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Bending of beams of arbitrary cross sections - optimal design by analytical formulae

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Abstract The paper deals with the conceptual design of a beam under bending. The common problem of design-ing a beam in a state of pure bending is discussed in the framework of Pareto-optimality theory. The analytical for-mulation of the Pareto-optimal set is derived by using a procedure based on the reformulation of the Fritz John Pareto-optimality conditions. The shape of the cross section of the beam is defined by a number of design variables pertaining to the optimization process by means of effi-ciency factors. Such efficiency factors are able to describe the bending properties of any beam cross section and can be used to derive analytical formulae. Design performance is determined by the combination of cross section shape, mate-rial and process. Simple expressions for the Pareto-optimal set of a beam of arbitrary cross section shape under bending are derived. This expression can be used at the very early stage of the design to choose a possible cross section shape and material for the beam among optimal solutions.

Keywords Multi-objective optimization · Pareto-optimal set · Ashby material maps · Material selection · Analytical solution · Beam structures · Size optimization

1 Introduction

At the very early stage of the design of a structure, the designer has to make a number of choices that will affect the whole project and that could be extremely costly and time consuming to be modified in a later design stage. Many pos-sible design solutions are initially available. A preliminary optimization can be useful to the designer to get insight into the problem, to estimate the attainable performances.

The conceptual model of the structure, at this initial stage, is generally very simple and a few design variables have to be considered. In Gobbi et al. (2015), a simple and efficient tool is presented to derive the Pareto-optimal set for design problems described by a limited number of design variables and objective functions.

The availability of analytical expressions for the Pareto-optimal set allows the designer to quickly select many possible solutions and to choose the most promising system configurations for the subsequent refined design.

Referring to the simple, but very common, case of the design of a beam under bending, the first decision that has to be made is the material selection. The material selection is not trivial as it involves many conflicting requirements. The beam should be able to sustain the load without failing (structural safety) or becoming unstable (elastic stability). Additionally mass and compliance should be minimized to obtain a light and stiff structure (lightweight design) (Arora 2004; Gobbi and Mastinu 2001).

The comparison of different materials cannot be sepa-rated from the analysis of the cross sections that the beam can assume (Ashby 2011). In Ashby (2011), the effects of the cross section shape on the material selection for beams

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and axles is analysed. Procedures are presented in the book (Ashby 2011) for the correct choice of material and shape. The Pareto-optimality theory is applied to balance cost and performance.

The analytical solution for a single objective optimisation (minimum compliance) of a beam under different and separated loads is derived in Pedersen and Pedersen (2009). The considered design variable is the area of the beam. In Banichuk (1990) the shape optimization of beams under bending or torsion is performed analytically for the different cases of stress, stiffness or stability driven design. The optimization of cylindrical bar cross sections with regular polygonal contours under stiffness and strength constraints is discussed in Banichuk et al. (2002). The problem of finding a minimum area thin-walled closed symmetrical cross section subject to bending with prescribed constant thickness and bending stiffness is solved in Ragnedda and Serra (2005). In Gobbi et al. (2005), Mastinu et al. (2006), Bendsoe and Sigmund (2003), and Erfani et al. (2013), the multi-objective problem referring to the derivation of an optimal beam has been introduced and solved by applying numerical optimization methods. In Gobbi et al. (2015) the multi-objective optimisation of a beam of fixed cross section (rectangular) is solved analytically.

In none of the known literature a comprehensive analytical method is proposed to solve the basic problem of designing a beam subjected to bending. The paper aims at covering this topic. The paper is organized as follows. Firstly, the classical enunciation of the problem of designing a light and stiff beam is stated. Then, shape efficiency factors are defined to generalize the problem to arbitrary cross sections. Finally, in Sections 3 and 4 the analytical procedure for solving a multi-objective problem is presented and applied to the considered design problem.

2 Bending of a beam

In this section the problem of the design of a beam subjected to bending is stated.

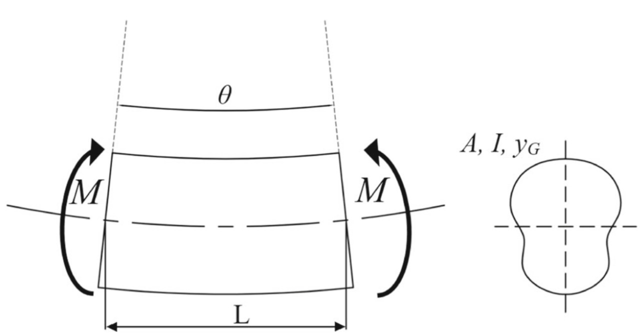


Fig. 1 Cantilever beam subjected to bending moment

Figure 1 shows a beam subjected to bending. The geometry of the cross section is characterized by the three quantities A, I and y_G representing, respectively, the area, the moment of inertia and the maximum distance from the neutral axis of a point of the cross section border, respectively. The cross section is constant along the length of the beam. L is an arbitrary length and, for sake of generality, in the following will be considered equal to 1. By this assumption, any computed value will represent the corresponding quantity per unit length. The beam material is described by its density ρ and its elastic modulus E. According to this notation, with $L = 1$, mass m and compliance c of the beam can be expressed as

$$m = \rho A \quad [kg/m] \tag{2.1}$$

$$c = \frac{\theta}{L} = \frac{M}{EI} \quad [rad/m] \tag{2.2}$$

Mass and compliance have to minimized.

The structural safety of the design is represented by the maximum admissible stress acting on the structure. The constraint can be written as

$$\sigma_{max} \leq \sigma_{adm} \tag{2.3}$$

where σ_{max} is the maximum stress in the structure and can be computed as Young and Budynas (2002)

$$\sigma_{max} = \frac{M y_G}{I} = \frac{M}{Z} \tag{2.4}$$

where Z is the section modulus.

Referring to buckling, a general formula taken from Ashby (2011) will be considered.

2.1 Shape factor for the elastic bending

Equation (2.2) shows that the compliance of a beam due to bending is defined by the elastic modulus E of the material and the moment of inertia of the section I. Considering a square section of edge length b_0 and area $A = b_0^2$, the moment of inertia I_0 of the section reads

$$I_0 = \frac{b_0^4}{12} = \frac{A^2}{12} \tag{2.5}$$

Let us consider the ratio between the compliance c_0 of the square section and the compliance c of an arbitrary cross section of moment of inertia I (Ashby 2011)

$$\phi^e = \frac{c_0}{c} = \frac{M}{EI_0} \cdot \frac{EI}{M} = \frac{I}{I_0} = \frac{12I}{A^2} \tag{2.6}$$

ϕ^e is the *shape factor for elastic bending* (called shape factor in the following) of the cross section and represents its efficiency, in terms of compliance, with respect to the square section. Elongated shapes in the direction orthogonal to the neutral axis have $\phi^e > 1$.

2.2 Stress factor for elastic bending

Be Z_0 the section modulus of a square section

$$Z_0 = \frac{b_0^3}{6} = \frac{A^{3/2}}{6} \tag{2.7}$$

the *stress factor for elastic bending* (called stress factor in the following) ϕ^f can be defined as the ratio between the section modulus of a arbitrary section Z and the section modulus of the square section Z_0

$$\phi^f = \frac{Z}{Z_0} = \frac{6Z}{A^{3/2}} \tag{2.8}$$

ϕ^f represents the efficiency of a cross section, in terms of maximum stress, with respect to the square section.

