a stationary solution as time goes to infinity. This last issue is not trivial since the set of steady states can be very complicated (see [32] for an analysis of the one dimensional stationary problem, cf. also [8]).

For equations involving a second order time derivative such as (1.1), second order time discretizations are very interesting because they do not regularize in finite time, unlike first order schemes: a fundamental property of the continuous model is therefore reproduced at the discrete level. In contrast with the second order two-step scheme in [3, 4], we loose the unconditional unique solvability, but one advantage is that our one-step scheme can be used with variable time steps. Moreover, we do not assume $\varepsilon < 1$ in (1.3), and we do not have any restriction on the initial value of u_t. Since energy solutions have a weaker regularity than the weak solutions, our convergence result as $(h, \tau) \to (0, 0)$ holds with the minimal regularity on the solution.

Our proofs are crucially based on the energy estimates. In order to establish the convergence to equilibrium, we also use the gradient-like structure of the problem and a Łojasiewicz inequality [30], as in the continuous case [25]. Related results have been obtained for first order schemes applied to second-order gradient-like equations in [1, 20], but the case of the second-order scheme here is more involved.

The paper is structured as follows. In Section 2 we introduce the functional setting and we recall useful results concerning the continuous problem. In Section 3 we consider the space semi-discrete problem. We show its well-posedness and establish energy estimates which allow us to prove the convergence of the semi-discrete solution to the energy solution of the MPFC problem (1.1)-(1.2). This gives a framework for the fully discrete problem which is treated in Section 4. Section 5 is concerned with the convergence to equilibrium for the fully discrete problem. In Section 6, numerical simulations in one and two space dimension illustrate the theoretical results.

2. The continuous problem

2.1. **Notation and functional spaces.** For a real Banach space V with dual $V^\star$, we indicate by $\langle \cdot, \cdot \rangle_{V^\star, V}$ the duality product between V and $V^\star$. We denote by H^m_{per}, $m \in \mathbb{N}$, the space of functions that are in $H^m_{loc}(\mathbb{R}^d)$ and periodic with period Ω. For any $m \in \mathbb{N}$, H^m_{per} is a Hilbert space for the scalar product

$$((u, v))_m = \sum_{|\kappa| \le m} \int_\Omega D^\kappa u(x) D^\kappa v(x) dx$$

(κ being a multi-index) and its associated norm $\|u\|_m = ((u, u))_m^{1/2}$.

For $m = 0$, $H^0_{per} = L^2(\Omega)$, the inner product as well as the norm on $L^2(\Omega)$ are simply indicated by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. For sake of simplicity, we assume that $\int_\Omega 1 dx = |\Omega| = 1$. The mean value of any function $u \in L^2(\Omega)$ is denoted by

$$\langle u \rangle = \int_\Omega u dx,$$

and we set $\dot{u} = u - \langle u \rangle$.

The dual space of H^m_{per} is denoted by H^{-m}_{per}, and it is equipped with the operator norm given by

$$\|\mathcal{T}\|_{-m} = \sup_{\|u\|_m = 1,\ u \in H^m_{per}} |\mathcal{T}(u)|.$$

For an operator $u \in H^{-m}_{per}$, we denote $\langle u\rangle = \langle u, 1\rangle_{H^{-m}_{per},H^m_{per}}$ and we set $\dot u = u - \langle u\rangle$. We denote by $\dot H^m_{per} = \{u \in H^m_{per} \ : \ \langle u\rangle = 0\}$ $(m \in \mathbb{Z})$ the Sobolev spaces for functions with zero mean. We will frequently use the fact that H^m_{per} is isomorphic to $\mathbb{R} \times \dot H^m_{per}$ $(m \in \mathbb{Z})$ through the decomposition $u = \langle u\rangle + \dot u$.

Using the dense and continuous inclusions $H^1_{per} \subset L^2(\Omega) \subset H^{-1}_{per}$, the semi-scalar product on H^1_{per},

$$(u, v) \mapsto (\nabla u, \nabla v),$$

defines a linear operator $\mathcal{A} = -\Delta : D(\mathcal{A}) \to L^2(\Omega)$ with domain $D(\mathcal{A}) = H^2_{per}$. We denote $\dot{\mathcal{A}} = -\Delta : D(\dot{\mathcal{A}}) \to L^2(\Omega)$ the restriction of $\mathcal{A}$ to $\dot L^2(\Omega)$, with domain $D(\dot{\mathcal{A}}) = \dot H^2_{per}$. We observe that $\dot{\mathcal{A}}$ is a positive self-adjoint operator with compact resolvent so that its powers $\dot{\mathcal{A}}^s$ $(s \in \mathbb{R})$ are well defined and it is possible to prove that $\dot H^m_{per} = D(\dot{\mathcal{A}}^{m/2})$ $(m \in \mathbb{Z})$.

For $m = -1$, we introduce an equivalent and more convenient norm $|\cdot|_{-1}$ on $\dot H^{-1}_{per}$ associated with the inner product

$$(\dot u, \dot v)_{-1} = (\dot{\mathcal{A}}^{-1/2}\dot u, \dot{\mathcal{A}}^{-1/2}\dot v),$$

so that for any $\dot u \in \dot H^{-1}_{per}$, we have

$$|\dot u|_{-1} = \|\dot{\mathcal{A}}^{-1/2}\dot u\| = \|\nabla \dot{\mathcal{A}}^{-1}\dot u\|.$$

Similarly, for $m = 1$, we will sometimes use the equivalent norm $|\cdot|_1$ in H^1_{per} associated with the inner product

$$(u, v)_1 = \langle u\rangle\langle v\rangle + (\nabla u, \nabla v).$$

Moreover, $\dot{\mathcal{A}}$ defines a continuous bijection from $\dot H^m_{per}$ onto $\dot H^{m-2}_{per}$. In particular, for $s = -1$,

$$(\dot u, \dot v)_{-1} = \langle \mathcal{A}^{-1}\dot u, \dot v\rangle_{H^1_{per},H^{-1}_{per}} = \langle \dot u, \mathcal{A}^{-1}\dot v\rangle_{H^{-1}_{per},H^1_{per}} = (\mathcal{A}^{-1/2}\dot u, \mathcal{A}^{-1/2}\dot v).$$

2.2. **Energy solutions.** The nonlinearity f is a polynomial of odd degree whose leading coefficient is positive and which vanishes at 0:

$$f(s) = \sum_{i=1}^{2p+1} a_i s^i \quad \forall s \in \mathbb{R} \quad (a_{2p+1} > 0), \tag{2.1}$$

with $p \in \mathbb{N}^\star$ if $d = 1$ or $d = 2$ and with $p \in \{1, 2\}$ if $d = 3$. We denote F the antiderivative of f which vanishes at 0, i.e.,

$$F(s) = \sum_{i=2}^{2p+2} \frac{a_{i-1}}{i} s^i \quad \forall s \in \mathbb{R}. \tag{2.2}$$

We will make use of the Sobolev injection $H^1_{per} \subset L^{2p+2}(\Omega)$. In particular, there is a constant $C_S = C_S(\Omega)$ such that

$$\|u\|_{L^{2p+2}(\Omega)} \le C_S|u|_1, \quad \forall u \in H^1_{per}, \tag{2.3}$$

and the map $v \mapsto f(v)$ is Lipschitz continuous on bounded sets of H^1_{per} with values into $L^{(2p+2)/(2p+1)}(\Omega) \subset H^{-1}_{per}$. We also have $H^2_{per} \subset C^0(\overline{\Omega})$ with continuous injection. Finally, we note that there exist constants $c_1 \ge 0$, $c_2 > 0$ and $c_3 \ge 0$ such that

$$F(s) \ge 2s^2 - c_1 \quad \forall s \in \mathbb{R}, \tag{2.4}$$

$$|f(s)| \le c_2 F(s) + c_3 \quad \forall s \in \mathbb{R}. \tag{2.5}$$

We point out that the expression (2.2) includes the standard quartic potential (obtained with $p = 1$)

$$F(s) = \frac{s^4}{4} + \frac{(1-\varepsilon)}{2} s^2. \tag{2.6}$$

In contrast to some authors, we do not assume that $\varepsilon < 1$ (we simply have $\varepsilon \in \mathbb{R}$).

In [25], a notion of energy solution was introduced. This is based on the following pseudo-energy:

$$\mathcal{E}(u, v) = E(u) + \frac{\beta}{2} |\dot{v}|_{-1}^2, \tag{2.7}$$

which is well defined for any $(u, v) \in H^2_{per} \times H^{-1}_{per}$.

Definition 2.1. A pair (u, u_t) is called an *energy solution* to problem (1.1)-(1.2) if

$$(u, u_t) \in L^\infty(\mathbb{R}_+; H^2_{per} \times H^{-1}_{per}), \quad u_{tt} \in L^\infty(\mathbb{R}_+; H^{-4}_{per}), \tag{2.8}$$

$$\mathcal{E}(u, u_t) \in L^\infty(\mathbb{R}_+), \tag{2.9}$$

and the following relations hold:

$$\langle \beta u_{tt} + u_t \rangle = 0, \tag{2.10}$$

$$\dot{\mathcal{A}}^{-1}(\beta u_{tt} + u_t) + \mathcal{A}^2 u - 2\mathcal{A}u + f(u) - \langle f(u) \rangle = 0,$$
$$\text{in } \dot{H}^{-2}_{per}, \quad \text{a.e. in } \mathbb{R}_+, \tag{2.11}$$

$$u(0) = u_0 \text{ in } H^2_{per}, \quad u_t(0) = v_0 \text{ in } H^{-1}_{per}. \tag{2.12}$$

Equation (2.10) can be interpreted as a conservation law for the mass. With this definition, we have

Theorem 2.2 ([25]). *For any intial data $(u_0, v_0) \in H^2_{per} \times H^{-1}_{per}$, problem* (1.1)-(1.2) *has a unique global energy solution (u, u_t). Moreover, any energy solution satifies the strong time continuity property*

$$u \in C^2(\mathbb{R}_+; H^{-4}_{per}) \cap C^1(\mathbb{R}_+; H^{-1}_{per}) \cap C(\mathbb{R}_+; H^2_{per}),$$

as well as the following energy identity, for all $s,\ t \in \mathbb{R}_+$ with $s < t$,

$$\mathcal{E}(u(t), u_t(t)) = \mathcal{E}(u(s), u_t(s)) - \int_s^t |\dot{u}_t(\tau)|_{-1}^2 d\tau + \int_s^t \langle v_0 \rangle e^{-\tau/\beta} \int_\Omega f(u(\tau)) dx d\tau. \tag{2.13}$$

In particular, whenever $\langle v_0 \rangle = 0$ then the pseudo-energy is nonincreasing.

3. The space semi-discrete problem

3.1. **The space semi-discrete scheme.** Our space discretization is based on two ideas: first, in view of the time discretization, we write the PDE (1.1) as a first order system; second, in order to use low order finite elements, we split the tri-Laplacian into three terms, in the spirit of a well-known splitting approach in Cahn-Hilliard type equations (see, e.g., [9, 21, 17]). We obtain the following system, which is (formally) equivalent to (1.1):

$$\begin{cases} u_t = v \\ \beta v_t = -v + \Delta w \\ z = -\Delta u \\ w = -\Delta z + 2\Delta u + f(u). \end{cases}$$

Now, let V_h denote a finite-dimensional subspace of H^1_{per} which contains the constants. In applications, V_h will be a space of conforming finite elements (see Section 6). The space V_h can also be obtained with a spectral basis.

The space semi-discrete scheme reads: find $u_h, v_h, z_h, w_h : \mathbb{R}_+ \to V_h$ such that

$$\begin{cases} (\partial_t u_h, \varphi_h) = (v_h, \varphi_h) \\ \beta(\partial_t v_h, \psi_h) = -(v_h, \psi_h) - (\nabla w_h, \nabla \psi_h) \\ (z_h, \zeta_h) = (\nabla u_h, \nabla \zeta_h) \\ (w_h, \xi) = (\nabla z_h, \nabla \xi_h) - 2(\nabla u_h, \nabla \xi_h) + (f(u_h), \xi_h), \end{cases} \tag{3.1}$$

for all $\varphi_h, \psi_h, \zeta_h, \xi_h$ in V_h. This problem is completed with initial conditions

$$u_h(0) = u_h^0, \quad v_h(0) = v_h^0, \tag{3.2}$$

where u_h^0 and v_h^0 are given in V_h.

It will be convenient to work with an appropriate basis of V_h. For this purpose, let $(e_h^i)_{1\le i\le N_h}$ denote an orthonormal basis of V_h for the $L^2(\Omega)$ scalar product, such that $e_h^1 \equiv 1$. The integer N_h is the dimension of V_h. To every function $r_h = \sum_{i=1}^{N_h} r_i e_h^i \in V_h$ corresponds a unique (column) vector $R = (r_1, \ldots, r_{N_h})^t$, represented by the corresponding capital letter. We seek

$$u_h(t) = \sum_{i=1}^{N_h} u_i(t) e_h^i \simeq (u_1(t), \ldots, u_{N_h}(t))^t = U(t), \quad v_h \simeq V, \quad z_h \simeq Z, \quad w_h \simeq W.$$

Define $A = (A_{ij})_{1\le i,j\le N_h}$, where

$$A_{ij} = (\nabla e_h^i, \nabla e_h^j), \quad 1 \le i, j \le N_h, \tag{3.3}$$

and let

$$F_h(U) = (F(u_h), 1), \qquad \text{so that} \qquad \nabla F_h(U) = \Big((f(u_h), e_h^1), \ldots, (f(u_h), e_h^{N_h}) \Big)^t.$$

By choosing the test functions φ_h, ψ_h, ζ_h, ξ_h in (3.1) as the basis functions e_h^i, we obtain the following equivalent system:

$$\begin{cases} U_t = V \\ \beta V_t = -V - AW \\ Z = AU \\ W = AZ - 2AU + \nabla F_h(U). \end{cases} \tag{3.4}$$

Eliminating V, Z and W, we see that (3.4) is equivalent to

$$\beta U_{tt} + U_t = -A[A^2 U - 2AU + \nabla F_h(U)], \quad t \ge 0. \tag{3.5}$$

Since A is a discretization of $-\Delta$, this is natural space semi-discrete version of (1.1).

Let U denote a solution of (3.5). We notice that the first line and the first column of A are filled with zeros (recall $e_h^1 \equiv 0$, so that $\nabla e_h^1 \equiv 0$). Thus, the first component of U, $u_1(t) = (u_h(t), 1)$, satisfies

$$\beta \partial_{tt} u_1 + \partial_t u_1 = 0, \quad t \ge 0. \tag{3.6}$$

Solving (3.6) with initial conditions $u_1(0) = (u_h^0, 1) =: u_1^0$ and $\partial_t u_1(0) = (v_h^0, 1) =: v_1^0$ yields

$$\partial_t u_1(t) = v_1^0 e^{-t/\beta} =: a(t) \quad \text{and} \quad u_1(t) = M - \beta a(t), \text{ with } M = \beta v_1^0 + u_1^0. \tag{3.7}$$

For every vector $R = (r_1, \dots, r_{N_h})^t \in \mathbb{R}^{N_h}$, we denote $\dot{R} = (r_2, \dots, r_{N_h})^t \in \mathbb{R}^{N_h-1}$. Then $\dot{U}$ satisfies

$$\beta \dot{U}_{tt} + \dot{U}_t = -\dot{A}[\dot{A}^2\dot{U} - 2\dot{A}\dot{U} + \dot{\nabla} F_h(U)], \quad t \geq 0, \tag{3.8}$$

where $\dot{A}$ is the submatrix $\dot{A} = (A_{ij})_{2\leq i,j\leq N_h}$, and

$$\dot{\nabla} F_h(U) = \Big((f(u_h), e_h^2), \dots, (f(u_h), e_h^{N_h}) \Big)^t .$$

We can also write $\dot{\nabla} F_h(U) = \dot{P}(\nabla F_h(u_1(t), \dot{U}))$, where $\dot{P} : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h-1}$ is the projection on the components $2, \dots, N_h$. This shows that $\dot{\nabla} F_h(U)$ is a "non autonomous" function of $\dot{U}$ (recall that $u_1(t)$ is determined by (3.7)). For later purpose, we note that by (3.3), $\dot{A}$ is symmetric positive definite: in particular, $\dot{A}$ is invertible.

Conversely, any solution U of (3.7)-(3.8) satisfies (3.4), i.e. that the second equation of (3.4) is satisfied with V, Z and W given by the three other equations of the system (3.4).