By combining (2.8), (2.6) and (2.4), ϕ^f can be related to ϕ^e as

$$\phi^f = \frac{\phi^e}{2y_G}\sqrt{A} \tag{2.9}$$

where y_G is the maximum distance from the neutral axis of a point laying on the border of the beam cross section.

2.3 Buckling

Several papers can be found providing accurate buckling formulae for different beams (Young and Budynas 2002; Rotter et al. 2014; Rondal et al. 1992). Buckling is often considered to be an elastic phenomenon, in many practical applications buckling can be caused by local defects (Ashby 2011). Standards (EN-1993-1-1 2005) and manuals (Ashby 2011) suggest simplified formulae that consider, beside the elastic properties of the materials, also the limit stress for the computation of the instability limit. In general, higher resistance material are more prone to local effects, thus the instability limit decreases as the admissible stress increases. A formulation suitable for an early stage design is given in Ashby (2011) and reads

$$\phi^e \le \phi^e_{cr} \simeq 2.3\sqrt{\frac{E}{\sigma_{adm}}} \tag{2.10}$$

Equation (2.10) is well in accordance with standards such as Eurocode (EN-1993-1-1 2005). Moreover, in Ashby (2011) it is shown that often commercially available semi-finished rods comply with the limit of (2.10). In Table 1, upper bounds for shape and stress efficiency factors for some materials taken from Ashby (2011) and based on the analysis of actually available cross sections are reported.

2.4 Optimization problem formulation

Introducing the shape factor into (2.2) and the stress efficiency factor into (2.3) , the problem of the mass and compliance optimization of a beam of arbitrary shape subjected to bending can be stated in mathematical form as follows

Table 1 Upper limits for the shape factor (data from Ashby (2011))

Material	ϕ^e_{cr}
Structural steel	65
6061 aluminum alloy	44
Glass fiber reinforced polymer (GCFRP)	39
Carbon fiber reinforced polymer (CFRP)	39
Nylon	12
Hard wood	5
Elastomers	<6

Given

M	the applied moment (see Fig. 1)	[Nm]
σ_{adm}	the admissible stress of the material	[MPa]
E	the material modulus of elasticity (Young's modulus)	[MPa]
ρ	the material density	[kg/m^3]

and defining

A	the beam cross section area	[m^2]
ϕ^e	the beam cross section shape factor	[-]
ϕ^f	the beam cross section stress factor	[-]
m	the beam mass (per unit of length)	[kg/m]
c	the compliance of the cantilever (rotation per unit of length) due to the moment M	[rad/m]
σ_{max}	the maximum stress	[MPa]
ϕ^e_{cr}	the maximum shape factor for stability	[-]

find A and ϕ^e such that

$$\min\begin{pmatrix} m(A,\phi^e) \\ c(A,\phi^e)\end{pmatrix} = \min\begin{pmatrix} \rho A \\ \frac{12M}{E\phi^e A^2}\end{pmatrix} \tag{2.11}$$

where

$$\begin{aligned} m &= \rho A \\ c &= \tfrac{12M}{E\phi^e A^2}\end{aligned} \tag{2.12}$$

subject to

$$A_{min} \le A \le A_{max} \tag{2.13}$$

$$\phi^e_{min} \le \phi^e \le \phi^e_{max} \tag{2.14}$$

$$\sigma_{max} = \frac{6M}{\phi^f A^{3/2}} \le \sigma_{adm} \tag{2.15}$$

$$\phi^e \le \phi^e_{cr} = 2.3\sqrt{\frac{E}{\sigma_{adm}}} \tag{2.16}$$

By inspecting (2.12), (2.15) and (2.16), it can be observed that the objective functions actually depend on the two design variables A and ϕ^e, but one of the constraint (2.15) depends also on the additional variable ϕ^f (and thus on

y_G). Table 2 reports the expressions of A, ϕ^e and ϕ^f for some commonly used beam cross sections. Referring to such different cross sections, three groups can be identified.

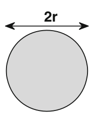
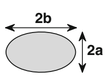
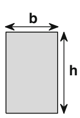
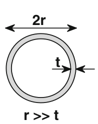
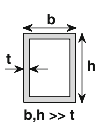
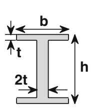
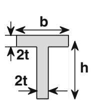
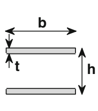
Group 1) Only one parameter is required to define the section. In this case, the number of design variables is less than the number of the objective functions. If A is the free parameter, ϕ^e and ϕ^f can be computed as functions of A (i.e. $\phi^e = \phi^e(A)$ and $\phi^f = \phi^f(A)$). In this case, the solution can be found by simply substituting the constraints expressions into the objective functions (see Section 3 and (Meyer 2001; Mastinu et al. 2006)). Design optimisation is not required. This case will not be further discussed in the paper.

Group 2) Two parameters are required to define the section. In this case, the number of design variables is equal to the number of the objective functions. A and ϕ^e are the free parameters, ϕ^f can be computed as functions of A and ϕ^e (i.e. $\phi^f = \phi^f(A, \phi^e)$).

Group 3) Three (or more) parameters are required to define the section. In this case, the number of design variables is still equal to the number of the objective functions (only A and ϕ^e appear in the expressions of the objective function m and c, see (2.12)). The maximum distance from the neutral axis y_G is an additional parameter that changes the structural safety constraint of the problem ((2.4) and (2.15)) and $\phi^f = \phi^f(A, \phi^e, y_G)$ (2.9).

It can be observed that for Group 3 the number of parameters required to define the beam cross section is higher than the number of objective functions. In this case, each Pareto-optimal solution can be obtained by more than one combination of the cross section parameters. In Appendix A,

Table 2 Formulas for some commonly used beam cross sections

#	Section shape	A	I	y_G	ϕ^e	ϕ^f
1		πr^2	$\frac{\pi r^4}{4}$	r	$\frac{3}{\pi}$	$\frac{3}{2\sqrt{\pi}}$
2		πab	$\frac{\pi a^3 b}{4}$	a	$\frac{3a}{\pi b}$	$\frac{3\sqrt{a}}{2\sqrt{\pi b}}$
3		bh	$\frac{bh^3}{12}$	$\frac{h}{2}$	$\frac{h}{b}$	$\sqrt{\left(\frac{h}{b}\right)}$
4		$2\pi rt$	$\pi r^3 t$	r	$\frac{3r}{\pi t}$	$\frac{3\sqrt{r}}{\sqrt{2\pi t}}$
5		$2t(h+b)$	$\frac{h^2 t(h+3b)}{6}$	$\frac{h}{2}$	$\frac{h^2(h+3b)}{2t(h+b)^2}$	$\frac{h(h+3b)}{\sqrt{2t(h+b)^3}}$
6		$2t(h+b)$	$\frac{h^2 t(h+3b)}{6}$	$\frac{h}{2}$	$\frac{h^2(h+3b)}{2t(h+b)^2}$	$\frac{h(h+3b)}{\sqrt{2t(h+b)^3}}$
7		$2t(h+b)$	$\frac{h^3 t(h+4b)}{6(h+b)}$	$\frac{h(h+2b)}{2(h+b)}$	$\frac{h^3(h+4b)}{2t(h+b)^3}$	$\frac{\sqrt{2}h^2(h+4b)}{2\sqrt{t(h+b)^3}(h+2b)}$
8		$2bt$	$\frac{bth^2}{2}$	$\frac{h}{2}$	$\frac{3h^2}{2bt}$	$\frac{3h}{\sqrt{2bt}}$

this scenario is analysed with reference to a I-shaped cross section beam.

3 Analytical derivation of the Pareto-optimal set

The following theoretical procedure for the analytical derivation of the Pareto-optimal set has been presented in Gobbi et al. (2015) and Levi and Gobbi (2006) along with mathematical examples and the analytical solution of the optimal design of a beam of rectangular section subjected to bending. Here, the procedure is summarized.

Let us consider a general constrained multi-objective optimisation (minimisation) problem

$$\begin{array}{ll} min & \mathbf{F}(\mathbf{x}) = \mathbf{F}(f_1(\mathbf{x}), f_2(\mathbf{x}, ..., f_k(\mathbf{x})) \\ s.t. & \mathbf{G}(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), ..., g_w(\mathbf{x})), \quad \mathbf{x} \in \mathbf{R}^n \end{array} \tag{3.1}$$

where $\mathbf{F}$ is the vector of the objective functions, $\mathbf{x}$ is the vector of the design variables and $\mathbf{G}$ is the vector of the constraints.

The optimal solution of a multi-objective problem of the kind of (3.1) can be found according to the following definition (Pareto-optimal solution)

Definition 3.1 (Pareto-optimal solution) Given a MOP (Multi-Objective Programming) problem with n design variables and k objective functions a Pareto optimal solution (vector) $\mathbf{x}^*$ is that for which there does not exist another solution $\mathbf{x} \in X$ such that:

$$\begin{array}{l} f_j(\mathbf{x}) \leq f_j(\mathbf{x}^*) \quad j = 1, 2, ..., k \\ \exists l: \ f_l(\mathbf{x}) < f_l(\mathbf{x}^*) \end{array} \tag{3.2}$$

For the analytical derivation of the Pareto optimal set, the following approach based on the Fritz John necessary condition can be applied (Gobbi et al. 2015).

Fritz John necessary condition (Miettinen 1999; Mastinu et al. 2006) Let the objective function and the constraint vector of (3.1) be continuously differentiable at a decision vector $\mathbf{x}^* \in S$. A necessary condition for $\mathbf{x}^*$ to be Pareto optimal is that there exist vectors $\boldsymbol{\lambda} \in \mathbf{R}^n \geq \mathbf{0}$ and $\boldsymbol{\mu} \in \mathbf{R}^w \geq \mathbf{0}$ $(\boldsymbol{\lambda}, \boldsymbol{\mu} \neq (\mathbf{0}, \mathbf{0}))$ such that

$$\begin{array}{l} \sum_{i=1}^{k} \lambda_i \nabla f_i(\mathbf{x}^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(\mathbf{x}^*) = \mathbf{0} \\ \mu_j g_j(\mathbf{x}^*) = \mathbf{0} \end{array} \tag{3.3}$$

The condition is also sufficient if the objective functions and the constraints are convex or pseudoconvex (Kim et al. 2001; Askar and Tiwari 2009; Marusciac 1982).

Equation (3.3) can be rearranged in a matrix form as (Levi and Gobbi 2006; Gobbi et al. 2015)

$$\mathbf{L} \cdot \boldsymbol{\delta} = \mathbf{0} \tag{3.4}$$

where $\mathbf{L}$ is a $[(n+w) \times (k+w)]$ matrix defined as

$$\mathbf{L} = \begin{bmatrix} \nabla\mathbf{F} & \nabla\mathbf{G} \\ \mathbf{O} & \mathbf{G} \end{bmatrix} \tag{3.5}$$

with

$$\nabla\mathbf{F} = [\,\nabla f_1 \ \nabla f_2 \ ... \ \nabla f_k] \tag{3.6}$$

$$\nabla\mathbf{G} = [\,\nabla g_1 \ \nabla g_2 \ ... \ \nabla g_w] \tag{3.7}$$

$$\mathbf{G} = diag(g_1, g_2, ..., g_w) \tag{3.8}$$

and $\mathbf{O}$ the null matrix of dimensions $[w \times k]$. $\boldsymbol{\delta}$ is a vector containing $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ ($\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\lambda} & \boldsymbol{\mu} \end{bmatrix}^T \geq \mathbf{0}$).

The Fritz John conditions (see (3.3)) can be relaxed by removing $\boldsymbol{\delta} \geq \mathbf{0}$. This relaxation implies that we are dealing with necessary conditions also in presence of convex objective functions and constraints.

For $n \geq k$, i.e. the number of design variables is equal or greater than the number of objective functions, (3.4) admits non-trivial solution if (Bjorck 1996)

$$\det(\mathbf{L}^{\mathbf{T}}\mathbf{L}) = 0 \tag{3.9}$$

This condition states that the solutions of the optimization problem are those values of the decision vector $\mathbf{x}^* \in S$ for which the $det(\mathbf{L}^{\mathbf{T}}\mathbf{L})$ is equal to zero.

For square $\mathbf{L}$ matrix (i.e. $n = k$, the number of design variables is equal to the number of objective functions) this condition reduces to

$$det(\mathbf{L}) = 0 \tag{3.10}$$

By inspecting (3.10), one may notice that in this case the gradient of the constraints has no influence on the solution. Furthermore, being $\mathbf{G}$ a diagonal matrix, (3.10) can be rewritten as

$$det(\mathbf{L}) = 0 \Rightarrow (\prod_{j=1}^{m} g_j) \cdot det(\nabla\mathbf{F}) = 0 \tag{3.11}$$

and therefore the solution is either an active constraint or the Pareto optimal set of the unconstrained problem (Levi and Gobbi 2006). In fact, if the problem is unconstrained, the $\mathbf{L}$ matrix is $\mathbf{L} = \nabla\mathbf{F}$ and the solution is given by

$$det(\nabla\mathbf{F}) = 0 \tag{3.12}$$

If $n < k$, i.e. the number of design variables is smaller than the number of objective functions, $det(\mathbf{L}^{\mathbf{T}}\mathbf{L})$ is always equal to zero and the problem is no longer a minimization problem. The solution can be found by simply substituting the constraints expressions into the objective functions as explained in Meyer (2001) and Mastinu et al. (2006).

According to this procedure an optimization problem in the form of (3.1) can be analytically solved by the following steps.

1) *Pseudo-convexity check.* The Hessian matrix has to be computed for any objective function or active constraint. If all the computed Hessian matrices are positive

semi-definite, objective functions and constraints are convex, the method can be applied. If this condition is not satisfied, the pseudo-convexity of the functions can be checked as explained in Kim et al. (2001), Askar and Tiwari (2009), and Gobbi et al. (2015). If also this check fails, the method cannot be applied.