3.2. **Existence, uniqueness, and discrete energy estimate.** The standard Euclidean norm in $\mathbb{R}^{N_h}$ or $\mathbb{R}^{N_h-1}$ will be denoted $|\cdot|$. We also use the following quadratic norm:

$$|\dot{R}|_{-1} = \Big(\dot{R}^t \dot{A}^{-1} \dot{R} \Big)^{1/2}, \tag{3.9}$$

defined for all $\dot{R} \in \mathbb{R}^{N_h-1}$. Notice that $|A^s U| = |\dot{A}^s \dot{U}|$ ($s > 0$, $U \in \mathbb{R}^{N_h}$). We set

$$E_h(U) = \frac{1}{2}|AU|^2 - |A^{1/2}U|^2 + F_h(U), \tag{3.10}$$

$$\mathcal{E}_h(U, V) = E_h(U) + \frac{\beta}{2}|\dot{V}|_{-1}^2. \tag{3.11}$$

As a shortcut, for a solution (U, U_t) of (3.5), we will write

$$\mathcal{E}_h(t) = \mathcal{E}_h(U(t), U_t(t)).$$

Notice that by the Cauchy-Schwarz inequality we have

$$|A^{1/2}U|^2 = U^t A U \leq \frac{1}{4}|AU|^2 + |U|^2. \tag{3.12}$$

Then, using also (2.4), we find that

$$E_h(U) \geq \frac{1}{4}|AU|^2 + |U|^2 - c_1. \tag{3.13}$$

We first prove the following

Lemma 3.1. *Any solution $U \in C^2([0, T); \mathbb{R}^{N_h})$ of (3.5) satisfies the energy equality*

$$\frac{d}{dt}\mathcal{E}_h(t) + |\dot{U}_t|_{-1}^2 = v_1^0 e^{-t/\beta}(f(u_h), 1), \tag{3.14}$$

and the energy estimate

$$\mathcal{E}_h(t) + \int_0^t |\dot{U}_t(s)|_{-1}^2 ds \leq \mathcal{E}_h(0) e^{2c_2|v_1^0|\beta} + (c_1c_2 + c_3)|v_1^0|\beta e^{2c_2|v_1^0|\beta}, \tag{3.15}$$

for all $t \in [0, T)$, where c_1, c_2 and c_3 depend only on f (see (2.4)-(2.5)), and where $v_1^0 = \partial_t u_1(0)$.

Proof. Recall that $\partial_t u_1(t) = v_1^0 e^{-t/\beta}$ (see (3.7)). On multiplying (3.8) by $\dot{U}_t^t \dot{A}^{-1}$, and using

$$\frac{d}{dt}[F_h(U(t))] = (\nabla F_h(U), U_t(t)) = \sum_{i=1}^{N_h} \partial_t u_i(t)(f(u_h), e_h^i),$$

we find the energy equality (3.14). Estimate (2.5) yields

$$\frac{d}{dt}\mathcal{E}_h(t) + |\dot{U}_t|_{-1}^2 \leq |v_1^0| e^{-t/\beta}(c_2 F_h(U) + c_3).$$

By the Cauchy-Schwarz inequality, we have

$$2|A^{1/2}U|^2 = 2U^t AU \leq |AU|^2 + |U|^2.$$

Thus we get

$$2E_h(U) = (|AU|^2 - 2|A^{1/2}U|^2 + |U|^2) + F_h(U) + (F_h(U) - |U|^2) \geq F_h(U) - c_1,$$

so that

$$F_h(U) \leq 2\mathcal{E}_h(U, V) + c_1. \tag{3.16}$$

Therefore we obtain

$$\frac{d}{dt}\mathcal{E}_h(t) + |\dot{U}_t|_{-1}^2 \leq |v_1^0| e^{-t/\beta}\left(2c_2\mathcal{E}_h(t) + c_1 c_2 + c_3\right).$$

Letting $\eta(t) = \int_0^t 2c_2 |v_1^0| e^{-t/\beta}$ and $c_3' = c_1 c_2 + c_3$, Gronwall's lemma yields

$$\mathcal{E}_h(t) + \int_0^t |\dot{U}_t(s)|_{-1}^2 e^{\eta(t)-\eta(s)} ds \leq \mathcal{E}_h(0) e^{\eta(t)} + \int_0^t c_3' |v_1^0| e^{-s/\beta} e^{\eta(t)-\eta(s)} ds.$$

Since $\eta(t) = 2c_2|v_1^0|\beta(1-e^{-t/\beta}) \leq 2c_2|v_1^0|\beta$, we deduce the energy estimate (3.15). □

Theorem 3.2. *For every U^0, V^0 in $\mathbb{R}^{N_h}$, there exists a unique solution $U \in C^2(\mathbb{R}_+, \mathbb{R}^{N_h})$ of* (3.5) *such that $U(0) = U^0$ and $U_t(0) = V^0$.*

Proof. By the Cauchy-Lipschitz theorem, there exists a unique maximal solution $U \in C^2([0, T^+); \mathbb{R}^{N_h})$ of (3.5) satisfying the given initial conditions. The energy estimate (3.15) shows that $\mathcal{E}_h$ is uniformly bounded for $t \geq 0$. By (3.13), $|U|$ and $|\dot{U}_t|_{-1}$ are uniformly bounded for $t \geq 0$. This, together with the estimate (3.7) on the mass, implies that $T^+ = +\infty$. □

3.3. **Some additional notation.** We assume now that $(V_h)_{h>0}$ is a family of subspaces of H^1_{per} such that:

- (H1) for all $h > 0$, V_h has finite dimension and contains all the constants;
- (H2) for any $\varphi \in H^1_{per}$, there exists $\varphi_h \in V_h$ such that $\varphi_h \to \varphi$ (strongly) in H^1_{per}, as h tends to 0.

For the convergence result as $h \to 0$, it will be useful to have h-dependent operators and norms. We denote $\mathcal{A}_h : V_h \to V_h$ the linear operator such that for any $q_h \in V_h$, $\mathcal{A}_h q_h$ solves

$$(\mathcal{A}_h q_h, \zeta_h) = (\nabla q_h, \nabla \zeta_h), \quad \forall \zeta_h \in V_h. \tag{3.17}$$

The operator $\mathcal{A}_h$ is a discrete Laplacian, $\mathcal{A}_h \simeq -\Delta_h$. Notice that if q_h is constant, then $\mathcal{A}_h q_h = 0$ so that $\mathcal{A}_h$ is not invertible. In order to define a discrete version of $\dot{\mathcal{A}}^{-1}$, we introduce the subspace

$$\dot{V}_h = \{\varphi_h \in V_h \; : \; \langle \varphi_h \rangle = 0\}.$$

The bilinear form $(\nabla\cdot,\nabla\cdot)$ is symmetric positive definite on $\dot{V}_h$. We can define the operator $\dot{\mathcal{S}}_h : \dot{V}_h \to \dot{V}_h$ such that for any $\dot{r}_h \in \dot{V}_h$, $\dot{\mathcal{S}}_h \dot{r}_h$ is the unique solution of

$$(\nabla\dot{\mathcal{S}}_h\dot{r}_h, \nabla\dot{\varphi}_h) = (\dot{r}_h, \dot{\varphi}_h),\ \forall\dot{\varphi}_h \in \dot{V}_h. \tag{3.18}$$

By choosing $\zeta_h \equiv 1$ in (3.17), we see that $\mathcal{A}_h(V_h) \subset \dot{V}_h$, so that the restriction $\dot{\mathcal{A}}_h : \dot{V}_h \to \dot{V}_h$ of $\mathcal{A}_h$ is well defined. Using (3.17)-(3.18), it is easily seen that $\dot{\mathcal{S}}_h = \dot{\mathcal{A}}_h^{-1}$.

We also define the L^2-orthogonal projector $P_h : L^2(\Omega) \to V_h$, i.e.,

$$(P_h q, \varphi_h) = (q, \varphi_h), \quad \forall q \in L^2(\Omega),\ \forall\varphi_h \in V_h.$$

By Pythagoras' theorem and assumption $(\mathcal{H}2)$, for any $q \in L^2(\Omega)$, we have

$$\|q - P_h q\| = \inf_{r_h\in V_h} \|q - r_h\| \to 0,\ \text{as } h \to 0. \tag{3.19}$$

Since $V_h \subset H^1_{per}$, the operator P_h has a natural extension to H^{-1}_{per} (also denoted P_h), by setting

$$P_h q \in V_h, \quad (P_h q, \varphi_h) = \langle q, \varphi_h\rangle_{H^{-1}_{per},H^1_{per}}, \quad \forall q \in H^{-1}_{per},\ \forall\varphi_h \in H^1_{per}.$$

The H^1-orthogonal projector $\Pi_h : H^1_{per} \to V_h$ is defined as follows: for any $q \in H^1_{per}$, $\Pi_h q \in V_h$ is uniquely defined by

$$\langle\Pi_h q\rangle = \langle q\rangle \quad \text{and} \quad (\nabla\Pi_h q, \nabla\varphi_h) = (\nabla q, \nabla\varphi_h), \quad \forall q \in H^1_{per},\ \forall\varphi_h \in V_h. \tag{3.20}$$

By Pythagora's theorem and assumption $(\mathcal{H}2)$, for any $q \in H^1_{per}$, we get

$$|q - \Pi_h q|_1 = \inf_{r_h\in V_h} |q - r_h|_1 \to 0,\ \text{as } h \to 0. \tag{3.21}$$

We point out that the space $\dot{V}_h$ is invariant by Π_h and by P_h.

By using a L^2-orthonormal basis $(e^i_h)_{1\le i\le N_h}$ of V_h as in Section 3.1, with $e^1_h \equiv 1$, we see that the matrix of $\mathcal{A}_h$ is A, so that the energy E_h from (3.10) can be rewritten

$$E_h(u_h) = \frac{1}{2}\|\mathcal{A}_h u_h\|^2 - |\dot{u}_h|_1^2 + (F(u_h), 1);$$

the norm $|\cdot|_{-1}$ from (3.9) becomes for any element $\dot{r}_h \in \dot{V}_h$

$$|\dot{r}_h|_{-1,h} := (\dot{r}_h, \dot{\mathcal{A}}_h^{-1}\dot{r}_h) = |\dot{\mathcal{A}}_h^{-1}\dot{r}_h|_1.$$

It is an Euclidean norm for the following scalar product on $\dot{V}_h$

$$(\dot{r}_h, \dot{q}_h)_{-1,h} := (\dot{r}_h, \dot{\mathcal{A}}_h^{-1}\dot{q}_h) = (\dot{\mathcal{A}}_h^{-1}\dot{r}_r, \dot{\mathcal{A}}_h^{-1}\dot{q}_h)_1.$$

Thus, the discrete pseudo-energy (3.11) reads

$$\mathcal{E}_h(u_h, v_h) := E_h(u_h) + \frac{\beta}{2}|\dot{v}_h|^2_{-1,h}, \quad \forall u_h, v_h \in V_h. \tag{3.22}$$

The Cauchy-Schwarz (3.12) inequality gives

$$|\dot{u}|_1^2 \le \frac{1}{4}\|\dot{\mathcal{A}}_h u_h\|^2 + \|u_h\|^2, \quad \forall u_h \in V_h, \tag{3.23}$$

and estimate (3.13) becomes

$$E_h(u_h) \ge \frac{1}{4}\|\mathcal{A}_h u_h\|^2 + \|u_h\|^2 - c_1, \quad \forall u_h \in V_h. \tag{3.24}$$

3.4. **Convergence as $h \to 0$.**

Theorem 3.3. *Let $(u_0, v_0) \in H^2_{per} \times H^{-1}_{per}$. Assume that $(u^0_h, v^0_h)_{h>0}$ is a family of functions in $V_h \times V_h$ such that*

$$u^0_h \to u_0 \text{ in } H^1_{per}, \quad \mathcal{A}_h u^0_h \to \mathcal{A} u_0 \text{ in } L^2(\Omega), \tag{3.25}$$

$$\langle v^0_h \rangle \to \langle v_0 \rangle \text{ in } \mathbb{R}, \quad \dot{\mathcal{A}}_h^{-1} \dot{v}^0_h \to \dot{\mathcal{A}}^{-1} \dot{v}_0 \text{ in } \dot{H}^1_{per}, \tag{3.26}$$

as $h \to 0$. Then the solution $(u_h, \partial_t u_h)$ of the space semi-discrete scheme (3.1)-(3.2) *tends to the energy solution (u, u_t) of* (1.1)-(1.2) *in the following sense:*

$$\begin{aligned} u_h &\to u \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; H^1_{per}), \\ u_h &\to u \text{ strongly in } C([0,T]; L^2(\Omega)), \text{ for all } T > 0, \\ \mathcal{A}_h u_h &\to \mathcal{A} u \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; L^2(\Omega)), \\ \dot{\mathcal{A}}_h^{-1} \partial_t \dot{u}_h &\to \dot{\mathcal{A}}^{-1} \partial_t \dot{u} \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; \dot{H}^1_{per}) \text{ and weakly in } L^2(\mathbb{R}_+; \dot{H}^1_{per}). \end{aligned}$$

Proof. The idea is to use a priori estimates on the mass and on the discrete energy, and to pass to the limit in the equation by a compactness argument. We first consider the conservation law for the mass. By (3.7), we get

$$a_h(t) := \langle \partial_t u_h(t) \rangle = \langle v^0_h \rangle e^{-t/\beta}, \quad t \geq 0,$$

and

$$\langle u_h(t) \rangle = \beta \langle v^0_h \rangle + \langle u^0_h \rangle - \beta a_h(t), \quad t \geq 0. \tag{3.27}$$

By assumption, $\langle u^0_h \rangle \to \langle u_0 \rangle$ and $\langle v^0_h \rangle \to \langle v_0 \rangle$ in $\mathbb{R}$, so that a_h converges uniformly on $\mathbb{R}_+$ to the function $a(t) := \langle v_0 \rangle e^{-t/\beta}$, and $\langle u_h \rangle$ converges uniformly on $\mathbb{R}_+$ to the function $\beta \langle v_0 \rangle + \langle u_0 \rangle - \beta a(t)$. The estimates below show that $(u_h)_{h>0}$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\Omega))$, so that, up to a subsequence, u_h converges weakly $\star$ in $L^\infty(\mathbb{R}_+; L^2(\Omega))$ to some u and so $\langle u_h \rangle \to \langle u \rangle$ weakly $\star$ in $L^\infty(\mathbb{R}_+)$. By uniqueness of the limit, we find

$$\langle u \rangle = \beta \langle v_0 \rangle + \langle u_0 \rangle - \beta a(t).$$

By differentiating, we recover the conservation law for the mass:

$$\beta \partial_{tt} \langle u \rangle + \partial_t \langle u \rangle = 0, \quad t \geq 0.$$

We now turn to the energy estimate. As pointed out in Section 3.1, the (unique) solution $(u_h, \partial_t u_h)$ of (3.1)-(3.2) is in fact a solution (u_h, v_h, z_h, w_h) of (3.1)-(3.2). In particular, $v_h = \partial_t u_h$ and $z_h = \mathcal{A}_h u_h$. We have (recall (3.22)):

$$\mathcal{E}(u^0_h, v^0_h) = \frac{1}{2} \|\mathcal{A}_h u^0_h\|^2 - |\dot{u}^0_h|_1^2 + (F(u^0_h), 1) + \frac{\beta}{2} |\dot{\mathcal{A}}_h^{-1} \dot{v}^0_h|_1^2.$$

By using assumptions (3.25)-(3.26) and the Sobolev injection $H^1_{per} \hookrightarrow L^{2p+2}(\Omega)$, we see that $\mathcal{E}(u^0_h, v^0_h)$ is uniformly bounded as h tends to 0. The energy estimate (3.15) shows that there exists a constant C independent of h such that

$$\mathcal{E}_h(u_h(t), \partial_t u_h(t)) + \int_0^t |\partial_t \dot{u}_h(s)|^2_{-1,h} ds \leq C,$$

for all $t \geq 0$. By (3.24) and (3.23), we obtain that $z_h = \mathcal{A}_h u_h$ and u_h are uniformly bounded in $L^2(\Omega)$, that $\dot{u}_h$ and $\dot{r}_h := \dot{\mathcal{A}}_h^{-1} \partial_t \dot{u}_h$ are uniformly bounded in $\dot{H}^1_{per}$, and that

$$\int_0^\infty |\dot{\mathcal{A}}_h^{-1} \partial_t \dot{u}_h(t)|_1^2 dt \leq C.$$

This implies that $(u_h)_{h>0}$ is precompact in the space $C([0,T];L^2(\Omega))$, for all $T>0$, as a consequence of the Ascoli-Arzelà Theorem. Indeed, let $T>0$. The family $(u_h)_{h>0}$ is uniformly bounded from $[0,T]$ with values in H^1_{per}, and H^1_{per} is compactly embedded into $L^2(\Omega)$ by Rellich's Theorem. Moreover, for all $0\le s\le t\le T$, we have

$$\begin{aligned}\|\dot u_h(t)-\dot u_h(s)\|^2 &= 2\int_s^t(\partial_t\dot u_h(\sigma),\dot u_h(\sigma)-\dot u_h(s))d\sigma\\ &= 2\int_s^t(\nabla\dot r_h(\sigma),\nabla[\dot u_h(\sigma)-\dot u_h(s)])d\sigma\\ &\le 4\|\dot r_h\|_{L^\infty(\mathbb{R}_+;H^1_{per})}\|\dot u_h\|_{L^\infty(\mathbb{R}_+;H^1_{per})}|t-s|. \end{aligned}\tag{3.28}$$