2) *Select the equation.* If the system is unconstrained, use (3.11). If the system is constrained, use (3.9) if $n > k$ or (3.10) if $n = k$.
3) *Synthesize the Pareto-optimal set.* The equation computed at the previous step contains the Pareto-optimal set along with non optimal solutions that have to be excluded. Compute the minimum of each objective function separately. These will give the limits to the Pareto-optimal set.

4 Pareto-optimal set for the beam of arbitrary cross section shape subjected to bending

For the analytical solution of the stated optimization problem, let us start by considering the unconstrained problem, i.e. structural and stability constraint ((2.15) and (2.16)) are removed from the problem formulation.

The two objective functions are continuous and differentiable. The beam mass m, being function only of A and monotonically increasing, is pseudo-convex (Gobbi et al. 2015). The Hessian matrix of the beam compliance c reads

$$\mathbf{H} = \frac{24M}{A^2\phi^e}\begin{bmatrix} \frac{3}{A^2} & \frac{1}{A\phi^e} \\ \frac{1}{A\phi^e} & \frac{1}{\phi^{e^2}} \end{bmatrix} \tag{4.1}$$

and has both eigenvalues real and positive. Thus, the beam compliance c is convex (Arora 2004). For a unconstrained problem with the same number of design variables and objective functions, (3.12) can be applied and reads

$$det\,(\Delta\mathbf{F}) = det\left(\begin{bmatrix} \rho & -\frac{24M}{EA^3\phi^e} \\ 0 & -\frac{12M}{EA^2\phi^{e^2}} \end{bmatrix}\right) = -\frac{12M\rho}{EA^2\phi^{e^2}} = 0 \tag{4.2}$$

which has solution only if $A^2\phi^{e^2} \rightarrow \infty$. Such solution not belonging to the set of the positive and finite numbers has no physical meaning. The nonexistence of the Pareto-optimal set can be proved by considering the ϵ-constraint method (Miettinen 1999) and by applying the monotonicity principles (Papalambros and Wilde 2000). The two monotonicity principles are violated, so the problem is not well constrained. In fact, if we apply the ϵ-constraint method and we consider the beam mass (m) as objective function and the beam compliance (c) as (active) design constraint, the objective function m is monotonically increasing with the design variable A, but the design variable ϕ^e, that doesn't occur in m, is not bounded above and below by the active constraint on beam compliance (c) (second monotonicity principle), moreover the design variable A is not bounded below by the active constraint (c) (first monotonicity principle). So, there must exist at least one active constraint (Papalambros and Wilde 2000) and the Pareto-optimal set is defined by the active constraints.

This fact is conceptually very important. In the early design of a beam subjected to bending, constraints play a crucial role.

In other words, the optimal design of a beam subjected to bending is always a constrained optimal solution.

When the buckling constraint is active, the shape factor ϕ^e reads (see Eq. 2.16)

$$\phi^e = 2.3\sqrt{\frac{E}{\sigma_{adm}}} \tag{4.3}$$

By replacing (4.3) into (2.12) and eliminating A, the expression of the beam mass m as function of the beam compliance c can be derived as

$$m = 2\rho\sqrt[4]{\frac{9M^2\sigma_{adm}}{2.3^2E^3}}\frac{1}{\sqrt{c}} \tag{4.4}$$

Equation (4.4) is always decreasing and convex, thus, when active, the buckling constraint is part of the Pareto-optimal set. On the buckling constraint, $m \rightarrow \infty$ if $c \rightarrow 0$ and $m \rightarrow 0$ if $c \rightarrow \infty$.

Referring to the stress constraint, the problem of how to relate ϕ^f to the design variables has to be addressed. For a generic cross section, A, ϕ^e and y_G are independent. By considering A, ϕ^e and y_G as independent, a functional relationship between A, ϕ^e, y_G and ϕ^f can be assumed.

The considered relationship between ϕ_f and A, ϕ_e and y_G reads

$$\phi^f = k\sqrt{\phi^e} \tag{4.5}$$

where k is a positive expression and depends on the cross section shape (see Table 3). For simple cross section shapes, k is a constant. For more complex cross sections shapes, k is function of the cross section parameters.

By replacing (4.5) in (2.15), for any A, ϕ^e is limited by the stress constraint

$$\phi^e \geq \frac{36M^2}{k^2\sigma_{adm}^2A^3} \tag{4.6}$$

By substituting (4.6) in (2.12) and eliminating out A, the expression of the stress constraint, i.e. beam mass m as function of the beam compliance c, can be obtained

$$m \geq \frac{3\rho ME}{k^2\sigma_{adm}^2}c \tag{4.7}$$

Table 3 Computed values of k and r_G for the cross sections of Table 2

Section #	$k = \frac{\phi^f}{\sqrt{\phi^e}}$	$r_G = \sqrt{\frac{I}{A}}$
1	$\frac{\sqrt{3}}{2}$	$\frac{r}{2}$
2	$\frac{\sqrt{3}}{2}$	$\frac{a}{2}$
3	1	$\frac{h}{2\sqrt{3}}$
4	$\sqrt{\frac{3}{2}}$	$\frac{r}{\sqrt{2}}$
5	$\sqrt{\frac{h+3b}{h+b}}$	$\frac{h}{2\sqrt{3}}\sqrt{\frac{(h+3b)}{(h+b)}}$
6	$\sqrt{\frac{h+3b}{h+b}}$	$\frac{h}{2\sqrt{3}}\sqrt{\frac{(h+3b)}{(h+b)}}$
7	$\frac{\sqrt{h(h+4b)}}{h+2b}$	$\frac{h}{2\sqrt{3}}\sqrt{\frac{h(h+4b)}{(h+b)^2}}$
8	$\sqrt{3}$	$\frac{h}{2}$

Section numbers refer to Table 2

The constraint boundary of (4.7) can be obtained by replacing the "$\geq$" with "$=$"

$$m = \frac{3\rho M E}{k^2 \sigma_{adm}^2} c = B \frac{c}{k^2} \tag{4.8}$$

where B is a positive constant. In case of a constant value of k (e.g. for cross sections number 1, 2, 3, 4 and 8 in Table 3), (4.8) is a straight line with positive slope. The boundary condition in (4.8) intersects the buckling constraint.

The intersection between buckling and stress constraints represents the extremum of the Pareto-optimal set for the minimum value of mass.

If k is function of the cross section parameters, the slope of (4.8) has to be studied.

By considering (2.9) and the definition of ϕ^e (2.6), ϕ^f can be rewritten as

$$\phi^f = \sqrt{\phi^e} \frac{\sqrt{\phi^e}}{2 y_G} \sqrt{A} = \sqrt{\phi^e} \cdot \sqrt{3} \frac{r_G}{y_G} \tag{4.9}$$

where $r_G = \sqrt{\frac{I}{A}}$ is the radius of gyration of the cross section and its expressions for the considered cross sections are reported in Table 3.

By replacing (4.9) into (4.5), k can be rewritten as

$$k = \frac{\phi^f}{\sqrt{\phi^e}} = \sqrt{3} \frac{r_G}{y_G} \tag{4.10}$$

In general, the following approximations of y_G and r_G can be considered (see Tables 2 and 3)

$$\begin{aligned} y_G &= \tfrac{h}{2} + \epsilon \propto h \\ r_G &\propto h \end{aligned} \tag{4.11}$$

where ϵ is the distance between the neutral axis and the middle of the height of the cross section.

From (4.10) and (4.11), it follows that k is almost constant and the ratio $\frac{c}{k^2}$ is increasing in the plane $c - m$. This boundary condition intersects the buckling constraint.

For the purpose of the preliminary design, its reasonable to assume that the intersection between buckling and stress constraints represents the extremum of the Pareto-optimal set for the minimum value of mass. Given the sign of the inequality in (2.15), only the portion of the buckling constraint above the stress constraint gives feasible solutions.

The intersection between the buckling and structural safety constraints can be easily computed by simply using (4.4) and (4.7). The coordinates of the intersection in the objective function domain read

$$c = \frac{1}{E} \sqrt[3]{\frac{4k^4 \sigma_{adm}^4}{3M\phi_{cr}^e}} \tag{4.12}$$

$$m = \rho \sqrt[3]{\frac{36M^2}{k^2 \phi_{cr}^e \sigma_{adm}^2}} \tag{4.13}$$

and in the design variables domain read

$$A = \sqrt[3]{\frac{36M^2}{k^2 \phi_{cr}^e \sigma_{adm}^2}} \tag{4.14}$$

$$\phi^e = \phi_{cr}^e \tag{4.15}$$

From the designer perspective, not all materials can assume all the possible shapes. For each material, the minimum attainable mass can be estimated by (4.13) for the maximum k value that the cross section can assume for that material.

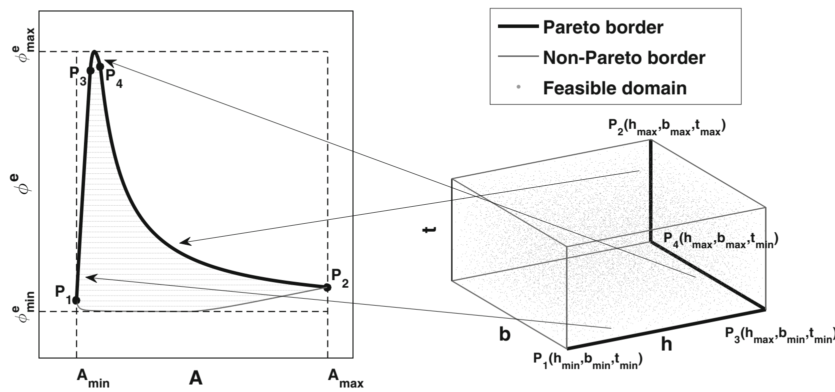
The last boundary constraint is related to the maximum available room for the cross section. To discuss this constraint, the limits A_{min} and A_{max} of (2.13) and of ϕ_{min}^e and ϕ_{max}^e of (2.14) have to be analysed.

For each cross section, A_{max} represents the maximum area of the cross section that can be fitted in the allowable room. In fact, from a engineering point of view, the maximum allowable room is not described by the area, but, in general, by the maximum height and the maximum width that the section can assume. Depending on the shape of the section, a maximum value of area can be derived. The minimum value of A (A_{min}) is given by the minimum dimensions that can be realized (depending for instance on material and technological constraints). Each of these two limits is related to a value of ϕ^e given by the corresponding values of the section parameters.

In a similar way, the values of ϕ_{min}^e and ϕ_{max}^e are given by the possible combinations of the parameters of each cross section. A corresponding value of A can be computed for these two values. Fig. 2 depicts this situation for a I shaped cross section.

If complying with the other constraints, the two solutions P_1 (minimum area) and P_2 (maximum area) of Fig. 2 are the points at minimum mass and minimum compliance

Fig. 2 Pareto-optimal set in the design variables domain for a I shaped cross section when only the available room boundary constraint is considered. *Left*: $A-\phi^e$ plane. *Right*: cross section parameters plane. Parameters definition in Table 2



respectively and are the extrema of the Pareto-optimal set for the considered shape.

According to (4.2), the Pareto-optimal set can be only on the border of the design variables space. The Pareto-optimal set is connected (Naccache 1978; Benoist 1998). Therefore, to connect P_1 to P_2, only two options are available. The Pareto-optimal set is one of the two border lines connecting P_1 and P_2. From (2.12), it can immediately be seen that the upper border line dominates the bottom border line of Fig. 2 and therefore represents the Pareto-optimal set.

Another consequence of (4.2) is that in the n-dimensional space of the cross section parameters (i.e. the actual design parameters of any cross section), the Pareto-optimal set can be only on the edges of the domain. This can be easily proved by fixing all the parameters but one. To obtain a unbounded value of the expression $A^2\phi^{e^2}$, the remaining parameter has to become infinity or zero. But, being each parameter bounded, it follows that the considered parameter must assume either its minimum or maximum value. In other words, on the Pareto-optimal set cross section parameters vary one at a time from their minimum to their maximum value (or vice-versa).

If the problem has solutions for a given cross section, point P_2 must be reachable for that cross section. At P_2 the area of the cross section is at its maximum. It is of clear engineering interest the knowledge of the only cross section parameter to be varied starting from P_2 to reduce the area of the cross section (on the Pareto-optimal set, the cross section parameters vary one at a time).

In order to remain on one edge of the domain and on the upper bound in the plane $A-\phi^e$, the parameter to change is the one that minimizes the derivative of ϕ^e with respect to A at P_2. Calling $\mathbf{p}$ the vector of the parameters of the cross section, the minimum of the derivative can be computed as

$$\min\left(\left.\frac{d\phi^e\left(A\left(\mathbf{p}\right)\right)}{dA\left(\mathbf{p}\right)}\right|_{@P_2}\right)=\min\left(\left[\left.\frac{\frac{\partial\phi^e(\mathbf{p})}{\partial p_i}}{\frac{\partial A(\mathbf{p})}{\partial p_i}}\right|_{@P_2}\right]_{i=1\ldots n}\cdot\mathbf{v}\right) \tag{4.16}$$

where p_i is one of the n parameters of the cross section and $\mathbf{v}$ is a unit direction vector in the space of the cross section parameters. Since from (4.2) the Pareto-optimal set can be only on the edges of the domain, the minimum of the derivative can be found only when $\mathbf{v}$ coincides with one of such directions. Therefore, $\mathbf{v}$ can assume only combinations

Table 4 Computed values of the derivatives of ϕ^e with respect to A for the solution with maximum mass and minimum compliance for the cross sections reported in Table 2