Moreover, by (3.27) and the mean value theorem, we find

$$|\langle u_h(t)\rangle-\langle u_h(s)\rangle|\le|\langle v_h^0\rangle||t-s|.$$

Thus, $(u_h)_{h>0}$ is uniformly equicontinuous from $[0,T]$ with values into $L^2(\Omega)$, and therefore pre-compact in $C([0,T];L^2(\Omega))$, as claimed. Up to a subsequence, we have the following convergence results

$$\begin{aligned} u_h &\to u \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+;H^1_{per}),\\ u_h &\to u \text{ strongly in } C([0,T];L^2(\Omega)),\ \text{for all } T>0,\\ u_h &\to u \text{ a.e. in } \Omega\times\mathbb{R}_+,\\ f(u_h) &\to f(u) \text{ weakly in } L^q(0,T;L^q(\Omega)),\ \text{for all } T>0,\\ z_h &\to z \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+;L^2(\Omega)),\\ \dot r_h &\to \dot r \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+;\dot H^1_{per}),\\ \dot r_h &\to \dot r \text{ weakly in } L^2(\mathbb{R}_+;\dot H^1_{per}),\end{aligned}$$

where $q=(2p+2)/(2p+1)>1$. Let now $\dot\psi\in\dot H^1_{per}$ and let $\dot\psi_h=\Pi_h(\dot\psi)$ so that $\dot\psi_h\to\dot\psi$ strongly in $\dot H^1_{per}$. We have

$$(\partial_t\dot u_h,\dot\psi_h)=(\dot{\mathcal{A}}_h\dot r_h,\dot\psi_h)=(\nabla\dot r_h,\nabla\dot\psi_h)\to(\dot r,\dot\psi)$$

weakly $\star$ in $L^\infty(\mathbb{R}_+)$. On the other hand, $\partial_t(\dot u_h,\dot\psi_h)\to\partial_t(\dot u,\dot\psi)$ in $\mathcal{D}'(0,\infty)$ (i.e. in the sense of distributions), since $(\dot u_h,\dot\psi_h)\to(\dot u,\dot\psi)$ in $L^\infty(\mathbb{R}_+)$ weakly $\star$. Thus,

$$\partial_t(\dot u,\dot\psi)=(\nabla\dot r,\nabla\dot\psi)=\langle\dot v,\dot\psi\rangle_{H^{-1}_{per},H^1_{per}},\tag{3.29}$$

with $\dot v=\dot{\mathcal{A}}\dot r\in L^\infty(\mathbb{R}_+;\dot H^{-1}_{per})$. This shows that $\partial_t\dot u=\dot v\in L^\infty(\mathbb{R}_+;\dot H^{-1}_{per})$.

Next, we set $\varphi\in H^2_{per}$ and we let $\varphi_h=\Pi_h(\varphi)$, so that $\varphi_h\to\varphi$ strongly in H^1_{per}. Let $(e_h^i)_{1\le i\le N_h}$ be an orthonormal basis of V_h with $e_h^1\equiv 1$. We let $\varphi_h=\sum_{i=1}^{N_h}\varphi_i e_h^i$ and $\Phi=(\varphi_1,\dots,\varphi_{N_h})^t$ be the vector associated to φ_h, as in Section 3.1. On multiplying (3.8) by $\dot\Phi^t\dot A^{-1}$ and using $z_h=\mathcal{A}_h u_h$, we find

$$\beta(\partial_{tt}\dot u_h,\dot\varphi_h)_{-1,h}+(\partial_t\dot u_h,\dot\varphi_h)_{-1,h}+(\nabla z_h,\nabla\varphi_h)-2(\nabla u_h,\nabla\varphi_h)+(f(u_h),\dot\varphi_h)=0,\tag{3.30}$$

for all $t\ge 0$. We have that

$$(\partial_t\dot u_h,\dot\varphi_h)_{-1,h}=(\dot r_h,\dot\varphi_h)\to(\dot r,\dot\varphi)=(\partial_t\dot u,\dot\varphi)_{-1},$$

by (3.29). The convergence above holds in $L^\infty(\mathbb{R}_+)$ weak $\star$, so that

$$(\partial_{tt}\dot u_h, \dot\varphi_h)_{-1,h} = \partial_t(\partial_t \dot u_h, \dot\varphi_h)_{-1,h} \to \partial_t(\partial_t\dot u, \dot\varphi)_{-1} \text{ in } \mathcal{D}'(0,\infty).$$

Moreover,

$$(\nabla u_h, \nabla\varphi_h) \to (\nabla u, \nabla\varphi) \text{ in } L^\infty(\mathbb{R}_+) \text{ weak } \star.$$

Since $\varphi_h = \Pi_h\varphi$, we have

$$(\nabla z_h, \nabla\varphi_h) = (\nabla z_h, \nabla\varphi) = (z_h, \mathcal{A}\varphi) \to (z, \mathcal{A}\varphi),$$

in $L^\infty(\mathbb{R}_+)$ weak $\star$. Concerning the last term in (3.30), we have

$$(f(u_h), \dot\varphi_h) \to (f(u), \dot\varphi) \quad \text{weakly in } L^q(0,T), \quad \forall T > 0.$$

Summing up, we have proved that

$$\beta\partial_t(\partial_t\dot u, \dot\varphi)_{-1} + (\partial_t\dot u, \dot\varphi)_{-1} + (z, \mathcal{A}\varphi) - 2(\nabla u, \nabla\varphi) + (f(u), \varphi) = \langle f(u)\rangle\langle\varphi\rangle. \quad (3.31)$$

The equality holds in $\mathcal{D}'(0,\infty)$, for all $\varphi \in H^2_{per}$. Moreover, $\partial_t\dot u \in L^\infty(\mathbb{R}_+; H^{-1}_{per})$, $z \in L^\infty(\mathbb{R}_+; L^2(\Omega))$, $u \in L^\infty(\mathbb{R}_+; H^1_{per})$ and $f(u) \in L^\infty(\mathbb{R}_+; H^{-1}_{per})$ so that $\partial_{tt}u$ belongs to $L^\infty(\mathbb{R}_+; H^{-4}_{per})$ and

$$\dot{\mathcal{A}}^{-1}(\beta\partial_{tt}\dot u + \partial_t\dot u) + \mathcal{A}z - 2\mathcal{A}u + f(u) - \langle f(u)\rangle = 0$$

in $\dot H^{-2}_{per}$, a.e. in $\mathbb{R}_+$. Now, recall that $z_h = \mathcal{A}_h u_h$. Let $\zeta \in H^1_{per}$ and $\zeta_h = \Pi_h\zeta$, so that $\zeta_h \to \zeta$ strongly in H^1_{per}. We have

$$(z_h, \zeta_h) = (\mathcal{A}_h u_h, \zeta_h) = (\nabla u_h, \nabla\zeta_h) \to (\nabla u, \nabla\zeta),$$

on one hand, and $(z_h, \zeta_h) \to (z, \zeta)$, on the other hand. Thus, we deduce

$$(z, \zeta) = (\nabla u, \nabla\zeta),$$

in $L^\infty(\mathbb{R}_+)$. The equality holds for every $\zeta \in H^1_{per}$, so $z = \mathcal{A}u$, $u \in L^\infty(\mathbb{R}_+; H^2_{per})$ and (u, u_t) is an energy solution of (1.1)-(1.2). By uniqueness of the limit (u, u_t), the whole family $(u_h, \partial_t u_h)$ converges to (u, u_t). □

Remark 3.4. Let $(u_0, v_0) \in H^2_{per} \times H^{-1}_{per}$. If $u_h^0 = \Pi_h(u_0)$ and $v_h^0 = P_h(v_0)$, then assumptions (3.25)-(3.26) are satisfied. Indeed, for all $\varphi_h \in V_h$,

$$(\mathcal{A}u_0, \varphi_h) = (\nabla u_0, \nabla\varphi_h) = (\nabla\Pi_h u_0, \nabla\varphi_h) = (\mathcal{A}_h(\Pi_h u_0), \varphi_h).$$

Then $\mathcal{A}_h(\Pi_h u_0) = P_h(\mathcal{A}u_0)$ and by (3.19) we obtain

$$\mathcal{A}_h(\Pi_h u_0) \to \mathcal{A}u_0 \text{ in } L^2(\Omega), \text{ as } h \to 0.$$

By definition, observe that

$$\langle P_h(v_0)\rangle = (P_h(v_0), 1) = \langle v_0, 1\rangle_{H^{-1}_{per}, H^1_{per}} = \langle v_0\rangle.$$

Moreover, for all $\varphi_h \in V_h$, we have

$$\begin{aligned}(P_h(\dot v_0), \dot\varphi_h) &= \langle \dot v_0, \dot\varphi_h\rangle_{H^{-1}_{per}, H^1_{per}} = (\nabla\dot{\mathcal{A}}^{-1}\dot v_0, \nabla\dot\varphi_h) \\ &= (\nabla\Pi_h(\dot{\mathcal{A}}^{-1}\dot v_0), \nabla\dot\varphi_h) = (\dot{\mathcal{A}}_h(\Pi_h(\dot{\mathcal{A}}^{-1}\dot v_0)), \dot\varphi_h),\end{aligned}$$

so that

$$P_h(\dot v_0) = \dot{\mathcal{A}}_h(\Pi_h(\dot{\mathcal{A}}^{-1}\dot v_0)).$$

Thus, thanks to (3.21), we deduce

$$\dot{\mathcal{A}}_h^{-1}(P_h(\dot v_0)) = \Pi_h(\dot{\mathcal{A}}^{-1}\dot v_0) \to \dot{\mathcal{A}}^{-1}\dot v_0 \text{ in } H^1_{per}.$$

4. The fully discrete problem

4.1. **The fully discrete scheme.** For the time discretization, we use the following decomposition:

(H3) $F = F_+ + F_-$, where F_+ and F_- are polynomials such that $F_+^{(iv)} \geq 0$, $F_-^{(iv)} \leq 0$, and $deg(F_-) < deg(F)$ (here, deg denotes the degree of the polynomial).

As a consequence, $deg(F_+) = deg(F)$ and F_+, F have the same leading coefficient. We denote $f = F' = f_+ + f_-$, where $f_+ = F_+'$ and $f_- = F_-'$. For the energy estimate, we will use the fact that there exist two constants $c_5 > 0$ and $c_6 \geq 0$ which depend only on f and on the decomposition $f = f_+ + f_-$ such that

$$\frac{1}{2}(|f(r)| + |f(s)|) + \frac{1}{12}(s-r)^2(|f_+''(r)| + |f_-''(s)|) \leq c_5(F(r) + F(s)) + c_6, \quad (4.1)$$

for all $r, s \in \mathbb{R}$.

Remark 4.1. A decomposition (H3) is always possible for a polynomial potential such as (2.2). Indeed, for a quartic polynomial (for instance (2.6)), we can always choose $F_+ = F$ and $F_- = 0$. For a polynomial with higher degree, we notice that $F^{(iv)}$, being a polynomial of even degree with strictly positive leading coefficient, is bounded from below, i.e.

$$F^{(iv)}(s) \geq -c_4 \quad \forall s \in \mathbb{R},$$

for some constant $c_4 \geq 0$. A possible (but not unique !) choice is then $F_+^{(iv)} = F^{(iv)} + c_4$ and $F_-^{(iv)} = -c_4$, i.e. $F_+(s) = F(s) + c_4 s^4/24$ and $F_-(s) = -c_4 s^4/24$.

We use the same notation as in Section 3. In particular, V_h is a family of finite-dimensional subspaces of H^1_{per} which satisfies assumptions (H1)-(H2).

Let $\tau > 0$ denote the time step, and (u_h^0, v_h^0) in $V_h \times V_h$ the initial datum. The fully discrete scheme reads: for $n \geq 0$, find $(u^{n+1}, v^{n+1}, z^{n+1}, w^{n+1}) \in (V_h)^4$ such that

$$\begin{cases} ((u_h^{n+1} - u_h^n)/\tau, \varphi_h) = (v_h^{n+1/2}, \varphi_h) \\ \beta((v_h^{n+1} - v_h^n)/\tau, \psi_h) = -(v_h^{n+1/2}, \psi_h) - (\nabla w_h^{n+1}, \nabla \psi_h) \\ (z_h^{n+1}, \zeta_h) = (\nabla u_h^{n+1/2}, \nabla \zeta_h) \\ (w_h^{n+1}, \xi_h) = (\nabla z_h^{n+1}, \nabla \xi_h) - 2(\nabla u_h^{n+1/2}, \nabla \xi_h) + ((f(u_h^n) + f(u_h^{n+1}))/2, \xi_h) \\ \qquad -\dfrac{1}{12}\left((u_h^{n+1} - u_h^n)^2 \left(f_+''(u_h^n) + f_-''(u_h^{n+1})\right), \xi_h\right), \end{cases} \quad (4.2)$$

for all φ_h, ψ_h, ζ_h, ξ_h in V_h. Here, we have denoted

$$u_h^{n+1/2} = (u_h^{n+1} + u_h^n)/2 \quad \text{and} \quad v_h^{n+1/2} = (v_h^{n+1} + v_h^n)/2.$$

Notice that z_h^0 and w_h^0 do not need to be defined. In fact, z_h^{n+1} (resp. (w_h^{n+1})) is a second-order (in time) approximation of $z_h(t_{n+1/2})$ (resp. $w_h(t_{n+1/2})$), where $t_{n+1/2} = (n + 1/2)\tau$.

Let $(e_h^i)_{1 \leq i \leq N_h}$ be an L^2-orthonormal basis of V_h, with $e_h^1 \equiv 1$, so that we have the identification $V_h \ni u_h \simeq U \in \mathbb{R}^{N_h}$. In $\mathbb{R}^{N_h}$, the scheme reads: let U^0, V^0 in $\mathbb{R}^{N_h}$

and for $n \geq 0$ find $(U^{n+1}, V^{n+1}, Z^{n+1}, W^{n+1}) \in (\mathbb{R}^{N_h})^4$ which solves

$$\begin{cases} (U^{n+1} - U^n)/\tau = V^{n+1/2} \\ \beta(V^{n+1} - V^n)/\tau = -V^{n+1/2} - AW^{n+1} \\ Z^{n+1} = AU^{n+1/2} \\ W^{n+1} = AZ^{n+1} - 2AU^{n+1/2} + \left(\nabla F_h(U^n) + \nabla F_h(U^{n+1})\right)/2 \\ \qquad\qquad -G(U^n, U^{n+1}), \end{cases} \tag{4.3}$$

where

$$G(U^n, U^{n+1}) = \frac{1}{12}\left(\left((u_h^{n+1} - u_h^n)^2 \left(f''_+(u_h^n) + f''_-(u_h^{n+1})\right), e_h^i\right)\right)_{1 \leq i \leq N_h}. \tag{4.4}$$

On eliminating Z^{n+1} and W^{n+1}, the scheme becomes

$$\begin{cases} (U^{n+1} - U^n)/\tau - V^{n+1/2} = 0 \\ \beta(V^{n+1} - V^n)/\tau + V^{n+1/2} + A\left(A^2U^{n+1/2} - 2AU^{n+1/2}\right) \\ \qquad +A\left(\left(\nabla F_h(U^n) + \nabla F_h(U^{n+1})\right)/2 - G(U^n, U^{n+1})\right) = 0. \end{cases} \tag{4.5}$$

In Section 6.1, a numerical example indicates that our fully discrete scheme (4.2) has a second order *convergence error* in time (and also in space if V_h is the space of P^1 finite elements). By arguing as in Gomez and Hughes [18], we check here that:

Proposition 4.2. *The scheme has a second order consistency error in time, i.e. that any solution of the space semi-discrete problem* (3.4) *satisfies the fully discrete scheme* (4.5) *with order* $O(\tau^2)$.

Proof. Let (U, V) be a solution of (3.4) on a finite time interval $[0, T]$. Since f is a polynomial, by a bootstrap argument, we know that $(U, V) \in C^\infty([0, T]; V_h \times V_h)$. In the time discrete scheme (4.5), we replace U^n, U^{n+1}, V^n, V^{n+1} by $U(t_n)$, $U(t_{n+1})$, $V(t_n)$, $V(t_{n+1})$ respectively. The purpose of this replacement is to find a local truncation error (or consistency error) in $O(\tau^2)$ in the right-hand side of (4.5), instead of $(0, 0)^t$. Now consider the midpoint scheme, which is the same as (4.5) without the term $G(U^n, U^{n+1})$. Standard results show that the midpoint scheme has a second order consistency error, so that it is sufficient to show that $G(U(t_n), U(t_{n+1})) = O(\tau^2)$. This is obvious by definition (4.4). Indeed, using the assumption on p and appropriate Hölder inequalities and Sobolev injections, we have

$$|G(U(t_n), U(t_{n+1}))| \leq C\|u(t_{n+1}) - u(t_n)\|_1^2.$$

Moreover, by Taylor expansion, $\|(u_h(t_{n+1}) - u_h(t_n))^2\|_1 = O(\tau^2)$, so the local truncation error is $O(\tau^2)$ as well. Notice that the constant in the consistency error depends on h, on T, and on maximum norms of derivatives of U, V up to order 3. ☐

4.2. **Existence, discrete energy estimate and uniqueness.** Let us prove the following

Theorem 4.3 (Existence for any τ)**.** *For any* $(u_h^0, v_h^0) \in V_h \times V_h$*, there exists at least one sequence* $(u_h^n, v_h^n, z_h^n, w_h^n)_{n \geq 1}$ *in* $(V_h)^4$ *which complies with* (4.2).