Section #	Parameters	Derivatives
1	r	$\left.\frac{d\phi^e}{dr}\Big/\frac{dA}{dr}\right\|_{@P_2}=0$
2	a, b	$\left.\frac{d\phi^e}{da}\Big/\frac{dA}{da}\right\|_{@P_2}=\frac{3}{\pi^2 b_{max}^2}$
		$\left.\frac{d\phi^e}{db}\Big/\frac{dA}{db}\right\|_{@P_2}=-\frac{3}{\pi^2 b^2}$
3	h, b	$\left.\frac{d\phi^e}{dh}\Big/\frac{dA}{dh}\right\|_{@P_2}=\frac{1}{b_{max}^2}$
		$\left.\frac{d\phi^e}{db}\Big/\frac{dA}{db}\right\|_{@P_2}=-\frac{1}{b^2}$
4	r, t	$\left.\frac{d\phi^e}{dr}\Big/\frac{dA}{dr}\right\|_{@P_2}=\frac{3}{2\pi^2 t_{max}^2}$
		$\left.\frac{d\phi^e}{dt}\Big/\frac{dA}{dt}\right\|_{@P_2}=-\frac{3}{2\pi^2 t^2}$
5 & 6	h, b, t	$\left.\frac{d\phi^e}{dh}\Big/\frac{dA}{dh}\right\|_{@P_2}=\frac{h(6b_{max}^2+3b_{max}h+h^2)}{4t_{max}^2(h+b_{max})^3}$
		$\left.\frac{d\phi^e}{db}\Big/\frac{dA}{db}\right\|_{@P_2}=-\frac{h_{max}^2(3b-h_{max})}{4t_{max}^2(h_{max}+b)^3}$
		$\left.\frac{d\phi^e}{dt}\Big/\frac{dA}{dt}\right\|_{@P_2}=-\frac{h_{max}^2(3b_{max}+h_{max})}{4t^2(h_{max}+b_{max})^3}$
7	h, b, t	$\left.\frac{d\phi^e}{dh}\Big/\frac{dA}{dh}\right\|_{@P_2}=\frac{h^2(12b_{max}^2+4b_{max}h+h^2)}{4t_{max}^2(h+b_{max})^3}$
		$\left.\frac{d\phi^e}{db}\Big/\frac{dA}{db}\right\|_{@P_2}=-\frac{h_{max}^3(8b-h_{max})}{4t_{max}^2(h_{max}+b)^4}$
		$\left.\frac{d\phi^e}{dt}\Big/\frac{dA}{dt}\right\|_{@P_2}=-\frac{h_{max}^3(4b_{max}+h_{max})}{4t^2(h_{max}+b_{max})^4}$
8	b, t	$\left.\frac{d\phi^e}{db}\Big/\frac{dA}{db}\right\|_{@P_2}=-\frac{3h_{max}^2}{4b^2 t_{max}^2}$
		$\left.\frac{d\phi^e}{dt}\Big/\frac{dA}{dt}\right\|_{@P_2}=-\frac{3h_{max}^2}{4b_{max}^2 t^2}$

Section numbers refer to Table 2

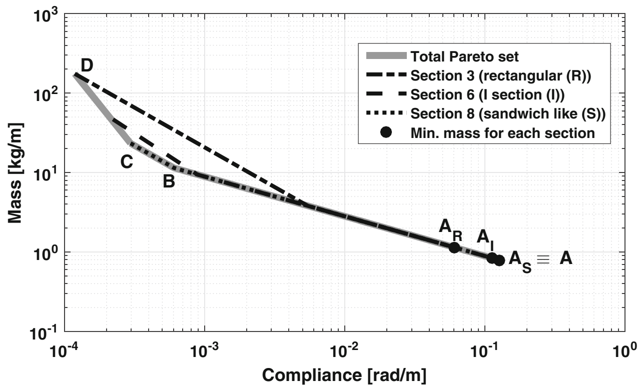


Fig. 3 Pareto-optimal set in the objective functions domain for a mild steel beam subjected to bending moment. Data in Table 5. Section numbers refer to Table 2. Applied moment 1000 Nm, $h_{max} = b_{max} = 0.15$ m, $t_{max} = 0.01$ m

where a single component is one and all the other components are zeros. By replacing this condition in (4.16), the minimum of the derivative of ϕ^e with respect to A at P_2 can be computed as

$$\min \left(\left. \frac{d\phi^e \left(A\left(\mathbf{p}\right)\right)}{dA\left(\mathbf{p}\right)} \right|_{@P_2} \right) = \min_{i=1\ldots n} \left(\left. \frac{\frac{\partial \phi^e(\mathbf{p})}{\partial p_i}}{\frac{\partial A(\mathbf{p})}{\partial p_i}} \right|_{@P_2} \right) \tag{4.17}$$

Equation (4.17) states the condition to find the unique cross section parameter that has to be changed to move from the point at minimum compliance along the Pareto-optimal set.

It can be observed that if we call h the parameter describing the height of the section, both $\frac{d\phi^e}{dh}$ and $\frac{dA}{dh}$ are positive for any combination of the other parameters. Therefore $\frac{d\phi^e}{dh} / \frac{dA}{dh}$ is positive and the height of the cross section is never the first parameter to be changed. Actually, being this ratio always positive, starting from P_2, the height of the cross section can be reduced only after the maximum value of ϕ^e has been reached.

In practical problems, a different constraint (elastic stability or structural safety) is usually reached before ϕ^e gets to its maximum values. The condition $h = h_{max}$ (being h_{max} the maximum value of h) is always part of the Pareto-optimal set. In most cases, the switch between the available room constraint and another constraint happens with $h = h_{max}$ still active. This conclusion is general, being always possible to parametrize a section in order to have its height as parameter.

In Table 4 the derivatives of ϕ^e with respect to A at point P_2 of maximum mass and minimum compliance for the cross sections of Table 2 are reported.

Referring to different cross section shapes, the absolute minimum value of compliance, corresponding to the maximum possible value of mass, is given by a rectangular section with height coincident with the maximum corss section height and width coincident with the maximum width. The rigorous way to compute the Pareto-optimal set at low values of compliance is to order all the possible cross sections for decreasing levels of efficiency up to the rectangular cross section. A reasonable way of approximating this curve, is to linearly connect the point at minimum compliance of the most efficient cross section to the absolute minimum of the compliance (rectangular cross section).

In Fig. 3, the Pareto-optimal set in the objective functions domain is shown for a steel beam (data in Table 5) considering shapes 3, 6 and 8 of Table 2. The applied moment is 1000 Nm and the allowable space is a square with side length 0.15 m. A maximum section thickness of 0.01 m is considered.

The Pareto-optimal set of Fig. 3 can be divided into three regions:

1) Buckling constraint: from point **A** to point **B**. In this region the buckling constraint is active. The curve is the same for all shapes, just the limits vary (points $\mathbf{A_R}$ for the rectangular section, $\mathbf{A_I}$ for the I section and $\mathbf{A_S}$ for the sandwich like section, see (4.13) and Table 3). The minimum absolute mass (point **A**) is given by the shape with the highest value of the constant k in Table 3 (in this case shape 8, see Table 2, i.e. $\mathbf{A} \equiv \mathbf{A_S}$). Solution **B** is given by the most efficient shape (in this case shape 8, see Table 2) when the maximum section height is reached.
2) Available room constraint on the most efficient cross section: from point **B** to point **C**. The curve is given by the most efficient shape along the maximum height curve up to the buckling constraint (in this case shape 8,

Table 5 Material data

Material	Limit stress [MPa]	Elastic modulus [MPa]	Density [Mg/m^3]	ϕ^{e}_{cr} *
Structural steel	500	200000	7.8	44
6061 aluminum alloy	200	70000	2.7	43
Hard wood	60	13500	0.9	5

(∗) minimum value between the value computed by (2.16) and the practical limits of Table 1 see Table 2). Point **C** represents the solution with minimum compliance, and maximum mass, for the most efficient shape, i.e. the solution with the most efficient shape when all the cross section parameters reach their maximum value.