Proof. We work with the $\mathbb{R}^{N_h}$ version (4.5). We will show that this problem is variational, and that we can find U^{n+1} by a minimization procedure. Let (U^n, V^n) be fixed in $\mathbb{R}^{N_h}$. Consider the polynomial of two variables

$$g(r, s) = \frac{1}{12}(s - r)^2(f''_+(r) + f''_-(s)) \quad (r, s \in \mathbb{R}).$$

By assumption (H3), we have $deg(f_-) < deg(f)$ and $deg(f_+) = deg(f)$ so $deg(f''_-) < deg(f) - 2$, and $deg(f''_+) = deg(f) - 2$. Thus, g is a polynomial of total degree less than or equal to $2p+1$, and its partial degree with respect to the variable s is strictly less than $2p+1$. We can write

$$g(r,s) = \sum_{k,l} b_{k,l} r^k s^l, \tag{4.6}$$

for coefficients $b_{k,l} \in \mathbb{R}$, where $0 \le k \le 2p+1$ and $0 \le l < 2p+1$. Let us set now

$$h(r,s) = \sum_{k,l} b_{k,l} r^k \frac{s^{l+1}}{l+1}, \tag{4.7}$$

so that $\partial_s h(r,s) = g(r,s)$. We define $H_h^n(U) = (h(u_h^n, u_h), 1)$ with $u_h \simeq U$, so that

$$\nabla H_h^n(U) = (g(u_h^n, u_h), e_i)_{1 \le i \le N_h} = G(U^n, U).$$

By (4.7) and Hölder's inequality, we get

$$|H_h^n(U)| \le C_n \left(\|u_h\|_{L^{2p+2}(\Omega)}^{2p+1} + 1 \right) \quad \forall U \in \mathbb{R}^{N_h}, \tag{4.8}$$

where the constant C_n depends on $\|u_h^n\|_{L^{2p+2}(\Omega)}$. Now, by eliminating V^{n+1}, we find that (4.5) is equivalent to

$$\begin{aligned} \frac{\beta}{\tau}\left(\frac{2(U^{n+1}-U^n)}{\tau} - V^n\right) + \frac{U^{n+1}-U^n}{\tau} + A\Bigg[A^2\frac{(U^{n+1}+U^n)}{2} \\ -2A\frac{(U^{n+1}+U^n)}{2} + \frac{\nabla F_h(U^n)+\nabla F_h(U^{n+1})}{2} - \nabla H_h^n(U^{n+1})\Bigg] \quad = \quad 0. \end{aligned}$$

Writing $U = (u_1, \dot{U})$, we see that this is equivalent to

$$\frac{\beta}{\tau}\left(\frac{2(u_1^{n+1}-u_1^n)}{\tau} - v_1^n\right) + \frac{u_1^{n+1}-u_1^n}{\tau} = 0, \tag{4.9}$$

$$\begin{aligned} \frac{\beta \dot{A}^{-1}}{\tau}\left(\frac{2(\dot{U}^{n+1}-\dot{U}^n)}{\tau} - \dot{V}^n\right) + \dot{A}^{-1}\frac{(\dot{U}^{n+1}-\dot{U}^n)}{\tau} + \dot{A}^2\frac{(\dot{U}^{n+1}+\dot{U}^n)}{2} \\ - 2\dot{A}\frac{(\dot{U}^{n+1}+\dot{U}^n)}{2} + \frac{\dot{\nabla} F_h(U^n)+\dot{\nabla} F_h(U^{n+1})}{2} - \dot{\nabla} H_h^n(U^{n+1}) = 0. \end{aligned} \tag{4.10}$$

The first equation determines u_1^{n+1} uniquely. The second equation can be solved by letting $\dot{U}^{n+1}$ be a minimizer on $\mathbb{R}^{N_h-1}$ of the function:

$$\begin{aligned} \mathcal{G} : \dot{U} \quad \mapsto \quad & \frac{\beta}{\tau^2}|\dot{U}-\dot{U}^n|_{-1}^2 - \frac{\beta}{\tau}(\dot{V}^n)^t \dot{A}^{-1}\dot{U} + \frac{1}{2\tau}|\dot{U}-\dot{U}^n|_{-1}^2 + \frac{1}{4}|\dot{A}(\dot{U}+\dot{U}^n)|^2 \\ & -\frac{1}{2}|\dot{A}^{1/2}(\dot{U}+\dot{U}^n)|^2 + \frac{(\dot{\nabla} F_h(U^n))^t}{2}\dot{U} + \frac{\tilde{F}_h^n(\dot{U})}{2} - \tilde{H}_h^n(\dot{U}), \end{aligned}$$

where

$$\tilde{F}_h^n(\dot{U}) = F_h(u_1^{n+1}, \dot{U}), \qquad \tilde{H}_h^n(\dot{U}) = H_h^n(u_1^{n+1}, \dot{U}).$$

By (2.2), we deduce

$$\tilde{F}_h^n(\dot{U}) \ge \frac{a_{2p+1}}{2p+2}\|u_h\|_{L^{2p+2}(\Omega)}^{2p+2} - C'_p(\|u_h\|_{L^{2p+2}(\Omega)}^{2p+1} + 1), \quad \forall \dot{U} \in \mathbb{R}^{N_h-1},$$

where the constant C'_p depends only on the coefficients of F. Thus, by (4.8), we find

$$\mathcal{G}(\dot{U}) \geq \frac{a_{2p+1}}{2(2p+2)}\|u_h\|_{L^{2p+2}(\Omega)}^{2p+2} - (C_n + \frac{C'_p}{2})\|u_h\|_{L^{2p+2}(\Omega)}^{2p+1} + c\|\dot{u}_h\|^2 - c'\|\dot{u}_h\| - c'',$$

where $c > 0$ and $c', c'' \geq 0$ depend on h, F and u_h^n. For the quadratic term, we used that all norms are equivalent in $\mathbb{R}^{N_h-1}$. As a consequence, $\mathcal{G}(\dot{U}) \to +\infty$ as $|\dot{U}| \to +\infty$: the continuous function $\mathcal{G}$ has a minimizer in $\mathbb{R}^{N_h-1}$, and the proof is complete. □

The behavior of (u_1^n, v_1^n) is straightforward, thanks to a discrete conservation law for the mass. Indeed, choosing $\psi = 1$ in the second equation of (4.2), we find

$$v_1^{n+1} = qv_1^n, \quad \text{with} \quad q = q(\beta,\tau) = \frac{2\beta - \tau}{2\beta + \tau}. \tag{4.11}$$

Thus, we obtain

$$v_1^n = q^n v_1^0. \tag{4.12}$$

We also have $v^{n+1/2} = qv^{n-1/2}$, so that $v^{n+1/2} = q^n v^{1/2}$. Notice that $|q| < 1$, since $\beta > 0$ and $\tau > 0$, so that $v_1^n \to 0$. If $\tau > 2\beta$, then $q < 0$.

On choosing $\varphi = 1$ in the first equation of (4.2), we find

$$u_1^{n+1} = u_1^n + \tau v_1^{n+1/2}.$$

By induction, we deduce

$$u_1^n = u_1^0 + \tau \left(\sum_{k=0}^{n-1} q^k\right) v_1^{1/2} = u_1^0 + \tau \frac{1-q^n}{1-q} v_1^{1/2}. \tag{4.13}$$

For the energy estimate, we will need a technical lemma, adapted from [18]:

Lemma 4.4. *Let $g \in C^3([0,1];\mathbb{R})$. Then the following identities hold*

$$\int_0^1 g(s)ds = \frac{1}{2}(g(0)+g(1)) - \frac{1}{12}g''(0) - \frac{1}{2}\int_0^1 k_2^+(\sigma)g^{(3)}(\sigma)d\sigma, \tag{4.14}$$

$$\int_0^1 g(s)ds = \frac{1}{2}(g(0)+g(1)) - \frac{1}{12}g''(1) + \frac{1}{2}\int_0^1 k_2^-(\sigma)g^{(3)}(\sigma)d\sigma, \tag{4.15}$$

where $k_2^+(\sigma) = (1-\sigma)^2(2\sigma+1)/6$ and $k_2^-(\sigma) = \sigma^2(3-2\sigma)/6$. In particular, $k_2^+(\sigma) \geq 0$ and $k_2^-(\sigma) \geq 0$ for all $\sigma \in [0,1]$.

Proof. We prove (4.15) (the proof of (4.14) is similar). For a function $\varphi \in C^2([0,1])$, let

$$Err(\varphi) = \int_0^1 \varphi(s)ds - \left(\frac{1}{2}(\varphi(0)+\varphi(1)) - \frac{1}{12}\varphi''(1)\right)$$

denote the error of the quadrature formula. If φ is a polynomial of degree ≤ 2, a direct computation shows that $Err(\varphi) = 0$. Now, let $g \in C^3([0,1])$. The Taylor formula of order 2 at $s = 0$ reads

$$g(s) = p_2(s) + \frac{1}{2}\int_0^1 (s-\sigma)_+^2 g^{(3)}(\sigma)d\sigma,$$

with $p_2(s) = g(0) + sg'(0) + s^2g''(0)/2$ and

$$(s-\sigma)_+ = \begin{cases} s-\sigma & \text{if } s \geq \sigma \\ 0 & \text{if } s \leq \sigma. \end{cases}$$

In particular, $Err(p_2) = 0$. By linearity of Err,

$$Err(g) \;=\; Err(p_2) + Err\left(s \mapsto \frac{1}{2}\int_0^1 (s-\sigma)_+^2 g^{(3)}(\sigma)d\sigma\right).$$

By inverting the integration signs, we obtain

$$Err(g) = \frac{1}{2}\int_0^1 k_2^-(\sigma) g^{(3)}(\sigma)d\sigma,$$

where $k_2^-(s) = Err(s \mapsto (s-\sigma)_+^2)$. Using the definition of Err, we find that for $\sigma \in [0,1)$,

$$\begin{aligned} k_2^-(\sigma) &= \int_0^1 (s-\sigma)_+^2 ds - \left(\frac{1}{2}[(0-\sigma)_+^2 + (1-\sigma)_+^2] - \frac{2}{12}\right), \\ &= \sigma^2/2 - \sigma^3/3. \end{aligned}$$

The claim is proved. □

We have (compare with Lemma 3.1):

Lemma 4.5 (Energy estimate for any τ)**.** *If $(U^n, V^n, Z^n, W^n)_{n\geq 1}$ is a sequence in $(\mathbb{R}^{N_h})^4$ which complies with (4.3), then for all $n \geq 0$,*

$$\mathcal{E}_h(U^{n+1}, V^{n+1}) - \mathcal{E}_h(U^n, V^n) + \tau|\dot{V}^{n+1/2}|_{-1}^2 \leq \tau v_1^{n+1/2} w_1^{n+1}. \tag{4.16}$$

As a consequence, for all $k \geq 0$, we have

$$\begin{aligned} &\mathcal{E}_h(U^{N_0+k}, V^{N_0+k}) + \sum_{j=0}^{k-1} \tau|\dot{V}^{N_0+j+1/2}|_{-1}^2 \\ &\leq \exp\left(16c_5\frac{\tau|q|^{N_0}}{1-|q|}|v_1^{1/2}|\right)\left(\mathcal{E}_h(U^{N_0}, V^{N_0}) + c_7\frac{\tau|q|^{N_0}}{1-|q|}|v_1^{1/2}|\right), \end{aligned} \tag{4.17}$$

where $N_0 = N_0(\beta, c_5, \tau, |v_1^0|) \in \mathbb{N}$ is such that

$$2c_5\tau|q|^{N_0}|v_1^{1/2}| \leq 1/2, \tag{4.18}$$

$c_7 = 2c_1c_5 + c_6$ depends only on f, f_+, f_- (see (2.4), (4.1)), and q is defined by (4.11).

Proof. Let $\delta u_h^n = u_h^{n+1} - u_h^n$. Since $f_+ = F_+'$ and $f_- = F_-'$, we have

$$F_+(u_h^{n+1}) - F_+(u_h^n) = \delta u_h^n \int_0^1 f_+(u_h^n + s\delta u_h^n)ds, \tag{4.19}$$

$$F_-(u_h^{n+1}) - F_-(u_h^n) = \delta u_h^n \int_0^1 f_-(u_h^n + s\delta u_h^n)ds. \tag{4.20}$$

Choosing $g(s) = f_+(u_h^n + s\delta u_h^n)$ in (4.14), we find

$$\begin{aligned} \int_0^1 f_+(u_h^n + s\delta u_h^n)ds \;&=\; \frac{1}{2}(f_+(u_h^n) + f_+(u_h^{n+1})) - \frac{(\delta u_h^n)^2}{12} f_+''(u_h^n) \\ &\quad -\frac{(\delta u_h^n)^3}{2}\int_0^1 k_2^+(\sigma) f_+'''(u_h^n + \sigma\delta u_h^n)d\sigma. \end{aligned} \tag{4.21}$$

Setting $g(s) = f_-(u_h^n + s\delta u_h^n)$ in (4.15), we find

$$\int_0^1 f_-(u_h^n + s\delta u_h^n)ds = \frac{1}{2}(f_-(u_h^n) + f_-(u_h^{n+1})) - \frac{(\delta u_h^n)^2}{12} f_-''(u_h^{n+1}) + \frac{(\delta u_h^n)^3}{2}\int_0^1 k_2^-(\sigma) f_-'''(u_h^n + \sigma\delta u_h^n)d\sigma. \tag{4.22}$$

Adding (4.19) and (4.20) leads to

$$F(u_h^{n+1}) - F(u_h^n) = \delta u_h^n \Big[\frac{1}{2}(f(u_h^n) + f(u_h^{n+1})) - \frac{(\delta u_h^n)^2}{12}(f_+''(u_h^n) + f_-''(u_h^{n+1}))\Big] - \alpha^n,$$

where

$$\alpha^n = \frac{(\delta u_h^n)^4}{2}\left(\int_0^1 k_2^+(\sigma) f_+'''(u_h^n + \sigma\delta u_h^n)d\sigma - \int_0^1 k_2^-(\sigma) f_-'''(u_h^n + \sigma\delta u_h^n)d\sigma\right) \geq 0.$$

by assumption (H3) on the decomposition. Next, we choose $\xi_h = \delta u_h^n$ in the last equation of (4.2). This gives

$$(w_h^{n+1}, \delta u_h^n) - (\nabla z_h^{n+1}, \nabla \delta u_h^n) + 2(\nabla u_h^{n+1/2}, \nabla \delta u_h^n) = F(u_h^{n+1}) - F(u_h^n) + (\alpha^n, 1).$$

Using the vector form with $\delta U^n = U^{n+1} - U^n$, and eliminating z_h^{n+1}, we obtain

$$F_h(U^{n+1}) - F_h(U^n) + (\alpha^n, 1) = (W^{n+1})^t \delta U^n - \frac{1}{2}(|AU^{n+1}|^2 - |AU^n|^2) + |A^{1/2}U^{n+1}|^2 - |A^{1/2}U^n|^2. \tag{4.23}$$

The second equation in (4.2) implies

$$-\dot{W} = \dot{A}^{-1}\left(\beta\frac{(\dot{V}^{n+1} - \dot{V}^n)}{\tau} + \dot{V}^{n+1/2}\right).$$

Plugging this in (4.23), together with $\delta U^n = \tau V^{n+1/2}$, we get

$$E_h(U^{n+1}) - E_h(U^n) + (\alpha^n, 1) + \frac{\beta}{2}\left(|\dot{V}^{n+1}|_{-1}^2 - |\dot{V}^n|_{-1}^2\right) + \tau|\dot{V}^{n+1/2}|_{-1}^2 = \tau v_1^{n+1/2} w_1^{n+1}.$$

This yields the energy estimate (4.16).