3) Approximation of the available room constraint on other possible cross sections: from point **C** to point **D**. The curve linearly connects the point at minimum compliance on the most efficient cross section to the absolute minimum of compliance (rectangular section, section number 3 in Table 2). This curve roughly interpolates the points at minimum compliance of the other cross sections. Not all points of this curve can be actually obtained. However, this part of the Pareto-optimal set is of limited practical use, benig this design region very close to the maximum mass. In this region, a small decrement of the compliance costs a large increase in mass.

In Fig. 4, the Pareto-optimal sets referring to three different materials (structural steel, aluminium alloy and hard wood, data in Table 5) for the same load and geometrical limits of the previous example are reported. For the wood, rectangular sections only have been considered.

As expected, aluminium shows better performances than the other considered materials for low values of mass, while steel is the best choice for low levels of compliance. Hard wood, although has the highest ratio E/ρ, shows the worst performance among the considered materials. This is due to the limitation in the cross section shapes that have been considered for wood, which strongly limits its performance. Topologically different cross section configurations are allowed by using metals.

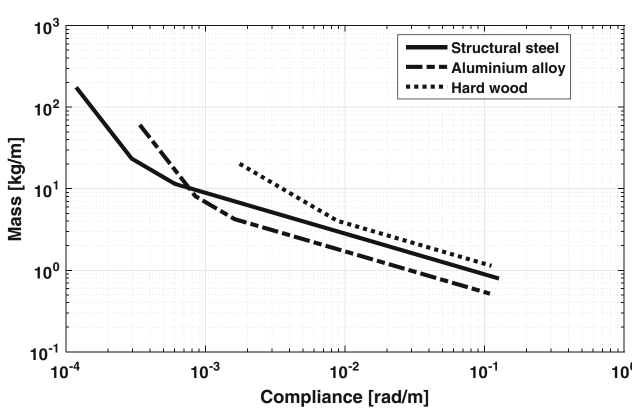


Fig. 4 Pareto-optimal set in the objective functions domain for mild steel, aluminium alloy and wood beams subjected to bending moment. Data in Table 5. Applied moment 1000 Nm, $h_{max} = b_{max} = 0.15$ m, $t_{max} = 0.01$ m

5 Conclusion

In this paper, Pareto-optimality theory has been applied to the study of a beam subjected to bending.

The Pareto-optimal set for a beam of arbitrary cross section shape and material has been derived analytically under the constraints of allowable room, structural safety (maximum stress) and elastic stability (buckling). The analytical derivation has proved that the Pareto-optimal set is composed by two connected regions. The first region is given by the elastic stability constraint while the second region is given by the available room constraint. In particular, the limit on the maximum height of the beam section is the leading parameter when the room constraint is active.

The structural safety constraint is not part of the Pareto-optimal set. Structural safety limits the minimum mass of the beam. In fact, the point at minimum mass is given by the intersection between the elastic stability and the structural safety constraints.

For the elastic stability, a simplified buckling formulation that includes also manufacturing considerations has been chosen. This formulation is quite conservative. In case it is required to exploit the limit performances of a material, the approach used in this paper can still be applied by changing the buckling formulation.

The proposed approach can be effectively used by the designer at a very early stage of the project when the material and the shape of the structure has still to be defined.

Appendix A: I-shaped cross section

A I-shaped cross section (Group 3 in Table 2) steel beam is considered in the following.

The Pareto-optimal solutions in the objective functions domain are reported in Fig. 7. Each optimal solution in the objective functions domain corresponds biunivocally to a single solution A, ϕ^e in the design variables domain, as shown in Fig. 6. Being the I-shaped cross section defined by three parameters, namely h, b, t (see Table 2), each Pareto-optimal solution in the design variables domain can be obtained by more than one combination of the cross section parameters. This is clearly shown in Fig. 5. The optimal design solutions with the same level of grey are defined by different vaues of h, b, t, but they have exactly the same performance in terms of mass m and compliance c (and the same value of A and ϕ^e).

Point A_I, in Figs. 5, 6, 7 represents the solution with minimum mass; point B_I, the switching point between the buckling constraint and the available room constraint; point D, the solution with minimum compliance.

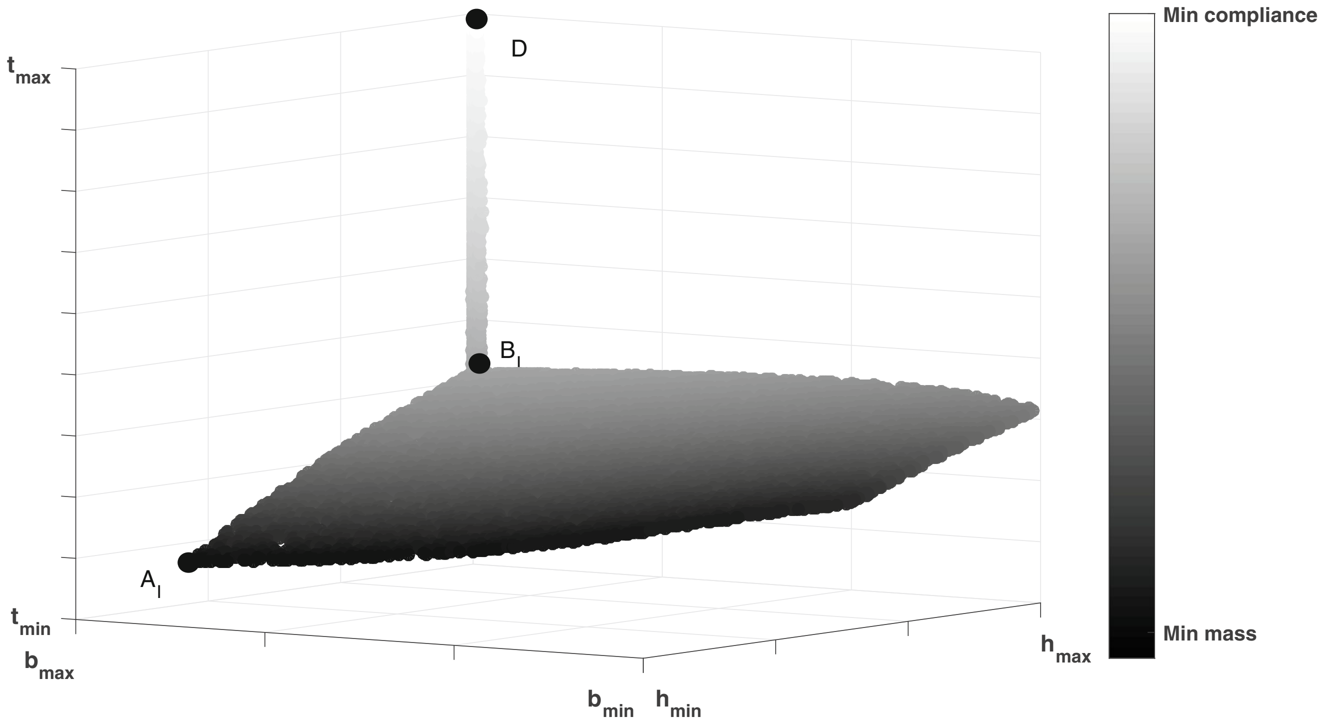


Fig. 5 Pareto-optimal set for a steel I beam in the cross section parameters space. Parameters definition in Table 2. Material data in Table 5. Applied moment 1000 Nm, $h_{max} = b_{max} = 0.15$ m, $t_{max} = 0.01$ m

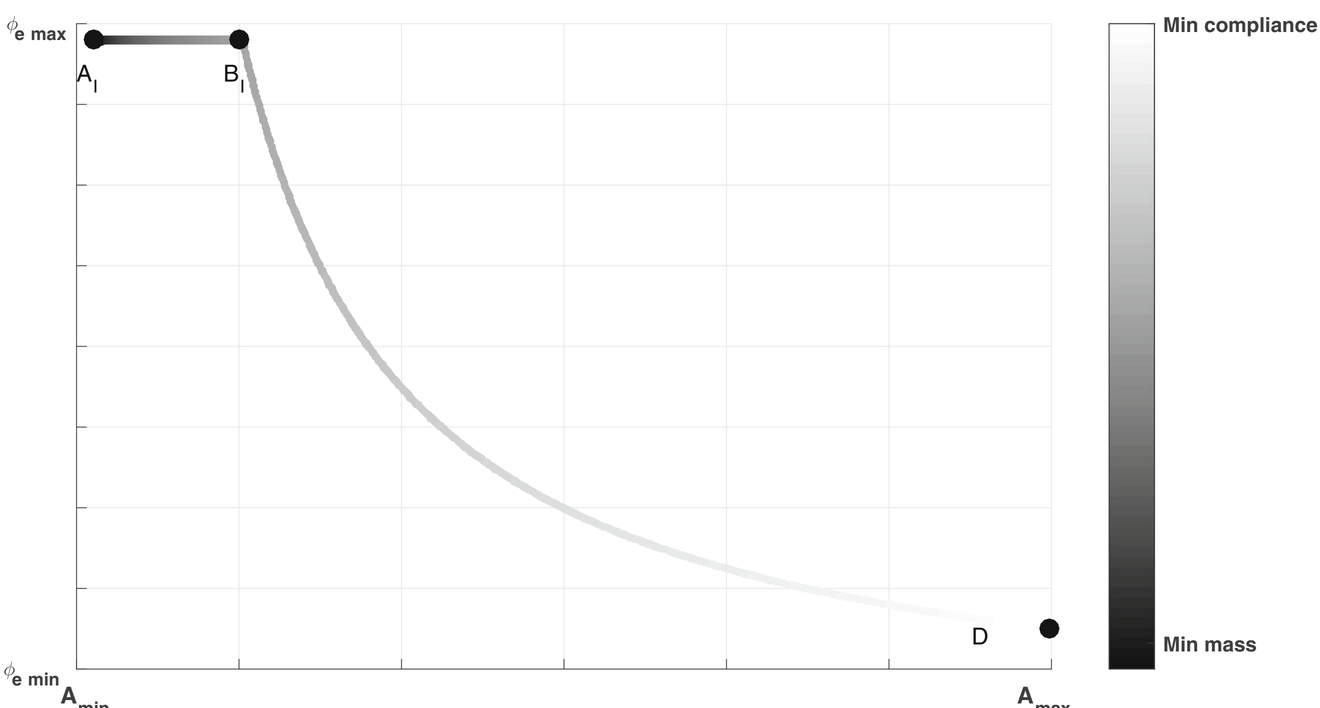


Fig. 6 Pareto-optimal set in the design variables domain for a I shaped steel cross section. Material data in Table 5. Applied moment 1000 Nm, $h_{max} = b_{max} = 0.15$ m, $t_{max} = 0.01$ m

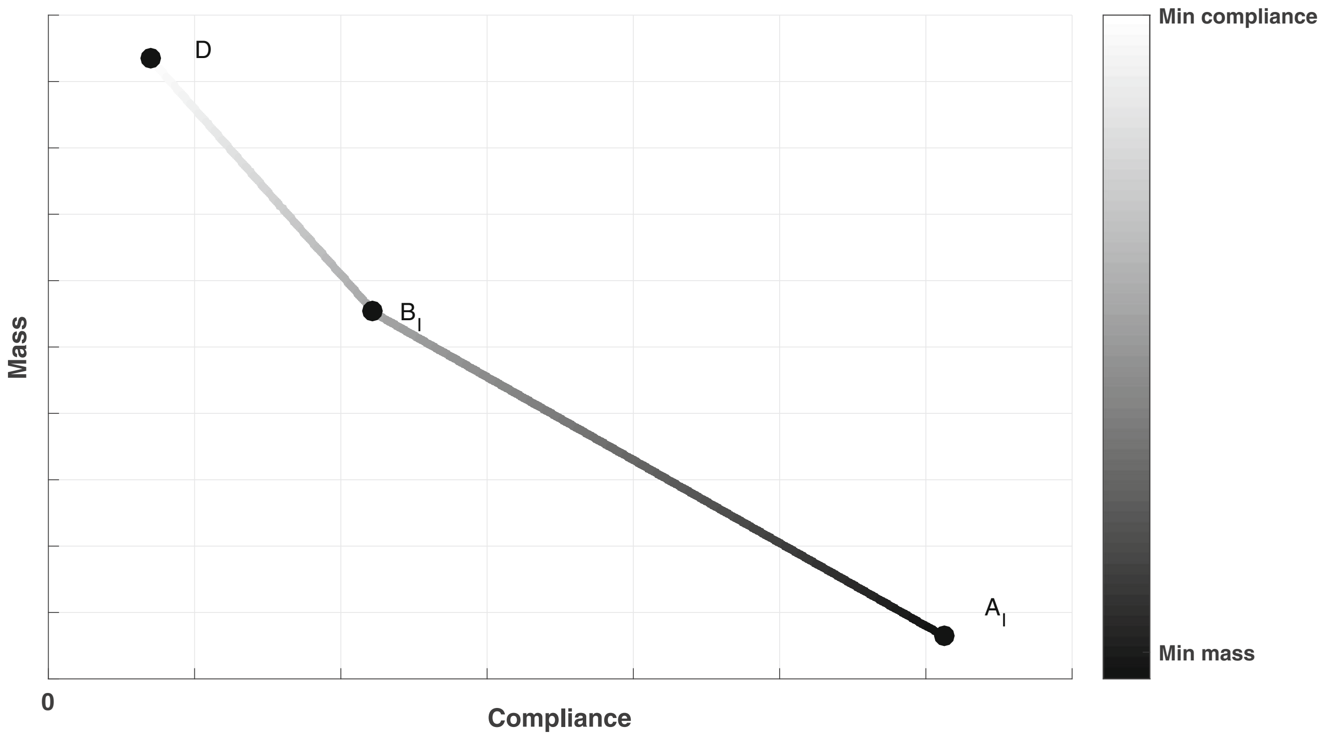


Fig. 7 Pareto-optimal set in the objective functions domain for a I shaped steel cross section. Material data in Table 5. Applied moment 1000 Nm, $h_{max} = b_{max} = 0.15$ m, $t_{max} = 0.01$ m

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