Choosing $r = 1$ in the last equation of (4.2), we find

$$w_1^{n+1} = \frac{1}{2}(f(u_h^n) + f(u_h^{n+1}), 1) - \frac{1}{12}\left((u_h^{n+1} - u_h^n)^2\left(f_+''(u_h^n) + f_-''(u_h^{n+1})\right), 1\right).$$

Thus, by (4.1), we deduce

$$\begin{aligned} |w_1^{n+1}| &\leq \big(\frac{1}{2}(|f(u_h^n)| + |f(u_h^{n+1})|) + \frac{1}{12}((u_h^{n+1} - u_h^n)^2(|f_+''(u_h^n)| + |f_-''(u_h^{n+1})|), 1)\big) \\ &\leq \big(c_5(F(u_h^n) + F(u_h^{n+1})) + c_6, 1\big), \end{aligned}$$

As a consequence, by (3.16), we get

$$\begin{aligned}\mathcal{E}_h(U^{n+1},V^{n+1}) - \mathcal{E}_h(U^n,V^n) + \tau|\dot V^{n+1/2}|_{-1}^2 \\ \leq \tau|v_1^{n+1/2}|\left(c_5(F(u_h^n),1) + c_5(F(u_h^{n+1}),1) + c_6\right) \\ \leq \tau|v_1^{n+1/2}|\left(2c_5E_h(U^{n+1}) + 2c_5E_h(U^n) + 2c_5c_1 + c_6\right)\end{aligned}$$

Let us set

$$\mathcal{E}_h^n = E_h(U^n) + \frac{\beta}{2}|\dot V^n|_{-1}^2.$$

So far, we have proved that

$$\mathcal{E}_h^{n+1} + \tau|\dot V^{n+1/2}|_{-1}^2 \leq \mathcal{E}_h^n + \tau|q|^n|v_1^{1/2}|\big(2c_5\mathcal{E}_h^n + 2c_5\mathcal{E}_h^{n+1} + c_7\big),$$

where $c_7 = 2c_5c_1 + c_6$. Let $N_0 = N_0(\beta, c_5, \tau, |v_1^0|) \in \mathbb{N}$ satisfy (4.18). Then for $n \geq N_0$, we have

$$(1 - 2c_5\tau|q|^n|v_1^{1/2}|)\mathcal{E}_h^{n+1} + \tau|\dot V^{n+1/2}|_{-1}^2 \leq (1 + 2c_5\tau|q|^n|v_1^{1/2}|)\mathcal{E}_h^n + c_7\tau|q|^n|v_1^{1/2}|.$$

We divide by this inequality $(1 - 2c_5\tau|q|^n|v_1^{1/2}|)$ and we use that (by the mean value inequality) for all $x \in [0, 1/2]$,

$$1 \leq \frac{1}{1-x} \quad \text{and} \quad \frac{1+x}{1-x} \leq 1 + 8x \leq \exp(8x).$$

We obtain

$$\mathcal{E}_h^{n+1} + \tau|\dot V^{n+1/2}|_{-1}^2 \leq \exp(16c_5\tau|q|^n|v_1^{1/2}|)\left(\mathcal{E}_h^n + c_7\tau|q|^n|v_1^{1/2}|\right),$$

for all $n \geq N_0$. By induction, for all $k \in \mathbb{N}$, we deduce

$$\begin{aligned}\mathcal{E}_h^{N_0+k} + \sum_{j=0}^{k-1}\tau|\dot V^{N_0+j+1/2}|_{-1}^2 \leq \exp\left(16c_5\tau|v_1^{1/2}|\sum_{j=0}^{k-1}|q|^{N_0+j}\right)\mathcal{E}_h^{N_0} \\ + \sum_{j=0}^{k-1}\exp\left(16c_5\tau|v_1^{1/2}|(|q|^{N_0} + \cdots + |q|^{N_0+k-1-j})\right)c_7\tau|q|^{N_0+j}|v_1^{1/2}|.\end{aligned}$$

Estimate (4.17) follows by using the inequality $\sum_{j=0}^{k-1}|q|^{N_0+j} \leq |q|^{N_0}/(1-|q|)$. □

Theorem 4.6 (Uniqueness for small τ)**.** *For any $(u_h^0, v_h^0) \in V_h \times V_h$, there exists $\tau^\star = \tau^\star(h) > 0$ such that for any $\tau \in (0, \tau^\star)$, there is a unique sequence $(u_h^n, v_h^n, z_h^n, w_h^n)_{n\geq 1}$ which complies with* (4.2)*. Moreover, $\tau^\star$ can be made independent of h if $(u_h^0, v_h^0)_{h>0}$ is a family such that*

$$|\langle v_h^0\rangle| + \mathcal{E}_h(u_h^0, v_h^0) \leq C_1, \tag{4.24}$$

for some constant C_1 independent of h.

Proof. Assume that (u_h^n, v_h^n) is uniquely determined for some $n \geq 0$. We have seen that $u_1^{n+1} = \langle u_h^{n+1}\rangle$ is uniquely determined (see (4.9)). It is sufficient to show that $\dot u_h^{n+1} \simeq \dot U^{n+1}$ is uniquely determined by (4.10), for τ sufficiently small and independent of n. Then v_h^{n+1} can be recovered by the first equation in (4.5).

Assume that (4.10) has two solutions $\dot{u}_h^{n+1} \simeq \dot{U}^{n+1}$ and $\underline{\dot{u}}_h^{n+1} \simeq \underline{\dot{U}}^{n+1}$. We subtract the two resulting systems (4.10), and we multiply by $\delta\dot{U} = \dot{U}^{n+1} - \underline{\dot{U}}^{n+1} \simeq \delta\dot{u}_h = \delta u_h$. We obtain

$$\frac{2\beta}{\tau^2}|\delta\dot{u}_h|_{-1,h}^2 + \frac{1}{\tau}|\delta\dot{u}_h|_{-1,h}^2 + \frac{1}{2}\|\mathcal{A}_h\delta\dot{u}_h\|^2 - |\delta\dot{u}_h|_1^2$$
$$+\frac{1}{2}(f(u_h^{n+1}) - f(\underline{u}_h^{n+1}), \delta u_h) - (g(u_h^n, u_h^{n+1}) - g(u_h^n, \underline{u}_h^{n+1}), \delta u_h) = 0. \quad (4.25)$$

By (2.1), f' is a polynomial of even degree with strictly positive leading coefficient, so that f' is bounded from below. There exists an (optimal) constant $c_f \geq 0$ such that

$$f'(s) \geq -c_f \quad \forall\, s \in \mathbb{R}. \quad (4.26)$$

By the mean value theorem,

$$(f(u_h^{n+1}) - f(\underline{u}_h^{n+1}), \delta u_h) \geq -c_f\|\delta\dot{u}_h\|^2.$$

On the other hand, by (4.6), we have

$$g(u_h^n, u_h^{n+1}) - g(u_h^n, \underline{u}_h^{n+1}) = \sum_{0\leq k+l\leq 2p+1} b_{k,l}(u_h^n)^k[(u_h^{n+1})^l - (\underline{u}_h^{n+1})^l],$$

so that by Hölder's inequality,

$$\left|(g(u_h^n, u_h^{n+1}) - g(u_h^n, \underline{u}_h^{n+1}), \delta u_h)\right| \leq C'_n\|\delta\dot{u}_h\|^2_{L^{2p+2}(\Omega)},$$

where

$$C'_n = C'\left(\|u_h^n\|_{L^{2p+2}(\Omega)}, \|u_h^{n+1}\|_{L^{2p+2}(\Omega)}, \|\underline{u}_h^{n+1}\|_{L^{2p+2}(\Omega)}\right),$$

and C' is a nondecreasing function of its arguments. Thus, by (2.3), equation (4.25) implies

$$\frac{2\beta}{\tau^2}|\delta\dot{u}_h|_{-1,h}^2 + \frac{1}{\tau}|\delta\dot{u}_h|_{-1,h}^2 + \frac{1}{2}\|\mathcal{A}_h\delta\dot{u}_h\|^2 \leq \frac{c_f}{2}\|\delta\dot{u}_h\|^2 + (C'_nC_S + 1)|\delta\dot{u}_h|_1^2. \quad (4.27)$$

Let (u_h^0, v_h^0) be a given initial data and let $\overline{\tau} = \min\{2\beta, (4c_5|v_1^0|)^{-1}\}$. Then for $\tau \in (0, \overline{\tau})$, $q = (2\beta - \tau)/(2\beta + \tau) \in (0, 1)$ and (4.18) is satisfied for $N_0 = 0$ since $|v_1^{1/2}| = |(1 + q)v_1^0/2| \leq |v_1^0|$. Moreover,

$$\frac{\tau}{1-q} = \beta + \frac{\tau}{2} \leq 2\beta.$$

By the energy estimate (4.17), C'_n is bounded by a constant independent of n and τ. Since all norms are equivalent in V_h, estimate (4.27) implies that for $\tau > 0$ small enough (but dependent on h !), $\delta u_h = 0$.

Now, assume that the bound (4.24) is satisfied, and replace $\overline{\tau}$ by

$$\overline{\tau} = \min\{2\beta, (4c_5C_1)^{-1}\}.$$

By the energy estimate (4.17), $\mathcal{E}_h(U^n, V^n)$ is bounded by a constant independent of h, n and τ. Thus, C'_n is bounded by a constant $\overline{C'}$ independent of h and n. We apply Lemma 4.7 below with $\varepsilon_1 = 1/(4(\overline{C'}C_S + 1))$ and $\varepsilon_2 = 1/(2c_f)$, and we obtain

$$\frac{2\beta}{\tau^2}|\delta\dot{u}_h|_{-1,h}^2 + \frac{1}{\tau}|\delta\dot{u}_h|_{-1,h}^2 \leq \left(\frac{c_f}{2}\left(\frac{1}{4\varepsilon_2^2} + \frac{1}{4}\right) + \frac{\overline{C'}C_S + 1}{4\varepsilon_1^2}\right)|\delta\dot{u}_h|_{-1,h}^2.$$

We see that for $\tau > 0$ small enough (independent of h now), $\delta\dot{u}_h = 0$ and the proof is complete. □

Lemma 4.7. *Let $\varepsilon_1, \varepsilon_2 > 0$. Then, for all $\dot{u}_h \in \dot{V}_h$, there hold*

$$|\dot{u}_h|_1^2 \quad \leq \quad \varepsilon_1 \|\dot{\mathcal{A}}_h \dot{u}_h\|^2 + \frac{1}{4\varepsilon_1^2} |\dot{u}_h|_{-1,h}^2, \tag{4.28}$$

$$\|\dot{u}_h\|^2 \quad \leq \quad \varepsilon_2 \|\dot{\mathcal{A}}_h \dot{u}_h\|^2 + \left(\frac{1}{4\varepsilon_2^2} + \frac{1}{4} \right) |\dot{u}_h|_{-1,h}^2. \tag{4.29}$$

Proof. By arguing as in (3.12), we see that

$$|\dot{A}^{1/2} \dot{U}| = (\dot{A}\dot{U})^t \dot{U} \leq \frac{\varepsilon_1}{2} |\dot{A}\dot{U}|^2 + \frac{1}{2\varepsilon_1} |\dot{U}|^2.$$

Let $\varepsilon > 0$. Similarly, we have

$$|\dot{U}|^2 = |(\dot{A}^{1/2}\dot{U})^t \dot{A}^{-1/2} \dot{U}| \leq \varepsilon |\dot{A}^{1/2} \dot{U}|^2 + \frac{1}{4\varepsilon} |\dot{A}^{-1/2} \dot{U}|^2. \tag{4.30}$$

Thus, we get

$$|\dot{A}^{1/2} \dot{U}| \leq \frac{\varepsilon_1}{2} |\dot{A}\dot{U}|^2 + \frac{1}{2\varepsilon_1} \left(\varepsilon |\dot{A}^{1/2} \dot{U}|^2 + \frac{1}{4\varepsilon} |\dot{A}^{-1/2} \dot{U}|^2 \right).$$

By choosing $\varepsilon = \varepsilon_1$, we obtain (4.28). Next, we plug (4.28) into (4.30), with $\varepsilon = 1$ and $\varepsilon_1 = \varepsilon_2$, and we deduce (4.29). □

4.3. **Convergence as** $(h, \tau) \to (0, 0)$**.** For a time step $\tau > 0$, let $(u_h^n, v_h^n, z_h^n, w_h^n)_{n \geq 1}$ be a solution of the fully discrete scheme (4.2). We define the following functions from $\mathbb{R}_+$ into V_h:

$$\begin{aligned}
u_h^\tau(t) &= ((n+1) - t/\tau) u_h^n + (t/\tau - n) u_h^{n+1}, \quad t \in [n\tau, (n+1)\tau) \quad (n \in \mathbb{N}), \\
\overline{u}_h^\tau(t) &= u_h^{n+1}, \quad t \in [n\tau, (n+1)\tau) \quad (n \in \mathbb{N}), \\
\underline{u}_h^\tau(t) &= u_h^n, \quad t \in [n\tau, (n+1)\tau) \quad (n \in \mathbb{N}), \\
\hat{u}_h^\tau(t) &= (u_h^n + u_h^{n+1})/2, \quad t \in [n\tau, (n+1)\tau) \quad (n \in \mathbb{N}).
\end{aligned}$$

We define similarly the functions v_h^τ, $\overline{v}_h^\tau$, $\underline{v}_h^\tau$, $\hat{v}_h^\tau$ associated to the sequence $(v_h^n)_{n \geq 0}$ and the functions $\overline{z}_h^\tau$, $\overline{w}_h^\tau$. Notice that $\hat{u}_h^\tau = (\overline{u}_h^\tau + \underline{u}_h^\tau)/2$ for all $t \in \mathbb{R}_+$ and that

$$\partial_t u_h^\tau = (u_h^{n+1} - u_h^n)/\tau \quad \text{in} \quad \mathcal{D}'\big((0, \infty); V_h\big).$$

The convergence results is as follows:

Theorem 4.8. *Let $(u_0, v_0) \in H_{per}^2 \times H_{per}^{-1}$. Assume that $(u_h^0, v_h^0)_{h>0}$ is a family of functions in $V_h \times V_h$ which satisfies assumptions (3.25)-(3.26) as $h \to 0$. Then the solution (u_h^τ, v_h^τ) associated to the fully discrete scheme (4.2) tends to the energy solution of problem (1.1)-(1.2) in the following sense, as $(h, \tau) \to (0, 0)$:*

$$\begin{aligned}
u_h^\tau &\to u \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; H_{per}^1), \\
u_h^\tau &\to u \text{ strongly in } C([0, T], L^2(\Omega)), \text{ for all } T > 0, \\
\mathcal{A}_h u_h^\tau &\to \mathcal{A} u \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; L^2(\Omega)), \\
\dot{\mathcal{A}}_h^{-1} \partial_t \dot{u}_h^\tau &\to \dot{\mathcal{A}}^{-1} \partial_t \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; \dot{H}_{per}^1) \text{ and weakly in } L^2(\mathbb{R}_+; \dot{H}_{per}^1).
\end{aligned}$$

Proof. We proceed as in the proof of Theorem 3.3. We first consider the conservation law for the mass. By choosing $\varphi_h = 1$ and $\psi_h = 1$ in (4.2), we find

$$\begin{cases} (\partial_t u_h^\tau, 1) = \langle \hat{v}_h^\tau \rangle \\ \beta \partial_t \langle v_h^\tau \rangle + \langle \hat{v}_h^\tau \rangle = 0, \end{cases} \tag{4.31}$$

in $\mathcal{D}'((0,\infty))$. The estimates below show that $(u_h^\tau)_{h>0,\tau>0}$ is bounded in $L^\infty(\mathbb{R}_+;H^1_{per})$, so that, up to a subsequence, $u_h^\tau \to u$ in $L^\infty(\mathbb{R}_+;H^1_{per})$ weakly $\star$, and so

$$(\partial_t u_h^\tau, 1) \to (\partial_t u, 1) \quad \text{ in } \mathcal{D}'\big((0,\infty)\big), \text{ as } (h,\tau)\to(0,0).$$

By (4.12), $|v_1^n| \leq |v_1^0|$ for all n. Thus, $\langle \hat{v}_h^\tau\rangle$ is bounded in $L^\infty(\mathbb{R}_+)$, and so, up to a subsequence, $\langle \hat{v}_h^\tau\rangle$ converges weakly $\star$ in $L^\infty(\mathbb{R}_+)$ to some function $a\in L^\infty(\mathbb{R}_+)$. Moreover, by (4.12), we have

$$\left|v_1^{n+1}-v_1^n\right| = |q|^n|1-q||v_1^0| = |q|^n\frac{2\tau}{2\beta+\tau}|v_1^0| \leq \frac{\tau}{\beta}|v_1^0|,$$

Observe now that

$$v_h^\tau - \hat{v}_h^\tau = (t/\tau-(n+1/2))(v_h^{n+1}-v_h^n), \quad t\in[n\tau,(n+1)\tau), \quad (n\in\mathbb{N}). \tag{4.32}$$

Therefore we get $|\langle v_h^\tau\rangle - \langle\hat{v}_h^\tau\rangle| \leq \tau|v_1^0|/(2\beta)$ and so $|\langle v_h^\tau\rangle - \langle\hat{v}_h^\tau\rangle|$ converges uniformly to 0 in $\mathbb{R}_+$, as $(h,\tau)\to(0,0)$. Hence $\langle v_h^\tau\rangle$ converges to a weakly $\star$ in $L^\infty(\mathbb{R}_+)$. We can pass to the limit in (4.31) in the sense of distributions on $(0,\infty)$ and we find

$$\begin{cases}(\partial_t u, 1) = a(t)\\ \beta\partial_t a(t) + a(t) = 0,\end{cases}$$

which is the conservation law for the mass.

We now turn to the energy estimate. Let

$$\tau^\star = \min\{2\beta, (4c_5\sup_{h>0}\left|\langle v_h^0\rangle\right|)^{-1}\}.$$

If $\tau\leq\tau^\star$, then (4.18) is satisfied for $N_0=0$ (and for all $h>0$). Since

$$\frac{\tau}{1-q} = \beta+\tau/2 \leq 2\beta,$$

the energy estimate (4.17) implies

$$\mathcal{E}_h(u_h^n, v_h^n) + \sum_{k=0}^{n-1}\tau|v_h^{k+1/2}|^2_{-1,h} \leq \exp(32\beta c_5|v_1^0|)\left(\mathcal{E}_h(u_h^0,v_h^0)+2\beta c_7|v_1^0|\right), \tag{4.33}$$

for all $n\geq 0$. Assumptions (3.25)-(3.26) imply that $\mathcal{E}_h(u_h^0,v_h^0)$ and $|v_1^0|$ are bounded by a constant independent of h. The right-hand side of (4.33) is bounded by a constant independent of h and τ. Thus, u_h^τ is uniformly bounded in H^1_{per}, $\overline{z}_h^\tau = \mathcal{A}_h\hat{u}_h^\tau$ is uniformly bounded in $L^2(\Omega)$, and

$$\dot{r}_h^\tau := \dot{\mathcal{A}}_h^{-1}\partial_t\dot{u}_h^\tau = \dot{\mathcal{A}}_h^{-1}\dot{\hat{v}}_h^\tau$$

is uniformly bounded in H^1_{per}. By arguing as in (3.28), we see that for all $0\leq s\leq t$,

$$\|\dot{u}_h^\tau(t)-\dot{u}_h^\tau(s)\|^2 \leq 4\|\dot{r}_h^\tau\|_{L^\infty(\mathbb{R}_+;H^1_{per})}\|\dot{u}_h^\tau\|_{L^\infty(\mathbb{R}_+;H^1_{per})}|t-s|.$$

Moreover, for all $0\leq s\leq t$, observe that

$$|\langle u_h^\tau(t)\rangle - \langle u_h^\tau(s)\rangle| = \left|\int_s^t\langle\hat{v}_h^\tau(\sigma)\rangle d\sigma\right| \leq |\langle v_h^0\rangle||t-s|.$$

Thus, for all $T>0$, there is a constant C_T independent of (h,τ) such that

$$\|u_h^\tau(t)-u_h^\tau(s)\| \leq C_T|t-s|^{1/2}, \tag{4.34}$$

for all $0 \leq s \leq t \leq T$. By the Ascoli-Arzelà Theorem, (u_h^τ) is precompact in the space $C([0,T];L^2(\Omega))$, for all $T>0$. Applying (4.34) with $s = n\tau$ and $t = (n+1)\tau$ yields $\|u_h^{n+1} - u_h^n\| \leq C_T \tau^{1/2}$, so that

$$\|u_h^\tau - \overline{u}_h^\tau\|_{L^\infty(0,T;L^2(\Omega))} \to 0 \quad \text{ and } \quad \|u_h^\tau - \underline{u}_h^\tau\|_{L^\infty(0,T;L^2(\Omega))} \to 0,$$

as $\tau \to 0$. Up to a subsequence, we have

$$\begin{aligned}
u_h^\tau, \hat{u}_h^\tau &\to u \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; H^1_{per}),\\
\overline{u}_h^\tau, \underline{u}_h^\tau &\to u \text{ strongly in } C([0,T];L^2(\Omega)), \text{ for all } T>0,\\
\overline{u}_h^\tau, \underline{u}_h^\tau &\to u \text{ a.e. in } \Omega \times \mathbb{R}_+,\\
\overline{z}_h^\tau &\to \ \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; L^2(\Omega)),\\
\dot{r}_h^\tau &\to \dot{r} \text{ weakly } \star \text{ in } L^\infty(\mathbb{R}_+; \dot{H}^1_{per}) \text{ and weakly in } L^2(\mathbb{R}_+; \dot{H}^1_{per}),
\end{aligned}$$

as $(h,\tau) \to (0,0)$. Let $\varphi \in H^1_{per}$ and set $\varphi_h = \Pi_h(\varphi)$, so that $\varphi_h \to \varphi$ strongly in H^1_{per}. The first equation in (4.2) reads

$$(\partial_t u_h^\tau, \varphi_h) = (\hat{v}_h^\tau, \varphi_h).$$

By arguing as in (3.29) and letting $(h,\tau) \to (0,0)$, we obtain that

$$\partial_t(\dot{u}, \dot{\varphi}) = (\nabla \dot{r}, \nabla \dot{\varphi}) = \langle \dot{v}, \dot{\varphi} \rangle_{H^{-1}_{per}, H^1_{per}}$$

in $\mathcal{D}'((0,\infty))$, with $\dot{v} = \dot{\mathcal{A}}\dot{r}$. This shows that $\partial_t \dot{u} = \dot{v} \in L^\infty(\mathbb{R}_+; H^1_{per})$.

Next, we set $\psi \in H^2_{per}$ and we let $\psi_h = \Pi_h(\psi)$ so that $\psi_h \to \psi$ strongly in H^1_{per}. We have $\psi_h = \sum_{i=1}^{N_h} \Psi_i e_h^i$ and $\Psi = (\psi_1, \ldots, \psi_{N_h})^t$ is the vector associated to ψ_h. On multiplying (4.5) by $\dot{\Psi}^t \dot{A}^{-1}$, we find

$$\begin{aligned}
&\beta(\partial_t \dot{v}_h^\tau, \dot{\psi}_h)_{-1,h} + (\dot{\hat{v}}_h^\tau, \dot{\psi}_h)_{-1,h} + (\nabla \overline{z}_h^\tau, \nabla \psi_h) - 2(\nabla \hat{u}_h^\tau, \nabla \psi_h)\\
&+\frac{1}{2}(f(\underline{u}_h^\tau) + f(\overline{u}_h^\tau), \dot{\psi}_h) - \frac{1}{12}((\overline{u}_h^\tau - \underline{u}_h^\tau)^2 (f''_+(\underline{u}_h^\tau) + f''_-(\overline{u}_h^\tau)), \dot{\psi}_h) = 0. \quad (4.35)
\end{aligned}$$

By arguing as in the proof of Theorem 3.3, we get

$$(\dot{\hat{v}}_h^\tau, \dot{\psi}_h)_{-1,h} \to (v, \psi)_{-1}, \ (\nabla \overline{z}_h^\tau, \nabla \psi_h) \to (z, \mathcal{A}\psi), \ (\nabla \hat{u}_h^\tau, \nabla \psi_h) \to (\nabla u, \nabla \psi)$$

in $\mathcal{D}'((0,\infty))$. Thanks to the Sobolev injection $H^1_{per} \hookrightarrow L^{2p+2}(\Omega)$, for all $T>0$, the terms

$$\|f(\underline{u}_h^\tau)\|_{L^q(0,T;L^q(\Omega))}, \ \|f(\overline{u}_h^\tau)\|_{L^q(0,T;L^q(\Omega))},$$

and

$$\|(\overline{u}_h^\tau - \underline{u}_h^\tau)^2 (f''_+(\underline{u}_h^\tau) + f''_-(\overline{u}_h^\tau))\|_{L^q(0,T;L^q(\Omega))}$$

are bounded by a constant independent of h and τ, for $q = (2p+2)/(2p+1) \in (1,2)$. We can therefore pass to the limit in the nonlinear terms, and we find that

$$\frac{1}{2}(f(\underline{u}_h^\tau) + f(\overline{u}_h^\tau), \dot{\psi}_h) \to (f(u), \dot{\psi})$$

and

$$\frac{1}{12}((\overline{u}_h^\tau - \underline{u}_h^\tau)^2 (f''_+(\underline{u}_h^\tau) + f''_-(\overline{u}_h^\tau)), \dot{\psi}_h) \to 0$$

in $\mathcal{D}'((0,\infty))$. Thus, the first term in equation (4.35) has a limit in $\mathcal{D}'((0,\infty))$,

$$\beta(\partial_t \dot{v}_h^\tau, \dot{\psi}_h)_{-1,h} \to \eta_\psi.$$

As a consequence,

$$(\dot{\overline{v}}_h^\tau - \dot{\underline{v}}_h^\tau, \dot\psi_h)_{-1,h} = \tau(\partial_t \dot v_h^\tau, \dot\psi_h)_{-1,h} \to 0$$

in $\mathcal{D}'((0,\infty))$. Thus, by (4.32), we have

$$(\dot v_h^\tau, \dot\psi_h)_{-1,h} \to (\dot v, \dot\psi)_{-1}.$$

Summing up, we have proved that

$$\beta\partial_t(\dot v, \dot\psi)_{-1} + (\dot v, \dot\psi)_{-1} + (z, \mathcal{A}\psi) - 2(\nabla u, \nabla\psi) + (f(u), \dot\psi) = 0,$$

with $v = \partial_t u$ and $z = \mathcal{A}u$. We conclude as in Theorem 3.3 that (u, u_t) is an energy solution of (1.1)-(1.2). Note that the whole family converges to (u, u_t) due to the uniqueness of the limit. □

5. Convergence to equilibrium

In this section, we prove that any solution of the fully discrete scheme converges to a single equilibrium, for any time step $\tau > 0$. The parameter h is fixed (so that assumption (H2) is not relevant). We adapt the proof from [25] in a discrete setting. The main idea is to use the gradient-like flow structure of the problem and a suitable Łojasiewicz inequality. In three space dimensions, in addition to (H1) and (H3), we need the following assumption:

(H4) If $d = 3$, then either $V_h \subset L^\infty(\Omega)$ or $p = 1$.

Theorem 5.1. *Let $\tau > 0$ denote the time step and let $(U^n, V^n)_{n\geq 0}$ denote any sequence in $\mathbb{R}^{N_h} \times \mathbb{R}^{N_h}$ which complies with (4.5). Then (U^n, V^n) converges to $(U^\infty, 0)$, where $U^\infty = (u_1^\infty, \dot U^\infty)$ is a stationary solution with average constraint, i.e.,*

$$\begin{cases} u_1^\infty = M = u_1^0 + \beta v_1^0, \\ \dot A^2 \dot U^\infty - 2\dot A \dot U^\infty + \dot\nabla F_h(U^\infty) = 0. \end{cases} \tag{5.1}$$

We first prove the following

Lemma 5.2. *Let the assumptions of Theorem 5.1 hold. Then $V^{n+1/2} \to 0$.*

Proof. Since $|q| < 1$, estimate (4.18) is satisfied for N_0 large enough. By the energy estimate (4.17), $\sum_{n=N_0}^\infty |\dot V^{n+1/2}|_{-1}^2 < \infty$. In particular, $\dot V^{n+1/2} \to 0$ in $\mathbb{R}^{N_h-1}$. Moreover, $v_1^{n+1/2} = q^n v_1^{1/2}$ by (4.12), so $V^{n+1/2} \to 0$, as claimed. □

For any $M \in \mathbb{R}$, we introduce the auxiliary function $F_M(y) = F(M + y)$ and the following functionals

$$F_{M,h}(\dot U) = (F_M(\dot u_h), 1), \tag{5.2}$$

$$E_{M,h}(\dot U) = \frac{1}{2}|\dot A \dot U|^2 - |\dot A^{1/2}\dot U|^2 + F_{M,h}(\dot U), \tag{5.3}$$

$$\mathcal{E}_{M,h}(\dot U, \dot V) = E_{M,h}(\dot U) + \frac{\beta}{2}|\dot V|_{-1}^2, \tag{5.4}$$

defined for every $\dot U \simeq \dot u_h$ and every $\dot V$ in $\mathbb{R}^{N_h-1}$.

For any $M \in \mathbb{R}$, we also consider

$$\mathfrak{S}_M = \left\{ U \in \mathbb{R}^{N_h} \ : \ U \text{ satisfies (5.1)} \right\}.$$

For any sequence $(U^n, V^n)_{n\geq 0}$ in $\mathbb{R}^{N_h} \times \mathbb{R}^{N_h}$, we define its ω-limit set in $\mathbb{R}^{N_h} \times \mathbb{R}^{N_h}$:

$$\omega\left((U^n, V^n)_{n\geq 0}\right) = \{(U^\star, V^\star) \ : \ \exists n_j \nearrow \infty, \ (U^{n_j}, V^{n_j}) \to (U^\star, V^\star)|\}.$$

Similarly, we set

$$\omega\big((U^n)_{n\geq 0}\big) = \{U^\star \ : \ \exists n_j \nearrow \infty, \ U^{n_j} \to U^\star\}.$$

We have:

Proposition 5.3. *Let the assumptions of Theorem 5.1 hold. Then* $\omega\left((U^n, V^n)_{n\geq 0}\right)$ *is a nonempty compact and connected set such that*

$$\omega\left((U^n, V^n)_{n\geq 0}\right) = \omega\big((U^n)_{n\geq 0}\big) \times \{0\} \subset \{(U^\star, 0) \ : U^\star \in \mathfrak{S}_M\},$$

with $M = u_1^0 + \beta v_1^0$. *Moreover,* $E_{M,h}$ *is constant on* $\omega\big((U^n)_n\big)$.

This result implies in particular that $V^n \to 0$, as proved below.

Proof. Since $q = (2\beta - \tau)/(2\beta + \tau)$, we can rewrite (4.13) as

$$u_1^n = u_1^0 + (1 - q^n)\beta v_1^0. \tag{5.5}$$

Let $M = u_1^0 + \beta v_1^0$. We introduce the auxiliary functions

$$f_M(y) = f(M + y) \quad \text{and} \quad \hat{f}_M(r, s) = \hat{f}(M + r, M + s),$$

where

$$\hat{f}(r, s) = \frac{1}{2}(f(r) + f(s)) - \frac{1}{12}(s - r)^2(f''_+(r) + f''_-(s)).$$

We also set

$$F_{M,\pm}(y) = F_\pm(M + y) \quad \text{and} \ f_{M,\pm}(y) = f_\pm(M + y),$$

so that $F_M = F_{M,+} + F_{M,-}$ and $F_{M,+}^{(iv)} \geq 0$, $F_{M,-}^{(iv)} \leq 0$. In particular, the function F_M satisfies the decomposition (H3), and we have

$$\hat{f}_M(r, s) = \frac{1}{2}(f_M(r) + f_M(s)) - \frac{1}{12}(s - r)^2(f''_{M,+}(r) + f''_{M,-}(s)).$$

Then we rewrite the second equation in (4.5) in the following form:

$$\begin{aligned} \beta(\dot{V}^{n+1} - \dot{V}^n)/\tau + \dot{V}^{n+1/2} + \dot{A}\big(\dot{A}^2\dot{U}^{n+1/2} - 2\dot{A}\dot{U}^{n+1/2} + \dot{J}_M(\dot{U}^n, \dot{U}^{n+1})\big) \\ = \dot{A}\big(\dot{J}_M(\dot{U}^n, \dot{U}^{n+1}) - \dot{J}(U^n, U^{n+1})\big), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \dot{J}_M(\dot{U}^n, \dot{U}^{n+1}) &= \big((\hat{f}_M(\dot{u}_h^n, \dot{u}_h^{n+1}), e_i)\big)_{2\leq i\leq N_h}, \\ \dot{J}(U^n, U^{n+1}) &= \big((\hat{f}(u_h^n, u_h^{n+1}), e_i)\big)_{2\leq i\leq N_h}. \end{aligned}$$

Multiplying (5.6) by $(\dot{U}^{n+1} - \dot{U}^n)^t\dot{A}^{-1}$, using that (4.5) implies

$$\dot{U}^{n+1} - \dot{U}^n = \tau\dot{V}^{n+1/2}, \tag{5.7}$$

we find

$$\begin{aligned} &\frac{\beta}{2}(|\dot{V}^{n+1}|_{-1}^2 - |\dot{V}^n|_{-1}^2) + \tau|\dot{V}^{n+1/2}|_{-1}^2 + \frac{1}{2}(|\dot{A}\dot{U}^{n+1}|^2 - |\dot{A}\dot{U}^n|^2) \\ &-(|\dot{A}^{1/2}\dot{U}^{n+1}|^2 - |\dot{A}^{1/2}\dot{U}^n|^2) + \langle \dot{J}_M(\dot{U}^n, \dot{U}^{n+1}), \dot{U}^{n+1} - \dot{U}^n\rangle \\ &= (\hat{f}_M(\dot{u}_h^n, \dot{u}_h^{n+1}) - \hat{f}(u_h^n, u_h^{n+1}), \dot{u}_h^{n+1} - \dot{u}_h^n). \end{aligned} \tag{5.8}$$

Using now Lemma 4.4 and arguing as in the proof of Lemma 4.5, we obtain

$$(F_M(\dot u_h^{n+1}),1)-(F_M(\dot u_h^n),1)\le \langle \dot J_M(\dot U^n,\dot U^{n+1}),\dot U^{n+1}-\dot U^n\rangle. \tag{5.9}$$

By (5.5), for any solution $u_h^n\simeq U^n$ of (4.5), we have

$$\hat f(u_h^n,u_h^{n+1})=\hat f_M(\dot u_h^n-\beta q^n v_1^0,\dot u_h^{n+1}-\beta q^{n+1}v_1^0).$$

Thus we get

$$\begin{aligned}&(\hat f_M(\dot u_h^n,\dot u_h^{n+1})-\hat f(u_h^n,u_h^{n+1}),\dot u_h^{n+1}-\dot u_h^n)\\&=(\hat f_M(\dot u_h^n,\dot u_h^{n+1})-\hat f_M(\dot u_h^n-\beta q^n v_1^0,\dot u_h^{n+1}-\beta q^{n+1}v_1^0),\dot u_h^{n+1}-\dot u_h^n)\\&=-\beta q^n v_1^0\left(\int_0^1\partial_r\hat f_M(\tilde u_h^n(s),\tilde u_h^{n+1}(s))+q\partial_s\hat f_M(\tilde u_h^n(s),\tilde u_h^{n+1}(s))ds,\dot u_h^{n+1}-\dot u_h^n\right)\end{aligned}$$

where $\tilde u_h^n(s)=\dot u_h^n-s\beta q^n v_1^0$ for $s\in[0,1]$. By the energy estimate (4.17), the sequence $(u_h^n)_{n\ge0}$ is bounded in H^1_{per}. Moreover, the function $\hat f_M$ is a polynomial in (r,s) of total degree equal to $2p+1$. Using the Sobolev injection $H^1_{per}\hookrightarrow L^{2p+1}(\Omega)$, we obtain

$$\begin{aligned}&\left|(\hat f_M(\dot u_h^n,\dot u_h^{n+1})-\hat f(u_h^n,u_h^{n+1}),\dot u_h^{n+1}-\dot u_h^n)\right|\\&\qquad\le\beta|q|^n|v_1^0|C(\|u_h^n\|_1,\|u_h^{n+1}\|_1)\|\dot u_h^{n+1}-\dot u_h^n\|_1 &(5.10)\\&\qquad\le\frac{1}{4\tau}|\dot U^{n+1}-\dot U^n|_{-1}^2+C_0|q|^{2n}. &(5.11)\end{aligned}$$

Here and in the following, C_k $(k=0,1,\dots)$ denotes a constant independent of n (but which may depend on τ, h and other parameters of the problem). In the last inequality we have used that all norms are equivalent in $\dot V_h$. Adding up (5.8), (5.9) and (5.11), we find

$$\mathcal E_{M,h}(\dot U^{n+1},\dot V^{n+1})-\mathcal E_{M,h}(\dot U^n,\dot V^n)+\frac{3\tau}{4}|\dot V^{n+1/2}|_{-1}^2\le C_0|q|^{2n}, \tag{5.12}$$

for all $n\ge0$.

Set now

$$\mathcal G^n=\langle \dot A^{-1}\dot V^n,\dot A^{-1}(\dot A^2\dot U^n-2\dot A\dot U^n+\dot\nabla F_{M,h}(\dot U^n))\rangle_{-1},$$

where $\langle \dot U,\dot V\rangle_{-1}=\dot U^t\dot A^{-1}\dot V$, for all $\dot U,\dot V\in\mathbb R^{N_h-1}$. We have

$$\begin{aligned}\mathcal G^{n+1}-\mathcal G^n&=\langle \dot A^{-1}(\dot V^{n+1}-\dot V^n),\dot A^{-1}(\dot A^2\dot U^n-2\dot A\dot U^n+\dot\nabla F_{M,h}(\dot U^n))\rangle_{-1}\\&+\langle \dot A^{-1}\dot V^{n+1},\dot A^{-1}((\dot A^2-2\dot A)(\dot U^{n+1}-\dot U^n)+\dot\nabla F_{M,h}(\dot U^{n+1})-\dot\nabla F_{M,h}(\dot U^n))\rangle_{-1}.\end{aligned}$$

Let $\delta\mathcal G_1^n$ denote the first term on the right-hand side of this equality and denote the second by $\delta\mathcal G_2^n$. Using (5.6), $\dot U^n=\dot U^{n+1/2}-(\dot U^{n+1}-\dot U^n)/2$ and $\dot J_M(\dot U^n,\dot U^n)=\dot\nabla F_{M,h}(\dot U^n)$, we obtain

$$\begin{aligned}\delta\mathcal G_1^n&=-\frac{\tau}{\beta}\langle \dot A^{-1}\dot V^{n+1/2}+\dot S^n-\big(\dot J_M(\dot U^n,\dot U^{n+1})-\dot J(U^n,U^{n+1})\big),\dot A^{-1}\dot S^n\rangle_{-1}\\&\frac{\tau}{2\beta}\langle \dot A^{-1}\dot V^{n+1/2}+\dot S^n-\big(\dot J_M(\dot U^n,\dot U^{n+1})-\dot J(U^n,U^{n+1})\big),\dot A^{-1}\dot T_1^n\rangle_{-1},\end{aligned}$$

where

$$\begin{aligned}\dot S^n&=\dot A^2\dot U^{n+1/2}-2\dot A\dot U^{n+1/2}+\dot J_M(\dot U^n,\dot U^{n+1}),\\\dot T_1^n&=(\dot A^2-2\dot A)(\dot U^{n+1}-\dot U^n)+2\dot J_M(\dot U^n,\dot U^{n+1})-2\dot J_M(\dot U^n,\dot U^n).\end{aligned}$$

Using that all norms are equivalent in $\dot{V}_h$, the Cauchy-Schwarz inequality, Young's inequality and (5.7), we deduce

$$\delta\mathcal{G}_1^n + \frac{3\tau}{4\beta}|\dot{A}^{-1}\dot{S}^n|^2 \leq C_1\tau|\dot{V}^{n+1/2}|_{-1}^2 + C_2\left|\dot{J}_M(\dot{U}^n,\dot{U}^{n+1}) - \dot{J}(U^n,U^{n+1})\right|^2 + C_3\left|\dot{J}_M(\dot{U}^n,\dot{U}^{n+1}) - \dot{J}_M(\dot{U}^n,\dot{U}^n)\right|^2.$$

By Bessel's inequality, we get

$$\left|\dot{J}_M(\dot{U}^n,\dot{U}^{n+1}) - \dot{J}(U^n,U^{n+1})\right|^2 \leq \left\|\hat{f}_M(u_h^n,u_h^{n+1}) - \hat{f}(u_h^n,u_h^{n+1})\right\|^2.$$

Arguing as in (5.10), and using assumption (H4), we find that

$$\left|\dot{J}_M(\dot{U}^n,\dot{U}^{n+1}) - \dot{J}(U^n,U^{n+1})\right|^2 \leq C_4|q|^{2n}. \tag{5.13}$$

Similarly, we have

$$\left|\dot{J}_M(\dot{U}^n,\dot{U}^{n+1}) - \dot{J}_M(\dot{U}^n,\dot{U}^n)\right|^2 \leq C_5\tau\left|\dot{V}^{n+1/2}\right|_{-1}^2. \tag{5.14}$$

Summing up, we have proved

$$\delta\mathcal{G}_1^n + \frac{3\tau}{4\beta}|\dot{A}^{-1}\dot{S}^n|^2 \leq (C_1 + C_3C_5)\tau|\dot{V}^{n+1/2}|_{-1}^2 + C_2C_4|q|^{2n},$$

for all $n \geq 0$. We now consider the term $\delta\mathcal{G}_2^n$. Using $V^{n+1} = V^{n+1/2} + (V^{n+1} - V^n)/2$, equation (5.6), and arguing as for δG_1^n, we obtain

$$\delta\mathcal{G}_2^n \leq \frac{\tau}{4\beta}|\dot{A}^{-1}\dot{S}^n|^2 + C_6\tau|\dot{V}^{n+1/2}|_{-1}^2 + C_7|q|^{2n}.$$

Thus, we get

$$\mathcal{G}^{n+1} - \mathcal{G}^n + \frac{\tau}{2\beta}|\dot{A}^{-1}\dot{S}^n|^2 \leq C_8\tau|\dot{V}^{n+1/2}|_{-1}^2 + C_9|q|^{2n}, \tag{5.15}$$

for all $n \geq 0$, with $C_8 = C_1 + C_3C_5 + C_6$ and $C_9 = C_2C_4 + C_7$.

Let us introduce the sequence

$$W^n = 2\mathcal{E}_{M,h}(\dot{U}^n,\dot{V}^n) + \nu\mathcal{G}^n,$$

where $\nu > 0$ is sufficiently small so that $\nu C_8 \leq 1/2$. From estimates (5.12) and (5.15), it follows that

$$W^{n+1} - W^n + \tau|\dot{V}^{n+1/2}|_{-1}^2 + \frac{\nu\tau}{2\beta}|\dot{A}^{-1}\dot{S}^n|^2 \leq C_{10}|q|^{2n}, \tag{5.16}$$

with $C_{10} = 2C_0 + \nu C_9$. By the energy estimate (4.17), the sequence (U^n,V^n) is bounded, so $(W^n)_{n\geq 0}$ is bounded. This implies that W^n converges to some real number W^∞ as n tends to ∞. Indeed, let

$$\widetilde{W}^n = W^n + \frac{C_{10}}{1-|q|^2}|q|^{2n}.$$

Using (5.16), we see that $\widetilde{W}^{n+1} - \widetilde{W}^n \leq 0$, i.e. $\widetilde{W}^n$ is nonincreasing. and since $\widetilde{W}$ is bounded, $\widetilde{W}^n$ has a limit $\widetilde{W}^\infty = W^\infty$.

Adding up estimate (5.16), we obtain that $\sum_{n=0}^{\infty}|\dot{A}^{-1}\dot{S}^n|^2 < \infty$. In particular, $\dot{S}^n \to 0$. Moreover, by (5.13), we have

$$\dot{J}_M(\dot{U}^n,\dot{U}^{n+1}) - \dot{J}(U^n,U^{n+1}) \to 0$$

as $n \to \infty$. From (5.6) and Lemma 5.2, it follows that $\dot{V}^{n+1} - \dot{V}^n \to 0$ and that

$$\dot{V}^n = \dot{V}^{n+1/2} - (\dot{V}^{n+1} - \dot{V}^n)/2 \to 0.$$

If $(\dot{U}^{n'})_{n'}$ is a subsequence which converges to some $\dot{U}^\star$, then $\dot{U}^{n'+1} \to \dot{U}^\star$ as well, since $\dot{U}^{n'+1} - \dot{U}^{n'} = \tau \dot{V}^{n'+1/2}$. Since $\dot{S}^{n'} \to 0$ and $\dot{J}_M$ is continuous at $(\dot{U}^\star, \dot{U}^\star)$ with $\dot{J}_M(\dot{U}^\star, \dot{U}^\star) = \nabla F_{M,h}(\dot{U}^\star)$, we obtain that

$$\dot{A}^2 \dot{U}^\star - 2\dot{A}\dot{U}^\star + \dot{\nabla} F_{M,h}(\dot{U}^\star) = 0.$$

Using the conservation law (5.5), we see that $U^\star \in \mathfrak{S}_M$. Finally, the sequence (U^n, V^n) is bounded, and we have seen that $U^{n+1} - U^n \to 0$, $V^n \to 0$ so the ω-limit set $\omega\big((U^n, V^n)_{n\geq 0}\big)$ is a nonempty compact and connected subset of $\mathfrak{S}_M \times \{0\}$, equal to $\omega\big((U^n)_n\big) \times \{0\}$.

Since $\dot{V}^n \to 0$ and $W^n \to W^\infty$, we have $\mathcal{G}^n \to 0$ and

$$\mathcal{E}_{M,h}(\dot{U}^n, \dot{V}^n) = (1/2)(W^n - \nu \mathcal{G}^n) \to W^\infty/2.$$

By definition, $E_{M,h}(\dot{U}^n) = \mathcal{E}_{M,h}(\dot{U}^n, \dot{V}^n) - \frac{\beta}{2}|\dot{V}^n|_{-1}^2$, so $E_{M,h}(\dot{U}^n) \to W^\infty/2$. This implies that $E_{M,h}$ is constant and equal to $E^\infty := W^\infty/2$ on $\omega\big((U^n)_n\big)$. The proof of Proposition 5.3 is complete. ☐

We notice that the functional $E_{M,h}$ is a polynomial of the variables $(u_2, \dots, u_{N_h})$ of total degree $2p+2$, so the following Łojasiewicz inequality holds:

Lemma 5.4 (Łojasiewicz' inequality [30])**.** *Let $\dot{U}^\star \in \mathbb{R}^{N_h - 1}$ be a critical point of $E_{M,h}$. Then there exist constants $\theta \in (0, 1/2)$ and $\delta > 0$ such that for any $\dot{U} \in \mathbb{R}^{N_h-1}$ satisfying $|\dot{U} - \dot{U}^\star| < \delta$, there holds*

$$|E_{M,h}(\dot{U}) - E_{M,h}(\dot{U}^\star)|^{1-\theta} \leq |\dot{A}^2 \dot{U} - 2\dot{A}\dot{U} + \dot{\nabla} F_{M,h}(\dot{U})|. \tag{5.17}$$

Proof of Theorem 5.1. Let $M = u_1^0 + \beta v_1^0$ as previously. By Lemma 5.4, for every $U^\infty \in \omega\big((U^n)_n\big)$, there exist some $\delta > 0$ and $\theta \in (0, 1/2)$ that may depend on $\dot{U}^\infty$ such that the inequality (5.17) holds for all $\dot{U}$ in

$$\mathbf{B}_\delta(\dot{U}^\infty) = \{\dot{U} \in \mathbb{R}^{N_h - 1} \ : \ |\dot{U} - \dot{U}^\infty| < \delta\}$$

and $\Big| E_{M,h}(\dot{U}) - E_{M,h}(\dot{U}^\infty)\Big| \leq 1$. The union of balls $\{\mathbf{B}_\delta(\dot{U}^\infty) \ : \dot{U}^\infty \in \omega\big((\dot{U}^n)_n\big)\}$ forms an open covering of $\omega\big((\dot{U}^n)_n\big)$. Due to the compactness of $\omega\big((\dot{U}^n)_n\big)$ in $\mathbb{R}^{N_h-1}$, we can find a finite sub-covering $\{\mathbf{B}_{\delta_i}(\dot{U}_i^\infty)\}_{i=1,2,\dots,m}$, where the constants δ_i, θ_i corresponding to $\dot{U}_i^\infty$ in Lemma 5.4 are indexed by i.

From the definition of $\omega\big((\dot{U}^n)_n\big)$, we know that there exists a sufficiently large n_0 such that $\dot{U}^n \in \mathcal{U} := \cup_{i=1}^m \mathbf{B}_{\delta_i}(\dot{U}_i^\infty)$ for $n \geq n_0$. Taking $\theta = \min_{i=1}^m \{\theta_i\} \in (0, 1/2)$, we infer from Lemma 5.4 that, for all $n \geq n_0$,

$$|E_{M,h}(\dot{U}^n) - E^\infty|^{1-\theta} \leq |\dot{A}^2 \dot{U}^n - 2\dot{A}\dot{U}^n + \dot{\nabla} F_{M,h}(\dot{U}^n)|, \tag{5.18}$$

where $E^\infty = W^\infty/2$ is the constant value of $E_{M,h}$ on $\omega\big((\dot{U}^n)_n\big)$.

Let us now set

$$a_n = \left(\frac{\tau}{2}|\dot{V}^{n+1/2}|_{-1}^2 + \frac{\nu\tau}{4\beta}|\dot{A}^{-1}\dot{S}^n|^2\right)^{1/2} + |q|^n. \tag{5.19}$$

From (5.16), it follows that

$$\sum_{k=n}^{\infty} a_k^2 \leq W^n - W^\infty + C_{11}|q|^{2n}.$$

On the other hand, using the Łojasiewicz inequality (5.18) and the fact $1/(1-\theta) < 2$, we deduce that, for all $n \geq n_0$ (changing n_0 into a larger integer if necessary),

$$\begin{aligned} |W^n - W^\infty| &\leq 2|E_{M,h}(\dot{U}^n) - E^\infty| + \beta|\dot{V}^n|_{-1}^2 + \nu|\mathcal{G}^n| \\ &\leq 2|\dot{A}^2\dot{U}^n - 2\dot{A}\dot{U}^n + \dot{\nabla}F_{M,h}(\dot{U}^n)|^{1/(1-\theta)} + \beta|\dot{V}^n|_{-1}^2 \\ &\quad + C_{12}|\dot{V}^n|_{-1}|\dot{A}^2\dot{U}^n - 2\dot{A}\dot{U}^n + \dot{\nabla}F_{M,h}(\dot{U}^n)| \\ &\leq C_{13}\left(|\dot{A}^2\dot{U}^n - 2\dot{A}\dot{U}^n + \dot{\nabla}F_{M,h}(\dot{U}^n)|^{1/(1-\theta)} + |\dot{V}^n|_{-1}^{1/(1-\theta)}\right) \end{aligned} \tag{5.20}$$

Using $\dot{V}^n = \dot{V}^{n+1/2} - (\dot{V}^{n+1} - \dot{V}^n)/2$, we deduce from (5.6) and (5.13) that

$$|\dot{V}^n|_{-1} \leq C_{13}\left(|\dot{V}^{n+1/2}|_{-1} + |\dot{A}^{-1}\dot{S}^n| + |q|^n\right). \tag{5.21}$$

Similarly, from

$$\begin{aligned} \dot{A}^2\dot{U}^n - 2\dot{A}\dot{U}^n + \dot{\nabla}F_{M,h}(\dot{U}^n) &= \dot{S}^n + (\dot{A}^2 - 2\dot{A})(\dot{U}^n - \dot{U}^{n+1/2}) \\ &\quad + \dot{J}_M(\dot{U}^n, \dot{U}^n) - \dot{J}_M(\dot{U}^n, \dot{U}^{n+1}), \end{aligned}$$

$\dot{U}^{n+1} - \dot{U}^n = \tau\dot{V}^{n+1/2}$ and (5.14), we infer that

$$|\dot{A}^2\dot{U}^n - 2\dot{A}\dot{U}^n + \dot{\nabla}F_{M,h}(\dot{U}^n)| \leq C_{14}\left(|\dot{A}^{-1}\dot{S}^n| + |\dot{V}^{n+1/2}|_{-1}\right). \tag{5.22}$$

Collecting (5.20), (5.21) and (5.22), we obtain

$$|W^n - W^\infty| \leq C_{15}\left(|\dot{V}^{n+1/2}|_{-1}^{1/(1-\theta)} + |\dot{A}^{-1}\dot{S}^n|^{1/(1-\theta)} + (|q|^n)^{1/(1-\theta)}\right),$$

for all $n \geq n_0$. This gives

$$\sum_{k=n}^{\infty} a_k^2 \leq C_{16} a_n^{1/(1-\theta)}, \quad \forall n \geq n_0.$$

From Lemma 5.5 below, we conclude that $\sum_{n=0}^{\infty} a_n < \infty$. In particular, we have

$$\tau \sum_{n=0}^{\infty} |\dot{V}^{n+1/2}|_{-1} = \sum_{n=0}^{\infty} |\dot{U}^{n+1} - \dot{U}^n| < \infty.$$

This shows that the whole sequence $(\dot{U}^n)_n$ has a limit $\dot{U}^\infty$ as $n \to \infty$. From (5.5), we know that $u_1^n \to M$. Thus, $(U^n)_n$ tends to some U^∞ in $\mathbb{R}^{N_h}$, and the proof is complete. □

For the following lemma and its proof, we adapt Lemma 4.1 in [29] in a discrete setting (see also Lemma 7.1 in [13]).

Lemma 5.5. *Let $0 < \theta < 1/2$. Assume that $(a_n)_{n \geq 0}$ is a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} a_n^2 < \infty$, and there are a constant $C > 0$ and an integer n_0 such that*

$$\sum_{k=n}^{\infty} a_k^2 \leq C a_n^{1/(1-\theta)} \text{ for all } n \geq n_0. \tag{5.23}$$

Then $\sum_{n=0}^{\infty} a_n < \infty$.

Proof. First replacing a_n by $\max\{a_n, 1\}$ for $0 \le n < n_0$, and then taking C large enough to ensure $C \ge \sum_{n=0}^{\infty} a_n^2$, we observe that (5.23) becomes valid for all $n \ge 0$. So we may assume $n_0 = 0$. Set now

$$\rho_n := \sum_{k=n}^{\infty} a_k^2 \quad \text{and} \quad \sigma_n = \sum_{k=0}^{n} a_k \quad \text{for } n \ge 0.$$

Given any $n \ge 0$, we first raise inequality (5.23) to the power $1 - \theta > 0$:

$$\rho_n^{1-\theta} \le C^{1-\theta} a_n.$$

Next, we sum this relation and we obtain

$$\sum_{k=0}^{n} \rho_k^{1-\theta} \le C_1 \sum_{k=0}^{n} a_k = C_1 \sigma_n.$$

We now apply a discrete integration-by-parts on the left-hand side

$$\sum_{k=0}^{n} [(k+2) - (k+1)] \rho_k^{1-\theta} = (n+2)\rho_n^{1-\theta} - \rho_0^{1-\theta} + \sum_{k=1}^{n} (k+1)(\rho_{k-1}^{1-\theta} - \rho_k^{1-\theta}).$$

Next, we notice that

$$\rho_{k-1}^{1-\theta} - \rho_k^{1-\theta} = \int_{\rho_k}^{\rho_{k-1}} (1-\theta) s^{-\theta} ds \ge (1-\theta) a_{k-1}^2 \rho_{k-1}^{-\theta},$$

since $\rho_{k-1} = \rho_k + a_{k-1}^2$. This gives

$$(n+2)\rho_n^{1-\theta} - \rho_0^{1-\theta} + (1-\theta) \sum_{k=1}^{n} (k+1) a_{k-1}^2 \rho_{k-1}^{-\theta} \le C_1 \sigma_n.$$

It follows that, for every $n \ge 0$,

$$(n+1)\rho_n^{1-\theta} \le C_2(1 + \sigma_n),$$

and

$$\sum_{k=1}^{n} (k+1) a_{k-1}^2 \rho_{k-1}^{-\theta} \le C_2(1 + \sigma_n),$$

where $C_2 > 0$ is a constant independent of n. Since the sequence (σ_n) is nondecreasing, the former estimate yields

$$\rho_{k-1} \le C_3(1 + \sigma_n)^{1/(1-\theta)} k^{-1/(1-\theta)}, \quad 1 \le k \le n < \infty,$$

which we insert into the latter one, thus arriving at

$$\sum_{k=1}^{n} k^{1+\theta/(1-\theta)} a_{k-1}^2 \le C_4(1 + \sigma_n)^{1+\theta/(1-\theta)}.$$

The constants C_3, C_4 are independent of $n \ge 0$. As a consequence, using the Cauchy-Schwarz inequality, we obtain for $n \ge 1$,

$$\begin{aligned} \sigma_{n-1} = \sum_{k=1}^{n} a_{k-1} &\le \left(\sum_{k=1}^{n} k^{1/(1-\theta)} a_{k-1}^2 \right)^{1/2} \left(\sum_{k=1}^{n} k^{-1/(1-\theta)} \right)^{1/2} \\ &\le C_5(1 + \sigma_n)^{1/2(1-\theta)}. \end{aligned}$$

We conclude that the sequence $(\sigma_n)_n$ must be bounded, since $2(1-\theta) > 1$. Indeed, assume by contradiction that $(\sigma_n)_n$ is unbounded, and set $r_n = 1 + \sigma_n \geq 1$. Then (r_n) is nondecreasing, $r_n \to \infty$ and

$$r_{n-1} \leq C_6 r_n^{1/2(1-\theta)},$$

so that $r_{n-1}/r_n \to 0$, and we deduce that $r_n \to 0$. Contradiction. Thus (σ_n) is bounded, i.e. $\sum_{n=0}^{\infty} a_n < \infty$, as claimed. ☐

Remark 5.6. In the proof of convergence to equilibrium, we have used the fact that all norms are equivalent in V_h. An interesting open question would be to prove a similar result for the time semi-discrete version of our problem.

Remark 5.7. By arguing as in the continuous case (see [25]), using the energy estimate (cf. Lemma 3.1) and the Łojasiewicz inequality 5.17, it is possible to prove that any solution (u_h, v_h) of the space semi-discrete scheme (3.1) converges to a single equilibrium, provided assumptions (H1) and (H4) hold.

6. Numerical results

We present some numerical results in one space dimension (obtained with the `Scilab` software[1] and in two space dimensions (obtained with the `Freefem++` software [27]). In every case, the nonlinearity f is given by (1.3) for some parameter ε, and we set $f_+ = f$, $f_- = 0$ in assumption (H3). The space V_h is the space of piecewise linear (P^1) finite elements.

6.1. **Simulations in one space dimension.** We first choose an interval $\Omega = (0, L)$ with $L = 4\pi$. In Table 1, we compute the error in the $C^0([0,T]; L^2(\Omega))$-norm (which appears in Theorem 4.8). The parameters are $\varepsilon = 0.5$, $\beta = 0.5$ and $T = 2$. We use a uniform grid with space stepsize $h = L/M$ and time stepsize $\tau = T/N$. The initial value (u_h^0, v_h^0) is the P^1-interpolate of $u_0(x) = \cos(x) + 0.3\cos(3x)$, $v_0(x) = 0.1$.

Table 1. Convergence error for the time (left) and for the space (right) discretization

$N = T/\tau$	$err_h(\tau)$	ratio
80	0.5018208	-
160	0.1455507	3.45
320	0.0368516	3.95
640	0.0091325	4.04
1280	0.0022163	4.12

$M = L/h$	$err^{\tau}(h)$	ratio
40	0.7770682	-
80	0.2735932	2.84
160	0.0706798	3.87
320	0.0175677	4.02
640	0.0041882	4.20

For the error of the time discretization, $h = L/160$ is fixed. Since the exact solution u_h of the space semi-discrete scheme (3.1)-(3.2) is unknown, we use instead the solution on a fine grid with stepsize $\tau_{sol} = T/5120$. Table 1 (left) shows the error

$$err_h(\tau) = \max_{0 \leq k \leq 5120} \|u_h^{\tau}(t_k) - u_h^{\tau_{sol}}(t_k)\|_{L^2(0,L)}$$

evaluated on the fine grid $t_k = k\tau_{sol}$ ($k = 0, 1, \ldots, 5120$), and the ratio

$$err_h(\tau)/err_h(\tau/2)$$

[1] `Scilab` is freely available at http://www.scilab.org/

of consecutive errors. The computed ratio is close to 4, which means that the convergence error for the time discretization is $O(\tau^2)$, as expected.

For the error of the space discretization, $\tau = T/160$ is fixed. We use again the solution $u^{\tau}_{h_{sol}}$ on a fine grid with stepsize $h = L/2560$ for the comparison. Table (1) (right) shows the error

$$err^{\tau}(h) = \max_{0\le k\le 160} \|u^{\tau}_h(t_k) - u^{\tau}_{h_{sol}}(t_k)\|_{L^2(0,L)}$$

evaulated on the grid $t_k = k\tau$ $(k = 0,\ 1,\ \dots,\ 160)$ and the ratio $err^{\tau}(h)/err^{\tau}(h/2)$ of consecutive errors. Again, the computed ratio is close to 4: the convergence error for the space discretization is $O(h^2)$, as expected.

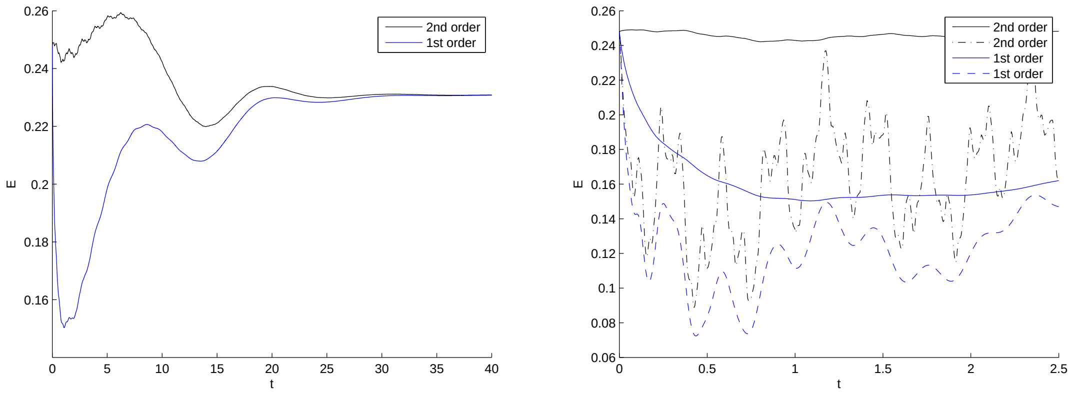

FIGURE 1. (Pseudo-)Energy vs time

Figure 1 shows the plot of the pseudo-energy $\mathcal{E}_h(u^n_h, v^n_h)$ (see (3.22)) versus the time t (in solid line). The domain is $\Omega = (0, L)$ with $L = 4\pi$. The parameters are $\varepsilon = 0.5$, $\beta = 5$, $h = L/320$ and $dt = 0.005$. The initial condition is the P^1-interpolate of $u_0(x) = 0.1/(1 + 0.7\cos(x))$, $v_0 = 0.034$. The black color corresponds to the second-order scheme (4.2); the blue color corresponds to a first-order scheme obtained by applying to the space semi-discrete scheme (3.1) the time discretization proposed by Wang and Wise [37, 38]. Both schemes are unconditionally stable.

The left figure shows the pseudo-energy on the interval $[0, 40]$. If we had $\langle v_0\rangle = 0$, then by (4.16), the pseudo-energy would be nonincreasing in both cases. Here, the pseudo-energy exhibits oscillations due to the fact that $\langle v_0\rangle \neq 0$. In both cases, the evolution is driven to a stationary state, as predicted by the theory (see Theorem 5.1). We notice that the first-order scheme has a smoothing effect which creates more dissipation, especially at the beginning of the evolution. This is seen in the right figure which shows the energy $E_h(u^n_h)$ on the interval $[0, 2.5]$ (in dashed and dashdot), in addition to the pseudo-energy $\mathcal{E}_h(u_h, v_h)$ (in solid line). The difference $\mathcal{E}_h(u^n_h, v^n_h) - E_h(u^n_h) = (\beta/2)|\dot{v}^n_h|^2_{-1,h}$ can be interpreted as a "kinetic energy".

6.2. **Simulations in two space dimensions.** The domain Ω is the square $(0, 6\pi)\times(0, 6\pi)$. It is decomposed in 50×50 squares, and each square is divided along the lower left/upper right diagonal, resulting in a uniform triangulation of ω. The parameters are $\beta = 0.1$, $\varepsilon = 2$ and the time step is $\tau = 0.25$. The initial condition is the P^1-interpolate of $u_0(x, y) = 0.2 + 0.2\cos(x)\cos(y)$ and $v_0 = 0$. Figures 2 and 3 show the evolution from stripes to a triangular distribution of drops. Numerical tests up to time $t = 1250$ indicate that the triangular distribution of drops is the steady state for this simulation.

For the continuous problem (1.1), using the translation invariance, from a triangular distribution of drops we easily build a two dimensional continuum of steady states. For the fully discrete scheme (4.2), the translation invariance is broken by the space discretization, but we expect a large number of steady states. This simulation illustrate the convergence to equilibrium result (Theorem 5.1) in a situation where the steady state is not unique.

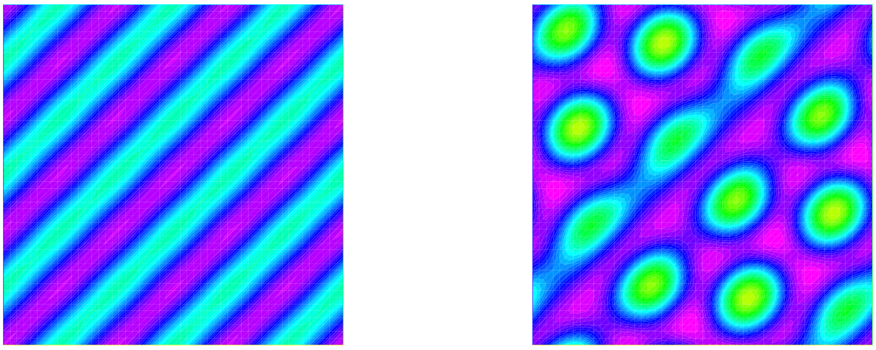

FIGURE 2. $t = 5$ (left) and $t = 26.25$ (right)

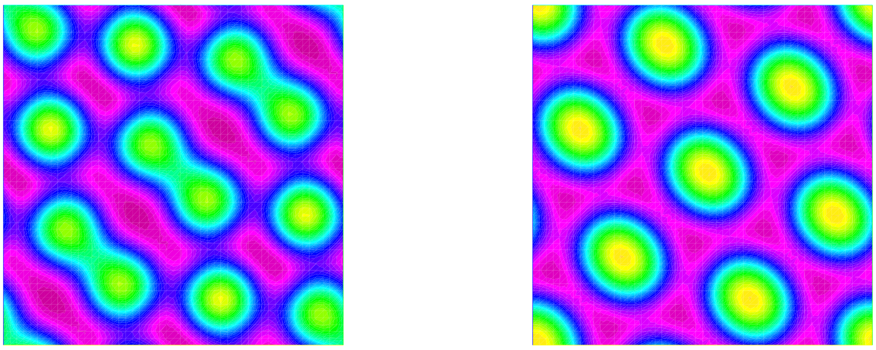

FIGURE 3. $t = 27.75$ (left) and $t = 50$ (right)

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